Systems of Equations with TI-Nspire™ CAS
Substitution and Elimination

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Attribution

Most of the examples in this article are from *A First Course in Linear Algebra an Open Text* by Lyrix Learning, base textbook version 2017 - revision A, by K. Kuttler.

1 Introduction

This is the first of several articles about solving systems of linear equations with TI-Nspire. This article describes two methods for solving these systems: the substitution method and the elimination method.

The TI-Nspire demonstrations and examples for this article require the CAS version of TI-Nspire.

2 Definitions and Terminology

2.1 Linear Equation

A *linear equation* in n variables is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where a_i are coefficients and x_i are variables. The coefficients are usually real numbers, but may be arbitrary expressions, as long as the expressions do not contain any of the variables. At least one of the coefficients must not be equal to zero. The variables must be of degree one and must not contain products of the variables.

An example of a linear equation in two variables is the standard linear equation

$$2x + 3y = 10$$

By solving the equation for y, the equation can be expressed as a function $y = f(x)$ whose graph is a line in the two-dimensional coordinate system.

An example of a linear equation in three variables is

$$2x + 3y + z = 10$$

Solving this equation for z results in a function of two variables $z = f(x, y)$ whose graph is a plane in the three-dimensional coordinate system.

Examples of equations which are *non-linear* are

$$2x^2 + 3y = 10$$
$$2xy + 3y = 10$$
$$2x + 3^xy = 10$$

2.2 Systems of Linear Equations

A *system of equations* consists of two or more equations, each containing one or more variables. If all the equations in a system of equations are *linear*, the system is a *system* *of linear equations*. The general form for a system of n linear equations in n unknowns is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

If all the equations in the system equal zero ($b_i = 0$), the system is called a *homogeneous system*.

The usual way to write a system of equations is by placing an open parenthesis to the left of the equations. A couple of examples of linear equations written with this notation are

$$\begin{cases} 3x + \ \ y &= 3 \\ \ x + 2y &= 1 \end{cases} \qquad \begin{cases} x + 3y + \ 6z &= 25 \\ 2x + 7y + 14z &= 58 \\ \qquad 2y + \ 5z &= 19 \end{cases}$$

Systems of equations are defined in a TI-Nspire **Calculator** page with the `system()` function or with the **system of equations** template in the **Math Templates** pane in the **Documents Toolbox**. The `system()` function is added to a calculator page with the keyboard/keypad or by selecting it from the **Catalog** pane. Examples of defining the above systems of equations in a calculator page are

$eq2\_1:=3\cdot x+2\cdot y=3$	$3\cdot x+2\cdot y=3$
$eq2\_2:=x+2\cdot y=1$	$x+2\cdot y=1$
$sys2:=\begin{cases} eq2\_1 \\ eq2\_2 \end{cases}$	$\{3\cdot x+2\cdot y=3,x+2\cdot y=1\}$
$eq3\_1:=x+2\cdot y+6\cdot z=25$	$x+2\cdot y+6\cdot z=25$
$eq3\_2:=2\cdot x+7\cdot y+14\cdot z=58$	$2\cdot x+7\cdot y+14\cdot z=58$
$eq3\_3:=2\cdot y+5\cdot z=19$	$2\cdot y+5\cdot z=19$
$sys3:=\begin{cases} eq3\_1 \\ eq3\_2 \\ eq3\_3 \end{cases}$	$\{x+2\cdot y+6\cdot z=25,2\cdot x+7\cdot y+14\cdot z=58,2\cdot y+5\cdot z=19\}$

Note: After typing the `system(...)` function and pressing the **enter** key, TI-Nspire replaces the entry with the *system of equations* template.

The system can also be defined by simply adding the equations to a list:

$sys:=\{eq2\_1,eq2\_2\}$	$\{3\cdot x+2\cdot y=3,x+2\cdot y=1\}$

2.3 Solutions of Systems of Linear Equations

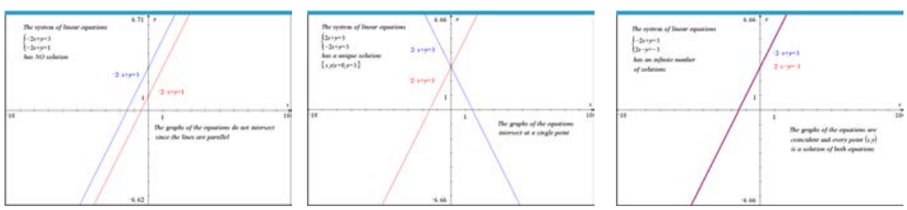
A solution to a system of linear equations consists of the point or set of points that satisfy all of the equations in the system.

A system of linear equations may have

- No solution,
- a unique solution,
- an infinite number of solutions.

In $\mathbb{R}^2$ (two-dimensional space), the graph of an equation in two variables is a line. Two lines in $\mathbb{R}^2$ either do not intersect (when the lines are parallel), intersect in one point, or intersect in an infinite number of points (when the lines coincide).

Figure 1 illustrates all three cases for two linear equations in two variables.



(a) No Solution (b) Unique Solution (c) Infinite Solutions

Figure 1: Solutions of Two Linear Equations in Two Variables

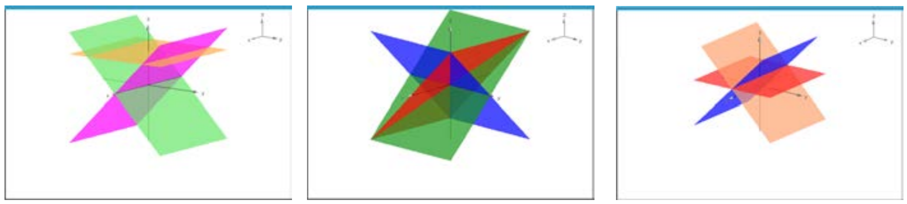
Figure 1a shows the case where the graphs of the equations are parallel lines. Since parallel lines never intersect, the equations have no points in common and thus no simultaneous solution. A system with no solution is *inconsistent*.

Figure 1b shows the case where the graphs of the equations intersect in a single point. The solution to the set of equations is the (x, y) intersection point. A system with a single solution is *consistent* and the equations are *independent*.

Figure 1c shows the case where the graphs of the equations are coincident lines. Since every point on the graph of the first line is also on the graph of the second line, the solution set for the simultaneous equations consists of an infinite number of solutions. A system with an infinite number of solutions is *consistent* and the equations are *dependent*.

In $\mathbb{R}^3$ (three-dimensional space), the graph of an equation of three variables is a plane. Analogous to lines in $\mathbb{R}^2$, three planes either have no points in common, have a single point in common, or have an infinite number of points in common.

Figure 2 illustrates all three cases for three linear equations in three variables.



(a) No Solution (b) Unique Solution (c) Infinite Solutions

Figure 2: Solutions of Three Linear Equations in Three Variables

Figure 2a shows the case where the graphs of the equations intersect, but none of the three equations have any points of intersection in common. This also occurs when the graphs of the planes are parallel and do not intersect at all. In this case, there is no simultaneous solution to the system and the system is *inconsistent*.

Figure 2b shows the case where the graphs of the equations intersect and only one of the intersection points, $(x, y, z) = (0, 0, 0)$, is common to all three planes. The solution to the set of equations is the single intersection point and the system is *consistent* and the equations are *independent*.

Figure 2c shows the case where the graphs of the equations intersect and the intersection points form a line. Since every point on the line of intersection is also a point on each of the planes, the solution set for the system of equations consists of an infinite number of solutions. This case also occurs when the graphs of the equations coincide. A system with an infinite number of solutions is *consistent* and the equations are *dependent*.

3 Solution Methods

3.1 Graphical Method

The TI-Nspire **Graphs Application** provides excellent functionality for analyzing two-dimensional equations. Determining where graphs of equations intersect is easily accomplished by selecting the **Document Tools - Analyze Graph - Intersection** menu item, then using the mouse to select the lower and upper bounds of the region containing the intersection coordinates. When this action is completed, the intersection point is displayed, along with the (x, y) coordinates of the point. The displayed coordinates where two or more graphs intersect are the x, y values that are the solution to the system of equations. Figure 3 illustrates solving the following system of linear equations in a graph page.

$$\begin{cases} 2x+y &= 3 \\ -2x+y &= 3 \\ 3x-y &= -3 \end{cases}$$

Solving systems of equations graphically works well only for equations that can

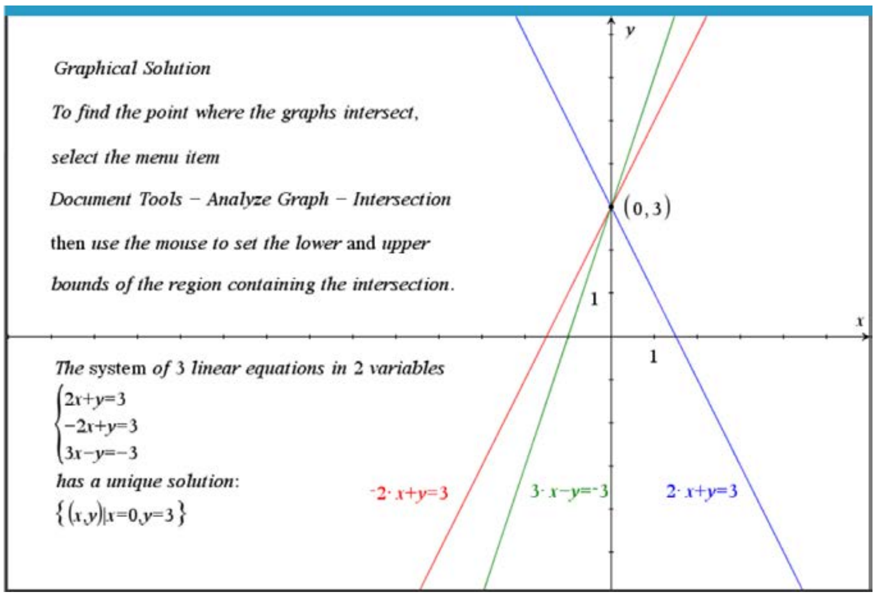


Figure 3: Graphical Solution to a System of Equations

be graphed in two-dimensional graph pages. TI-Nspire's three-dimensional graphing functionality does not support analyzing three-dimensional graphs, so this technique can not be used to find solutions for systems of equations with three variables.

3.2 Builtin TI-Nspire Functions

There are two TI-Nspire functions for solving systems of equations: `linSolve()` and `solve()` (or `cSolve()`). `solve()` is a general-purpose function for solving single equations and systems of equations, both linear and non-linear. `cSolve()` is a version of `solve()` that works with *complex* values. `linSolve()` is a special-purpose function specifically for solving single linear equations and systems of linear equations.

There are several different formats of input arguments for both these functions. The most convenient input format is a list of equations, followed by the solution variables separated by a comma:

```
 solution := solve({eqn1,eqn2,...},var1,var2,...)
solution := linSolve({eqn1,eqn2,...},var1,var2,...)
```

Refer to the *TI-Nspire™ CAS Reference Guide* for detailed descriptions of these two builtin functions.

The following examples demonstrate how to use these two functions to solve simple systems of linear equations in a calculator page.

Solving a system of linear equations that has a unique solution (see Figure 1b):

$\text{linSolve}(\{2\cdot x+y=3, {}^-2\cdot x+y=3\},x,y)$	$\{0,3\}$
$\text{solve}(\{2\cdot x+y=3, {}^-2\cdot x+y=3\},x,y)$	$x=0 \text{ and } y=3$

Solving a system of linear equations that has a no solution (see Figure 1a):

$\text{linSolve}(\{{}^-2\cdot x+y=3, {}^-2\cdot x+y=1\},x,y)$	"No solution found"
$\text{solve}(\{{}^-2\cdot x+y=3, {}^-2\cdot x+y=1\},x,y)$	false

Solving a system of linear equations that has multiple solutions (see Figure 1c):

$\text{linSolve}(\{{}^-2\cdot x+y=3, 2\cdot x-y={}^-3\},x,y)$	$\left\{\frac{c1-3}{2},c1\right\}$
$\text{solve}(\{{}^-2\cdot x+y=3, 2\cdot x-y={}^-3\},x,y)$	$x=\frac{c2-3}{2} \text{ and } y=c2$

The variables **C1** and **C2** in the last example indicate that the *y* variable can be any value, and that the value of the *x* variable is dependent on the value of the *y* variable (the system of equations is **consistent** and the equations are **dependent**). Because the value of the independent variable, *y*, can be any value, the independent variable is called a *free* variable. There may be more than one free variable in a dependent system. Free variables will be discussed in greater detail in a future article about matrices.

Although these builtin functions are convenient, most math instructors require that students explicitly solve systems of equations step-by-step. The following sections show how to do this with TI-Nspire using both the substitution method and the elimination method.

3.3 The Substitution Method

The substitution method is a simple algebraic method for solving systems of equations consisting of equations in two or three variables. For a system of equations in two variables, the method involves the following steps:

1. Pick one of the equations and solve it for one of the unknown variables in terms of the other unknown variable.

2. Substitute the solution in a different equation, resulting in an equation with only

one variable. Solve this equation for the actual value of the unknown variable.

3. Substitute the actual value of the variable in the other equation, then solve this equation for the value of the remaining unknown variable.

4. If the last substitution results in an invalid equality such as $a = b$ (a *false* statement), the system has no solution.

   If the last substitution results in an equality such as $a = a$ (a *true* statement), the system has an infinite number of solutions.

   Otherwise, the unique solution to the system consists of the values found for the two unknown variables.

3.3.1 Substitution Examples with TI-Nspire CAS

The following three examples demonstrate solving systems of two linear equations in two variables in a TI-Nspire calculator page. The examples use the builtin functions `solve()`, `linSolve()`, `left()`, `right()`, and the **constraint** operator (`|`). The `left()` and `right()` functions are used to extract the left-hand and right-hand sides of an equation. The constraint operator is used to perform **substitution**; *i.e.*, replace an expression with another expression.

Substitution Example 1. Use the substitution method to solve the system of equations

$$\begin{cases} 2x+y &= 3 \\ -2x+y &= 3 \end{cases}$$

Figure 4 shows the graphs of these two equations.

To solve this system using substitution in a calculator page, first define the equations:

example11:=2· *x*+*y*=3	2· *x*+*y*=3
example12:=⁻2· *x*+*y*=3	*y*−2· *x*=3

Solve the second equation for y in terms of x and substitute the expression in the first equation, resulting an equation in the single variable x:

© solve example12 for y	
sol12y:=solve(*example12*,*y*)	*y*=2· *x*+3
© substitute the solution for y in example11	
example11a:=*example11*|*sol12y*	4· *x*+3=3
© now solve for x	
sol1x:=solve(*example11a*,*x*)	*x*=0

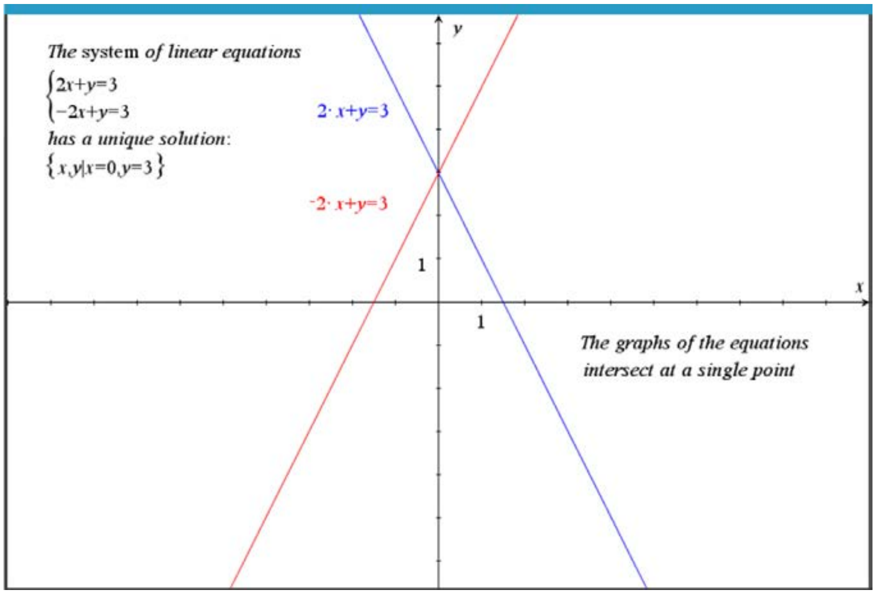


Figure 4: Graphs of Equations for Substitution Example 1

© substitute the solution for y in example11	
example11a:=*example11*|*sol12y*	4· *x*+3=3
© now solve for x	
sol1x:=solve(*example11a*,*x*)	*x*=0

The actual value for the variable *x* is now known and is used to find the value of *y*:

© substitute the solution for x in example12	
example12a:=*example12*|*sol1x*	*y*=3
© solve example12a for y (redundant, here for demonstration)	
sol1y:=solve(*example12a*,*y*)	*y*=3

The values of both variables have been found and can be verified by substituting the values in the two equations. A result of `true` means that when the values are substituted in the equations, the value of the equation's left-hand side equals the value of its right-hand side.

© Verify the solution by substituting the values in the equations	
example11|*sol1x* and *sol1y*	true
example12|*sol1x* and *sol1y*	true

The values of the left-hand and right-hand sides of the equations are obtained for comparison with the `constraint` operator and the `left()` and `right()` functions:

© examine the left−hand and right−hand sides of the equations with the solutions	
lhs11:=left(*example11*)	$2\cdot x+y$
lhs11|*sol1x* and *sol1y*	3
rhs11:=right(*example11*)	3
lhs12:=left(*example12*)	$y-2\cdot x$
lhs12|*sol1x* and *sol1y*	3
rhs12:=right(*example12*)	3

The solution can also be verified with the function `linSolve()`:

© solve the system with linSolve()	
linSolve({*example11*,*example12*},*x*,*y*)	{0,3}

The system of equations for Example 1 has the unique solution $(x, y) = (0, 3)$. The system is *consistent* and the equations are *independent*.

Substitution Example 2. Use the substitution method to solve the system of equations

$$\begin{cases} -2x+y &= 3 \\ -2x+y &= 1 \end{cases}$$

Figure 5 shows the graphs of these two equations.

This system is solved in a calculator page as follows:

example21:=−2· *x*+*y*=3	$y-2\cdot x=3$
example22:=−2· *x*+*y*=1	$y-2\cdot x=1$
© solve example22 for y	
sol22y:=solve(*example22*,*y*)	$y=2\cdot x+1$
© substitute the solution for y in example21	
example21a:=*example21*|*sol22y*	false
© The substitution results in −2x+2x+1=3 −> 1=3, which is false	
© Examine the left and right sides of example21	
lhs2:=left(*example21*)	$y-2\cdot x$
lhs2|*sol22y*	1

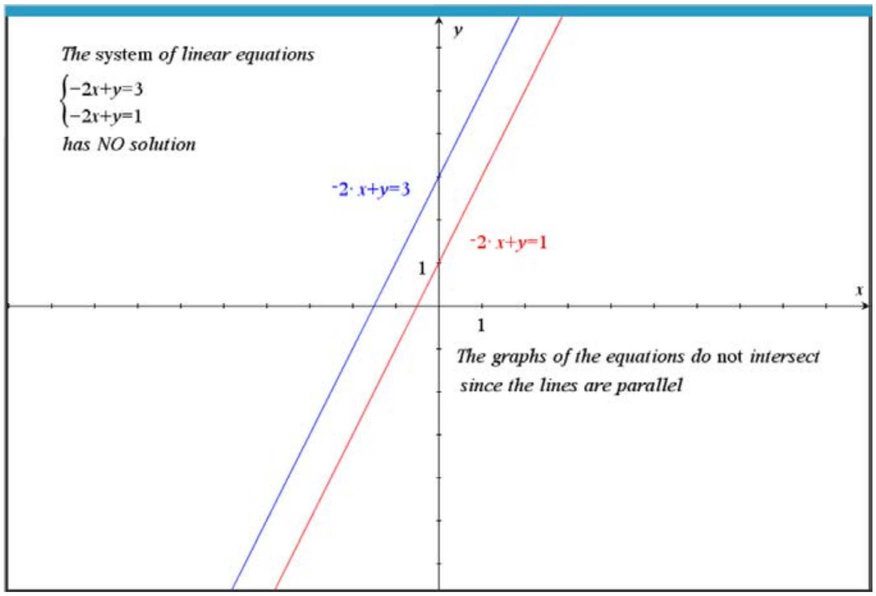


Figure 5: Graphs of Equations for Substitution Example 2

rhs2:=right(*example21*)	3
© 1=3 is false, therefore, the system of equations has no solution	
© verify the conclusion by calling linSolve	
linSolve({*example21*,*example22*},*x*,*y*)	"No solution found"

As shown, this system does not have a solution. The system is *inconsistent*.

Substitution Example 3. Use the substitution method to solve the system of equations

$$\begin{cases} 2x - y & = 3 \\ -2x + y & = -3 \end{cases}$$

Figure 6 shows the graphs of these two equations.

The solution to this system in a calculator page is:

© Substitution Example 3	
example31:=⁻2· *x*+*y*=3	*y*−2· *x*=3
example32:=2· *x*−*y*=⁻3	2· *x*−*y*=⁻3

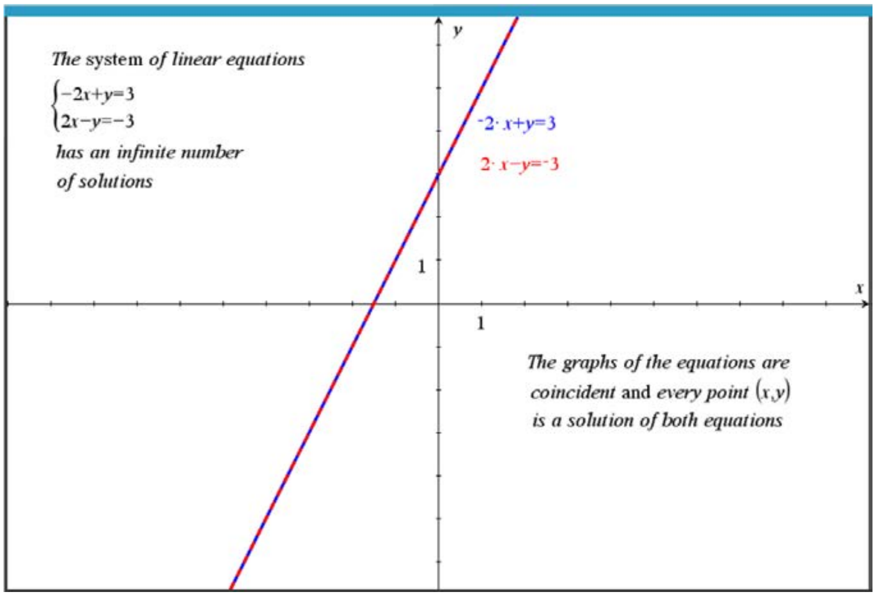


Figure 6: Graphs of Equations for Substitution Example 3

© solve example32 for y

sol32y:=solve(*example32*,*y*)	$y=2\cdot x+3$

© substitute the solution for y in example31

example31a:=*example31*|*sol32y*	true

© The substitution results in −2x+2x+3=3 −> 3=3, which is true

© examine the left and right−hand sides of example31

lhs3:=left(*example31*)	$y-2\cdot x$
lhs3|*sol32y*	3
rhs3:=right(*example31*)	3

© 3=3, a true statement − the system of equations has multiple solutions

© verify the conclusion by calling the linSolve function

linSolve({*example31*,*example32*},*x*,*y*)	$\left\{\frac{c1-3}{2},c1\right\}$

© @c1 means the value of x depends on the value of y. The equation is dependent.

The solution involves a free variable **C1** (y) that can be assigned any value, resulting in a solution to the system. For example, when $y = \mathbf{C1} = 3, x = 0$ and when $y = \mathbf{C1} = 0, x = \frac{-3}{2}$ are two of an infinite number of solutions to the system. This system is *consistent* and the two equations are *dependent*.

The substitution method is relatively easy to use for systems with two variables. This method can also be used to solve systems with three variables, although the technique involves more calculations and is cumbersome. Using TI-Nspire to perform the calculations makes it easier to solve these systems than performing the calculations by hand. The following examples demonstrate solving a system with three variables in a calculator page.

Substitution Example 4. Use the substitution method to solve the system of equations

$$\begin{cases} x+3y+\ \ 6z &= 25 \\ 2x+7y+14z &= 58 \\ \quad\ \ 2y+\ \ 5z &= 19 \end{cases}$$

Figure 7 shows the graphs of these three equations.

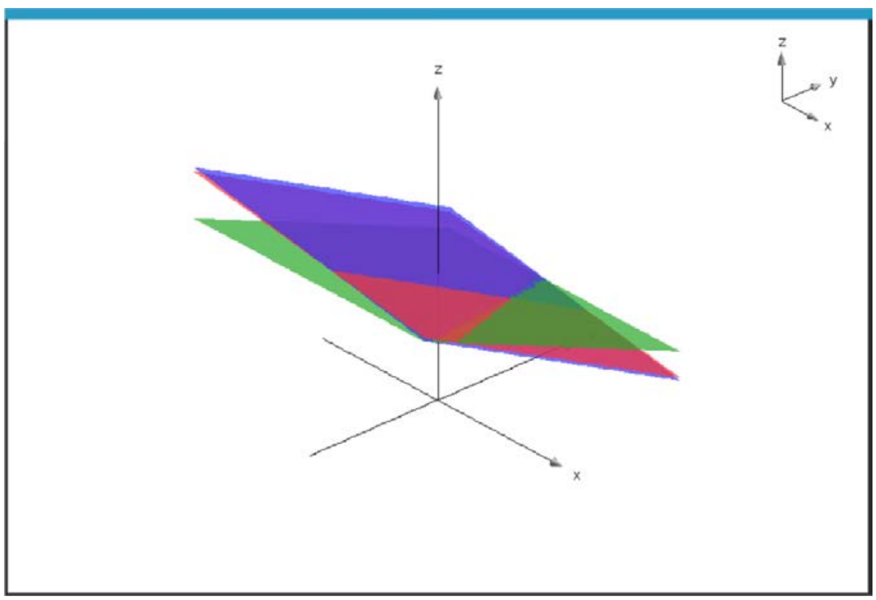


Figure 7: Graphs of Equations for Substitution Example 4

The solution to this system in a calculator page is:

example41:=x+3· y+6· z=25	x+3· y+6· z=25
example42:=2· x+7· y+14· z=58	2· x+7· y+14· z=58
example43:=2· y+5· z=19	2· y+5· z=19

© solve example43 for y	
sol43y:=solve(*example43*,*y*)	$y=\frac{-(5\cdot z-19)}{2}$
© substitute the solution for y in example42	
example42a:=*example42*|*sol43y*	$2\cdot x-\frac{7\cdot z}{2}+\frac{133}{2}=58$
© solve example42a for x	
sol42ax:=solve(*example42a*,*x*)	$x=\frac{7\cdot z-17}{4}$
© substitute the solution for x and y in example41	
example41a:=*example41*|*sol43y* and *sol42ax*	$\frac{z}{4}+\frac{97}{4}=25$
© now solve for z	
example4z:=solve(*example41a*,*z*)	$z=3$
© substitute z in sol43y and solve for y	
example4y:=solve(*sol43y*|*example4z*,*y*)	$y=2$
© substitute z in sol42ax and solve for x	
example4x:=solve(*sol42ax*|*example4z*,*x*)	$x=1$
© The solution is the 3d point (1,2,3)	
example4xyz:={*example4x*,*example4y*,*example4z*}	$\{x=1,y=2,z=3\}$
© verify that the solution satisfies all three equations	
example41|*example4x* and *example4y* and *example4z*	true
example42|*example4x* and *example4y* and *example4z*	true
example43|*example4x* and *example4y* and *example4z*	true
© Solve the system with linSolve	
linSolve({*example41*,*example42*,*example43*},*x*,*y*,*z*)	{1,2,3}

Substitution Example 5. Use the substitution method to solve the system of equations

$$\begin{cases} 2x+\ \ 4y-3z &= -1 \\ 5x+10y-7z &= -2 \\ 3x+\ \ 6y+5z &= \ \ 9 \end{cases}$$

Figure 8 shows the graphs of these three equations.

The solution to this system in a calculator page is:

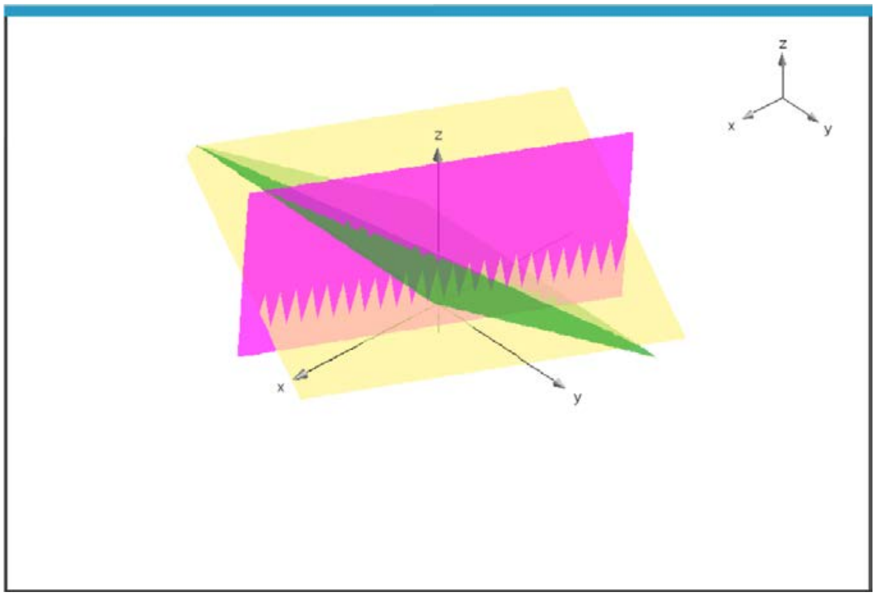


Figure 8: Graphs of Equations for Substitution Example 5

$example51 := 2\cdot x+4\cdot y-3\cdot z=^{-}1$	$2\cdot x+4\cdot y-3\cdot z=^{-}1$
$example52 := 5\cdot x+10\cdot y-7\cdot z=^{-}2$	$5\cdot x+10\cdot y-7\cdot z=^{-}2$
$example53 := 3\cdot x+6\cdot y+5\cdot z=9$	$3\cdot x+6\cdot y+5\cdot z=9$
© solve example53 for z	
$example53z := \text{solve}(example53, z)$	$z=\frac{^{-}3\cdot(x+2\cdot y-3)}{5}$
© substitute the value for z in example52	
$example52a := example52 \vert example53z$	$\frac{46\cdot x}{5}+\frac{92\cdot y}{5}-\frac{63}{5}=^{-}2$
© now solve example52a for y	
$example52ay := \text{solve}(example52a, y)$	$y=\frac{^{-}(46\cdot x-53)}{92}$
© now substitute expressions for z and y in example51	
$example51a := example51 \vert example53z$ and $example52ay$	false
© examine left and right−hand sides	
$lhs5 := \text{left}(example51)$	$2\cdot x+4\cdot y-3\cdot z$
$lhs5 \vert example53z$ and $example52ay$	$\frac{^{-}47}{46}$
$rhs5 := \text{right}(example51)$	$^{-}1$

© since $-\frac{47}{46}=-1$ is false, there is no solution to the system. The system is inconsistent.

© check the result with linSolve

linSolve({*example51*,*example52*,*example53*},*x*,*y*,*z*) "No solution found"

Substitution Example 6. Use the substitution method to solve the system of equations

$$\begin{cases} 3x-y-\ \ 5z &= \ \ 9 \\ \qquad\ \ y-10z &= \ \ 0 \\ -2x+y &= -6 \end{cases}$$

Figure 9 shows the graphs of these three equations.

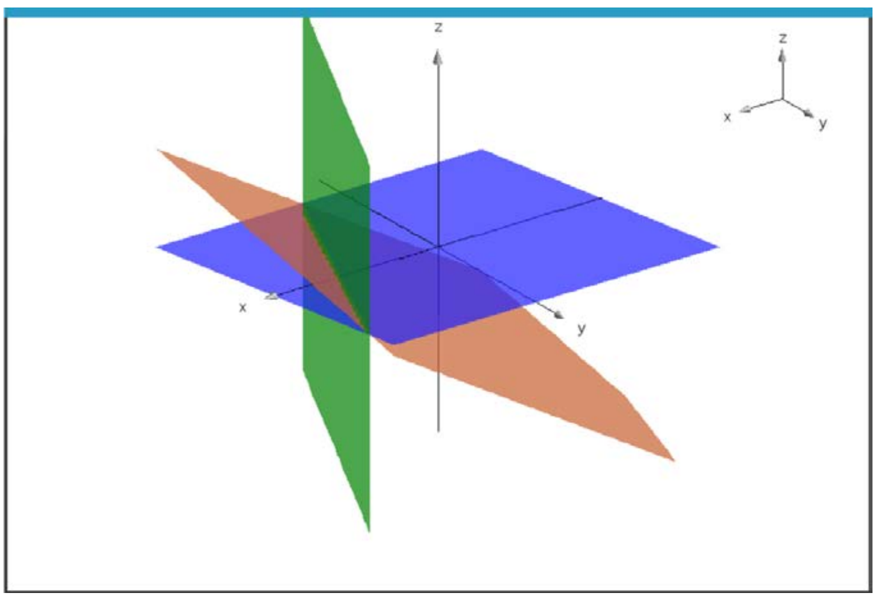


Figure 9: Graphs of Equations for Substitution Example 6

The solution to this system in a calculator page is:

example61:=3· *x*−*y*−5· *z*=9	3· *x*−*y*−5· *z*=9
example62:=*y*−10· *z*=0	*y*−10· *z*=0
example63:=⁻2· *x*+*y*=⁻6	*y*−2· *x*=⁻6

© solve example62 for y	
example62y:=solve(*example62*,*y*)	$y=10\cdot z$
© substitute example62y in example 63	
example63a:=*example63*|*example62y*	$10\cdot z-2\cdot x=-6$
© solve example63a for x	
example63x:=solve(*example63a*,*x*)	$x=5\cdot z+3$
© substitute example63x and example62y in example61	
example61a:=*example61*|*example63x* and *example62y*	true
© A value of true means there are an infinite number of solutions	
© Compare left−hand and right−hand sides of example61 with values of x,y	
lhs6:=left(*example61*)	$3\cdot x-y-5\cdot z$
lhs6|*example63x* and *example62y*	9
rhs6:=right(*example61*)	9
© 9=9 is a true statement, meaning there are an infinite # of solutions	
© The solution:	
example6xyz:={*example63x*,*example62y*,*z=any*}	$\{x=5\cdot z+3, y=10\cdot z, z=any\}$
© check the solution with linSolve	
linSolve({*example61*,*example62*,*example63*},*x*,*y*,*z*)	$\{5\cdot c1+3, 10\cdot c1, c1\}$
© The equations are dependent upon z=c1, which can be any value	

3.4 Substitution Method Summary

The substitution method is simple and easy to use for systems of equations with only two variables. The method also works for systems of equations with three variables, but is cumbersome, especially for hand calculations. A more logical and extensible method is needed to solve general systems of equations containing many variables and equations. The *elimination method* is such a method, and this method establishes the basis for solving systems of equations using *matrices* (to be discussed in a later article).

3.5 The Elimination Method

The elimination method transforms a system of equations into an *equivalent* system of equations that is easier to solve. Equivalent equations are equations that have the same solution. For example, $2x = 4$ and $4x = 8$ are equivalent since the solution to both equations is $x = 2$.

A system of equations is transformed into an equivalent system using three simple operations[3]:

1. Interchange any two equations of the system.

2. Multiply or divide each side of an equation by the same nonzero constant.
3. Replace any equation in the system by the sum or difference of that equation and a nonzero multiple of any other equation in the system.

These operations eliminate variables from succeeding equations one at a time until the last equation in the system has only a single variable remaining. This equation is then solved for the variable and the solution is *back-substituted* into the preceding equation. Each preceding equation is then solved and back-substitution continues until the first equation is solved. To illustrate how the elimination method works, examples of solving systems of equations are:

Elimination Method for a System of Two Equations: Solve the system

$$\begin{cases} 2x+y &= 3 \\ -2x+y &= 3 \end{cases}$$

Step 1. Add equation 1 to equation 2, eliminating x from equation 2. The result is the equivalent system:

$$\begin{cases} 2x+y &= 3 \\ 2y &= 6 \end{cases}$$

Step 2. The elimination process is complete. Solve equation 2 for y:

$$y = \frac{6}{2} = 3$$

Step 3. Back-substitute the value for y in equation 1:

$$2x+3 = 3$$

Step 4. Solve equation 1 for x:

$$x = \frac{3-3}{2} = 0$$

Back-substitution is complete. The solution is (x,y)=(0,3)

Elimination Method for a System of Three Equations: Solve the system

$$\begin{cases} x+3y+6z &= 25 \\ 2x+7y+14z &= 58 \\ 2y+5z &= 19 \end{cases}$$

Step 1. Add -2 times equation 1 to equation 2, eliminating x from equation 2. Equation 3 is already in the correct form. The result is the equivalent system:

$$\begin{cases} x+3y+6z &= 25 \\ y+2z &= 8 \\ 2y+5z &= 19 \end{cases}$$

Step 2. Add -2 times equation 2 to equation 3, eliminating y from equation 3. The result is the equivalent system:

$$\begin{cases} x+3y+\ 6z &= 25 \\ \quad\ \ y+\ 2z &= \ 8 \\ \qquad\quad\ \ z &= \ 3 \end{cases}$$

Step 3. The elimination process is complete. Back-substitute the value for z in equation 2:

$$y+2(3)=y+6=8$$

Step 4. Solve equation 2 for y:

$$y=8-6=2$$

Step 5. Back-substitute the value for z and y in equation 1:

$$x+3(2)+6(3)=x+6+18=x+24=25$$

Step 6. Back-substitution is complete. Solve equation 1 for x:

$$x=25-24=1$$

The process is complete. The solution is (x,y,z)=(1,2,3)

Elimination Method for a System of n Equations:

The exact same process as followed for systems of two and three equations is followed for systems of four or more equations. The only difference is that there are more elimination steps and more back-substitution steps.

End Result of the Elimination Process The elimination process ends when the last variable is eliminated. For systems with two variables, the last variable is y, for systems with three variables, the last variable is z, and for systems with n variables, the last variable is x_n. The result of eliminating the last variable indicates whether the system has a unique solution, no solution, or an infinite number of solutions, as follows:

1. If the result is an expression equating a variable with a value such as *variable* = *value*, the system has a unique solution. The values of the solution variables are found using back-substitution. The system is consistent and the equations are independent.

2. If the result is an invalid equality, $0=a$, where a is a constant, the system does not have a solution. The system is inconsistent.

3. If the result of eliminating a variable is a valid equality, $0=0$, the system has an infinite number of solutions. The eliminated variable is a free variable that can assume any value. The values of the solution variables are found using back-substitution with the values expressed in terms of the free variable(s). Assigning a value to the free variable(s) determines the values of the solution variables. The system is consistent and the equations are dependent.

3.5.1 Elimination Examples with TI-Nspire CAS

Following are examples of using the elimination method to solve two-variable and three-variable systems of linear equations. The systems of equations are the same systems used for the substitution method examples. The examples use the builtin functions `solve()`, `linSolve()`, and the `constraint` operator (|).

Elimination Example 1. Use the elimination method to solve the system of equations

$$\begin{cases} 2x+y &= 3 \\ -2x+y &= 3 \end{cases}$$

The graphs of these equations are shown above in Figure 4.

example11:=2· *x*+*y*=3	2· *x*+*y*=3
example12:=⁻2· *x*+*y*=3	*y*−2· *x*=3
© Add the two equations, eliminating x	
example12a:=*example11*+*example12*	2· *y*=6
© solve example12a for y	
example1y:=solve(*example12a*,*y*)	*y*=3
© Back substitute example1y into example11	
example11a:=*example11*|*example1y*	2· *x*+3=3
© solve example11a for x	
example1x:=solve(*example11a*,*x*)	*x*=0
© The unique solution to the system	
solution1:={*example1x*,*example1y*}	{*x*=0,*y*=3}
© solve the system with linSolve	
linSolve({*example11*,*example12*},*x*,*y*)	{0,3}

Elimination Example 2. Use the elimination method to solve the system of equations

$$\begin{cases} -2x+y &= 3 \\ -2x+y &= 1 \end{cases}$$

Figure 5 shows the graphs of these two equations.

example21:=⁻2· *x*+*y*=3	*y*−2· *x*=3
example22:=⁻2· *x*+*y*=1	*y*−2· *x*=1

© Multiply example22 by −1 and add the result to example21

example2a:=*example21*+⁻1· *example22*	0=2

© The result, 0=2, is an invalid equation, indicating the system does not have a solution

© Check the solution

linSolve({*example21*,*example22*},*x*,*y*)	"No solution found"

Elimination Example 3. Use the elimination method to solve the system of equations

$$\begin{cases} 2x - y &= 3 \\ -2x + y &= -3 \end{cases}$$

The graphs of these two equations are shown in Figure 6.

example31:=⁻2· *x*+*y*=3	*y*−2· *x*=3
example32:=2· *x*−*y*=⁻3	2· *x*−*y*=⁻3

© add the two equations

example3a:=*example31*+*example32*	0=0
example3a	true

© the result, 0=0, is an equality, indicating the system has an infinite number of solutions

© let y be the free variable and solve for x in terms of y

example3x:=solve(*example31*,*x*)	$x=\frac{y-3}{2}$

© the solution is

solution3:={*example3x*,*y*=*any*}	$\left\{x=\frac{y-3}{2},y=any\right\}$

© validate the solution with linSolve

linSolve({*example31*,*example32*},*x*,*y*)	$\left\{\frac{c3-3}{2},c3\right\}$

Elimination Example 4. Use the elimination method to solve the system of equations

$$\begin{cases} x + 3y + 6z &= 25 \\ 2x + 7y + 14z &= 58 \\ \phantom{2x+{}} 2y + 5z &= 19 \end{cases}$$

Figure 7 shows the graphs of these three equations.

example41:=*x*+3· *y*+6· *z*=25	*x*+3· *y*+6· *z*=25
example42:=2· *x*+7· *y*+14· *z*=58	2· *x*+7· *y*+14· *z*=58
example43:=2· *y*+5· *z*=19	2· *y*+5· *z*=19

© multiply example41 by −2 and add it to example42, eliminating x

example42a:=−2· *example41*+*example42* $\quad y+2\cdot z=8$

© now multiply example42a by −2 and add it to example43, eliminating y

example43a:=−2· *example42a*+*example43* $\quad z=3$

© the value for z is now known

example4z:=*example43a* $\quad z=3$

© back−substitute example4z in example42a and solve for y

example42b:=*example42a*|*example4z* $\quad y+6=8$

example4y:=solve(*example42b*,*y*) $\quad y=2$

© back−substitute example4y and example4z in example41 and solve for x

example41a:=*example41*|*example4y* and *example4z* $\quad x+24=25$

example4x:=solve(*example41a*,*x*) $\quad x=1$

© the unique solution is (1,2,3)

solution4:={*example4x*,*example4y*,*example4z*} $\quad \{x=1,y=2,z=3\}$

© check the solution with linSolve

linSolve({*example41*,*example42*,*example43*},*x*,*y*,*z*) $\quad \{1,2,3\}$

Elimination Example 5. Use the elimination method to solve the system of equations

$$\begin{cases} 2x+\ \ 4y-3z &= -1 \\ 5x+10y-7z &= -2 \\ 3x+\ \ 6y+5z &= \ \ 9 \end{cases}$$

Figure 8 shows the graphs of these three equations.

example51:=2· *x*+4· *y*−3· *z*=−1 $\quad 2\cdot x+4\cdot y-3\cdot z=-1$

example52:=5· *x*+10· *y*−7· *z*=−2 $\quad 5\cdot x+10\cdot y-7\cdot z=-2$

example53:=3· *x*+6· *y*+5· *z*=9 $\quad 3\cdot x+6\cdot y+5\cdot z=9$

© add −5· example51 and 2· example52

example52a:=−5· *example51*+2· *example52* $\quad z=1$

© add −3· example51 and 2· example53

example53a:=−3· *example51*+2· *example53* $\quad 19\cdot z=21$

© add −19· example52a and example53a

example53b:=−19· *example52a*+*example53a* $\quad 0=2$

© the result is an invalid equality − the system has no solution

© solve using linSolve

linSolve({*example51*,*example52*,*example53*},*x*,*y*,*z*) "No solution found"

Elimination Example 6. Use the elimination method to solve the system of equations

$$\begin{cases} 3x - y - \ 5z &= \ \ 9 \\ \qquad y - 10z &= \ \ 0 \\ -2x + y &= -6 \end{cases}$$

Figure 9 shows the graphs of these three equations.

example61:=3⋅ *x*−*y*−5⋅ *z*=9	3⋅ *x*−*y*−5⋅ *z*=9
example62:=*y*−10⋅ *z*=0	*y*−10⋅ *z*=0
example63:=⁻2⋅ *x*+*y*=⁻6	*y*−2⋅ *x*=⁻6
© interchange example62 and example63	
example61:=3⋅ *x*−*y*−5⋅ *z*=9	3⋅ *x*−*y*−5⋅ *z*=9
example62:=⁻2⋅ *x*+*y*=⁻6	*y*−2⋅ *x*=⁻6
example63:=*y*−10⋅ *z*=0	*y*−10⋅ *z*=0
© add 2⋅ example61 and 3⋅ example62, eliminating x	
example62a:=2⋅ *example61*+3⋅ *example62*	*y*−10⋅ *z*=0
© add −1⋅ example62a and example63 to eliminate y	
example63a:=⁻1⋅ *example62a*+*example63*	0=0
© the result is a true statement − the system has an infinite number of solutions	
© z is a free variable	
example6z:=*z*	*z*
© solve example62a for y in terms of z	
example6y:=solve(*example62a*,*y*)	*y*=10⋅ *z*
© back−substitute example6y in example61	
example61a:=*example61*|*example6y*	3⋅ *x*−15⋅ *z*=9
© solve example61a for x	
example61x:=solve(*example61a*,*x*)	*x*=5⋅ *z*+3
© the solution:	
solution6:={*example6x*,*example6y*,*example6z*}	{*x*=5⋅ *z*+3,*y*=10⋅ *z*,*z*}
© solve using linSolve	
linSolve({*example61*,*example62*,*example63*},*x*,*y*,*z*)	{5⋅ **c1**+3,10⋅ **c1**,**c1**}

3.6 Elimination Method Summary

The elimination method is a logical, well-defined process for solving a system of linear equations by transforming the system into an equivalent system that is simple to solve using back-substitution. The result of the elimination process identifies whether the system has a unique solution, no solution, or multiple solutions. For systems that have a unique solution or multiple solutions, the result of the back-substitution process is the solution to the system.

The method is applicable to systems with an arbitrary number of equations and variables. Because the rules for the method are simple and the process follows logical steps, the method is easy to automate.

4 Summary

This article described linear equations and systems of linear equations, described solutions of systems of equations, and demonstrated how to solve systems of equations. Examples of solving systems of equations using graphical techniques, the substitution method, and the elimination method were presented. The TI-Nspire documents accompanying this article, *substitution_examples.tns* and *elimination_examples.tns* show how to solve systems of linear equations with TI-Nspire CAS.

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