Introduction to Contour Integration

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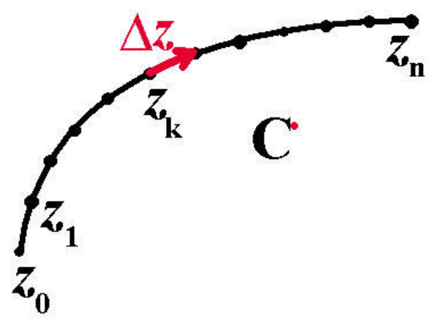
In first year calculus you learned that the definite integral is defined in terms of a limit of a sum, called a Riemann sum. We can apply the same definition to a function of a complex variable.

Let C be a curve in the complex plane. We shall refer to this as a *contour*. Subdivide it into subintervals and let z_0, z_1, z_2, . . ., z_n be the points on the boundaries of these subdivisions. Let $\Delta z = z_{k+1} - z_k$.



The product $f(z_k)\,\Delta z$ is a complex number for each value of k, so the sum of these complex numbers is also a complex number.

$$\sum_{k=1}^{n} f(z_k)\,\Delta z$$

The limit of this sum, as the number of subintervals goes to infinity, will still be a complex number. The limit of this sum is called a *contour integral.*

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(z_k)\,\Delta z$$

In Calculus I, the Riemann sum definition of the definite integral can be related to the area under a curve. However, if $f(z)$ is a function of a complex variable, then the contour integral $\int_C f(z)\,dz$ has no such area interpretation. If it doesn't represent area, then why bother definining it at all? As you will see later, contour integrals have applications to the integral transforms used to solve differential equations.

Since our definition of $\int_C f(z)\,dz$ is essentially the same as the one used in first year calculus, we should not be surprised to find that many of the integral properties encountered in first year calculus are still true for contour integrals.

Properties of the Integral

Suppose a is a constant

$$\int_C af(z)\,dz = \lim_{n\to\infty}\sum_{k=1}^{n} af\left(z_k\right)\,\Delta z = a\lim_{n\to\infty}\sum_{k=1}^{n} f\left(z_k\right)\,\Delta z$$

Therefore,

$$\int_C af(z)\,dz = a\int_C f(z)\,dz$$

In other words, constants can be factored out of contour integrals. Here's another familiar property:

$$\begin{aligned}\int_C (f(z)+g(z))\,dz &= \lim_{n\to\infty}\sum_{k=1}^{n}(f\left(z_k\right)+g\left(z_k\right))\,\Delta z\\ &= \lim_{n\to\infty}\sum_{k=1}^{n} f\left(z_k\right)\,\Delta z + \lim_{n\to\infty}\sum_{k=1}^{n} g\left(z_k\right)\,\Delta z\end{aligned}$$

Therefore,

$$\int_C (f(z)+g(z))\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

We can summarize both of these last two properties by the formula:

$$\int_C \left(af(z)+bg(z)\right)\,dz = a\int_C f(z)\,dz + b\int_C g(z)\,dz$$

where a and b are any constants.

Relationship of Contour Integrals to Line Integrals

If $f(z) = u(x,y) + i\,v(x,y)$ and $dz = dx + i\,dy$ then:

$$\begin{aligned}\int_C f(z)\,dz &= \int_C (u+iv)(dx+idy) = \int_C ((u\,dx - v\,dy) + i(v\,dx + u\,dy))\\ &= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)\end{aligned}$$

Both $\int_C (u\,dx - v\,dy)$ and $\int_C (v\,dx + u\,dy)$ are ordinary line integrals of the type we have already studied in MA 441. Therefore, we can use our knowledge of line integrals to calculate contour integrals of functions of a complex variable.

Example: Let C be the straight line path connecting $z = 0$ to $z = 1 + i$. Let $f(z) = \overline{z} = x - iy$. Calculate $\int_C \overline{z}\,dz$.

In this case, $u = x$ and $v = -y$. So, the formula

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

becomes, in this case,

$$\int_C \overline{z}\,dz = \int_C (x\,dx + y\,dy) + i\int_C (-y\,dx + x\,dy)$$

The straight line segment in this example can be described by the equation $y = x$ for $0 \le x \le 1$. If we make this substitution, we can express the line integrals in terms of ordinary integrals depending only on x.

$$\begin{aligned}
\int_C \overline{z}\,dz &= \int_C (x\,dx + y\,dy) + i\int_C (-y\,dx + x\,dy) \\
&= \int_0^1 (x\,dx + x\,dx) + i\int_0^1 (-x\,dx + x\,dx) \\
&= \int_0^1 2x\,dx + i \cdot 0 \\
&= \left[x^2\right]_0^1 = 1
\end{aligned}$$

Let's try a different path between $z = 0$ and $z = 1 + i$ and see if this makes any difference in the answer. Consider the following parametric equations:

$$x = \cos t \qquad y = 1 + \sin t \qquad -\frac{\pi}{2} \le t \le 0$$

These equations describe a quarter of a circle of radius 1 whose center is $(0, 1)$. In complex variable notation, the center is i. The initial point is $z = 0$ and the final point (at $t = 0$) is $1 + i$. So, this quarter circle path, which I will denote as Q, is another path connecting the same two points as the last example. If $x = \cos t$ and $y = 1 + \sin t$ then:

$$dx = -\sin t\, dt \qquad\qquad dy = \cos t\, dt$$

Let's substitute into the real and imaginary parts of $\int_Q \overline{z}\, dz$

$$\int_Q (x\, dx + y\, dy) = \int_{-\pi/2}^{0} (\cos t)(-\sin t)\, dt + (1 + \sin t)(\cos t\, dt)) = 1$$

$$\int_Q (-y\, dx + x\, dy) = \int_{-\pi/2}^{0} -(1 + \sin t)(-\sin t\, dt) + (\cos t)(\cos t\, dt) = \frac{\pi}{2} - 1$$

Therefore,

$$\int_Q \overline{z}\, dz = 1 + i\left(\frac{\pi}{2} - 1\right)$$

An interesting way to do this problem is to make use of the complex exponential.

$$z = x + iy = \cos t + i(1 + \sin t) = \cos t + i \sin t + i = e^{it} + i \qquad \text{so} \quad dz = ie^{it}\, dt$$

$$\overline{z} = x - iy = \cos t - i(1 + \sin t) = e^{-it} - i$$

$$\overline{z}\, dz = \left(e^{-it} - i\right)\left(ie^{it}\, dt\right) = \left(i + e^{it}\right)\, dt$$

$$\int_Q \overline{z}\, dz = \int_{-\pi/2}^{0} \left(i + e^{it}\right)\, dt = \left[it + \frac{1}{i}e^{it}\right]_{-\pi/2}^{0} = 1 + i\left(\frac{\pi}{2} - 1\right)$$

Notice that $\int_C \overline{z}\, dz$ is not the same as $\int_Q \overline{z}\, dz$ even though the both paths connect 0 to $1+i$. This contour integral is *path dependent.* This is consistent with our experience with line integrals. We have already seen that the value of a line integral may depend on the path connect the initial point to the final point.

Example

Let us define paths C and Q exactly as we did in the last example, but this time, take $f(z) = z^2$. Let's begin with path C, where $y = x$.

$$\begin{aligned}
\int_C z^2\,dz &= \int_C (x+iy)^2(dx + i\,dy) \\
&= \int_C (x+ix)^2(dx + i\,dx) \\
&= (1+i)^3 \int_0^1 x^2\,dx \\
&= \frac{1}{3}(1+i)^3 \\
&= -\frac{2}{3} + \frac{2}{3}i
\end{aligned}$$

Next, compare this answer with the integral along path Q, where $z = i + e^{it}$

$$\begin{aligned}
\int_Q z^2\,dz &= \int_{-\pi/2}^{0} \left(i + e^{it}\right)^2 ie^{it}\,dt \\
&= \int_{-\pi/2}^{0} \left(-1 + 2ie^{it} + e^{2it}\right) ie^{it}\,dt \\
&= \int_{-\pi/2}^{0} \left(-2e^{2it} - ie^{it} + ie^{3it}\right)\,dt \\
&= \left[-\frac{1}{i}e^{2it} - e^{it} + \frac{1}{3}e^{3it}\right]_{-\pi/2}^{0} \\
&= -\frac{2}{3} + \frac{2}{3}i
\end{aligned}$$

This time, the path between $z = 0$ and $z = 1+i$ did not make any difference in the answer. That is, the integral $\int_0^{1+i} z^2\,dz$ seems to be *path independent*.

Path Independence

We know that line integrals are path independent if closed loop integrals are zero. Under what circumstances will closed loop contour integrals be zero? Let's consider a typical closed loop integral of a function of a complex variable $f(z) = u + iv$. Let C be a closed loop and let $\mathcal{D}$ be the region inside the closed loop. If we apply Green's Theorem, we obtain:

$$\begin{aligned}\oint_C f(z)\,dz &= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy) \\ &= \iint_{\mathcal{D}} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\iint_{\mathcal{D}} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA\end{aligned}$$

Therefore, $\oint_C f(z)\,dz = 0$ when it's real and imaginary parts are zero:

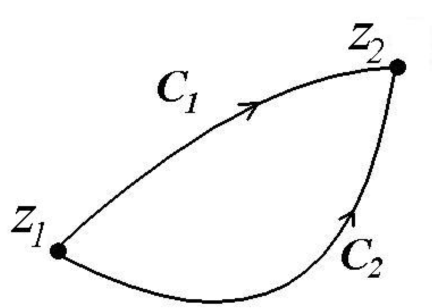
$$\iint_{\mathcal{D}} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA = 0 \qquad \iint_{\mathcal{D}} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA = 0$$

We are guaranteed that this will be the case if:

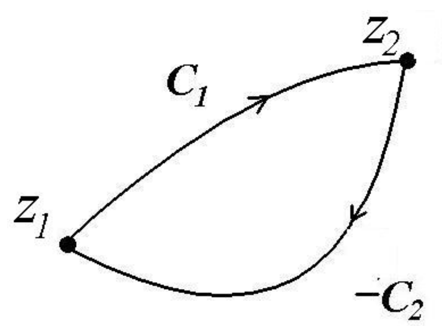
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are exactly the *Cauchy-Riemann* equations which are the conditions for a function $f(z)$ to be differentiable. Recall that a function is said to be *analytic* at a point if it is differentiable at that point and in a region surrounding that point. Thus, if $f(z)$ is analytic in some region then the contour integral of $f(z)$ around any closed loop inside that region will be zero.

Path independence is be an immediate consequence of the analyticity of a function. Let z_1 and z_2 be two points in the complex plane and let C_1 and C_2 be two different path connecting z_1 to z_2.



If we reverse direction along C_2, then we get a *closed loop*. Let $-C_2$ be the same path as C_2 but in the opposite direction. Let C be the closed loop traversed along the path C_1 followed by $-C_2$.



$\int_{-C_2} f(z)\,dz = -\int_{C_2} f(z)\,dz$ and therefore:

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = \oint_C f(z)\,dz$$

It follows that $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ if and only if $\oint_C f(z)\,dz = 0$. Thus, *path independence* of contour integrals occurs when we integrate analytic functions. This is consisent with our observation that in the preceding examples, $\int_C \overline{z}\,dz$ was path dependent ($\overline{z}$ is not analytic anywhere) and $\int_C z^2\,dz$ was path independent ($\frac{d}{dz}\left(z^2\right) = 2z$ for all z so z^2 is analytic).

Antiderivatives

A fact, usually proved in a complex variables course, is that analytic functions always have antiderivatives. When I say that $f(z)$ has an antiderivative, I mean that there is a function $F(z)$ such that $f(z) = F'(z)$. We know from first year calculus that $\int_a^b F'(x)\,dx = F(b) - F(a)$. Is this true for contour integrals of functions of a complex variable as well? Let us suppose that $F(z) = U + iV$ is the antiderivative of $f(z)$. Then

$$f(z) = F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$

If C is a path connecting z_1 to z_2 then:

$$\begin{aligned}
\int_C f(z)\,dz &= \int_C \left(\frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}\right)(dx + i\,dy) \\
&= \int_C \left(\frac{\partial U}{\partial x}\,dx - \frac{\partial V}{\partial x}\,dy\right) + i\int_C \left(\frac{\partial V}{\partial x}\,dx + \frac{\partial U}{\partial x}\,dy\right) \\
&= \int_C \left(\frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy\right) + i\int_C \left(\frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy\right)
\end{aligned}$$

The last line follows from the *Cauchy-Riemann* equations. Let's write this in more familiar vector notation using the gradient.

$$\begin{aligned}
\int_C f(z)\,dz &= \int_C \left(\frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy\right) + i\int_C \left(\frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy\right) \\
&= \int_C \nabla U \bullet d\vec{\mathbf{r}} + i\int_C \nabla V \bullet d\vec{\mathbf{r}}
\end{aligned}$$

Earlier in MA 441, we saw that $\int_C \nabla U \bullet d\vec{\mathbf{r}} = U(z_2) - U(z_1)$. This was referred to as the *Fundamental Theorem for Line Integrals*. It will also be true that $\int_C \nabla V \bullet d\vec{\mathbf{r}} = V(z_2) - V(z_1)$. It follows that:

$$\begin{aligned}
\int_C F'(z)\,dz &= U(z_2) - U(z_1) + i(V(z_2) - V(z_1)) \\
&= U(z_2) + iV(z_2) - (U(z_1) + iV(z_1)) \\
&= F(z_2) - F(z_1)
\end{aligned}$$

This formula can speed up many line integral calculations considerably. For example, we could have done $\int_C z^2\,dz$ this way.

$$\int_C z^2\,dz = \int_C \frac{d}{dz}\left(\frac{1}{3}z^3\right)\,dz = \left[\frac{1}{3}z^3\right]_0^{1+i} = \frac{1}{3}(1+i)^3 - \frac{1}{3}0^3 = -\frac{2}{3} + \frac{2}{3}i$$