Rules of Inference

Today's Menu

Rules of Inference

- Quantifiers: Universal and Existential
- Nesting of Quantifiers
- Applications

Old Example Re-Revisited

Our Old Example:

- Suppose we have:

  "All human beings are mortal."

  "Sachin is a human being."

- Does it follow that "Sachin is mortal?"

Solution:

- Let H(x): "x is a human being."
- Let M(x): "x is mortal."
- The domain of discourse U is all human beings.
- "All human beings are mortal." translates to $\forall x\,(H(x) \rightarrow M(x))$

  "Sachin is a human being." translates to H(Sachin)
- Therefore, for $H(Sachin) \rightarrow M(Sachin)$ to be true it must be the case that M(Sachin).

Arguments in Propositional Logic

- A argument in propositional logic is a sequence of propositions.
- All but the final proposition are called premises. The last statement is the conclusion.
- The argument is valid if the premises imply the conclusion.
- An argument form is an argument that is valid no matter what propositions are substituted into its propositional variables.
- If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology.
- Inference rules are all argument simple argument forms that will be used to construct more complex argument forms.

Next, we will discover some useful inference rules!

Modus Ponens or Law of Detachment

(Modus Ponens = mode that affirms)

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

Proof using Truth Table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example:

Let p be "It is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"It is snowing."

"Therefore , I will study discrete math."

Modus Tollens

aka Denying the Consequent

$$\frac{\begin{array}{c}\neg q \\ p \rightarrow q\end{array}}{\therefore \neg p}$$

Corresponding Tautology:

$$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$$

Proof using Truth Table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	**F**	**T**

Example:

Let *p* be "it is snowing."

Let *q* be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"I will not study discrete math."

"Therefore , it is not snowing."

Hypothetical Syllogism

aka Transitivity of Implication or Chain Argument

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology:

$$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Example:

Let *p* be "it snows."
Let *q* be "I will study discrete math."
Let *r* be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

aka Disjunction Elimination or OR Elimination

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$$((p \vee q) \wedge \neg p) \rightarrow q$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study English literature."

"I will study discrete math or I will study English literature."

"I will not study discrete math."

"Therefore , I will study English literature."

Addition

aka Disjunction Introduction

$$\frac{p}{\therefore (p \vee q)}$$

Corresponding Tautology:

$$p \rightarrow (p \vee q)$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Simplification

aka Conjunction Elimination

$$\frac{p \land q}{\therefore p}$$

Corresponding Tautology:

$$(p \land q) \rightarrow p$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

aka Conjunction Introduction

$$\begin{array}{c} p \\ q \\ \hline \therefore p \land q \end{array}$$

Corresponding Tautology:

$$((p) \land (q)) \to (p \land q)$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study English literature."

"I will study discrete math."

"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution

$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \hline \therefore q \lor r \end{array}$$

Resolution plays an important role in Artificial Intelligence and is used in the programming language Prolog.

Corresponding Tautology:

$$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$$

Example:

Let p be "I will study discrete math."

Let q be "I will study databases."

Let r be "I will study English literature."

"I will study discrete math or I will study databases."

"I will not study discrete math or I will study English literature."

"Therefore, I will study databases or I will English literature."

Proof by Cases

aka Disjunction Elimination

$$\begin{array}{c} p \to q \\ r \to q \\ p \lor r \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$$((p \to q) \land (r \to q) \land (p \lor r)) \to q$$

Example:

Let *p* be "I will study discrete math."
Let *q* be "I will study Computer Science."
Let *r* be "I will study databases."

"If I will study discrete math, then I will study Computer Science."
"If I will study databases, then I will study Computer Science."
"I will study discrete math or I will study databases."

"Therefore, I will study Computer Science."

Constructive Dilemma

$$p \to q$$
$$r \to s$$
$$p \vee r$$
$$\therefore q \vee s$$

Disjunction of modus ponens

Corresponding Tautology:

$$((p \to q) \wedge (r \to s) \wedge (p \vee r)) \to (q \vee s)$$

Example:

Let *p* be "I will study discrete math."
Let *q* be "I will study computer science."
Let *r* be "I will study protein structures."
Let *s* be "I will study biochemistry."

"If I will study discrete math, then I will study computer science."
"If I will study protein structures, then I will study biochemistry."
"I will study discrete math or I will study protein structures."

"Therefore, I will study computer science or biochemistry."

Destructive Dilemma

$$\begin{array}{l} p \to q \\ r \to s \\ \neg q \vee \neg s \\ \hline \therefore \neg p \vee \neg r \end{array}$$

Disjunction of modus tollens

Corresponding Tautology:

$$(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s) \to (\neg p \vee \neg r)$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study computer science."

Let *r* be "I will study protein structures."

Let *s* be "I will study biochemistry."

"If I will study discrete math, then I will study computer science."
"If I will study protein structures, then I will study biochemistry."
"I will not study computer science or I will not study biochemistry."

"Therefore, I will not study discrete math
or I will not study protein structures."

Absorption

$$\frac{p \to q}{\therefore p \to (p \wedge q)}$$

q is absorbed by p in the conclusion!

Corresponding Tautology:

$$(p \to q) \to (p \to (p \wedge q))$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study computer science."

"If I will study discrete math, then I will study computer science."

"Therefore, if I will study discrete math, then I will study discrete mathematics and I will study computer science."

Building Valid Arguments

- A valid argument is a sequence of statements where each statement is either a premise or follows from previous statements (called premises) by rules of inference. The last statement is called conclusion.
- A valid argument takes the following form:

$$\begin{array}{l} \text{Premise 1} \\ \text{Premise 2} \\ \quad \bullet \\ \quad \bullet \\ \quad \bullet \\ \text{Premise } n \\ \hline \therefore \text{ Conclusion} \end{array}$$

Valid Arguments

Example: From the single proposition

$$p \wedge (p \rightarrow q)$$

Show that q is a conclusion.

Solution:

Step	**Reason**
1. $p \wedge (p \rightarrow q)$	Premise
2. p	Conjunction using (1)
3. $p \rightarrow q$	Conjunction using (1)
4. q	Modus Ponens using (2) and (3)

Valid Arguments

Example:

- With these hypotheses:
  "It is not sunny this afternoon and it is colder than yesterday."
  "We will go swimming only if it is sunny."
  "If we do not go swimming, then we will take a canoe trip."
  "If we take a canoe trip, then we will be home by sunset."
- Using the inference rules, construct a valid argument for the conclusion:
  "We will be home by sunset."

Solution:

1. Choose propositional variables:
   p : "It is sunny this afternoon."
   q : "It is colder than yesterday."
   r : "We will go swimming."
   s : "We will take a canoe trip."
   t : "We will be home by sunset."
2. Translation into propositional logic:

$$\text{Hypotheses: } \neg p \land q,\ r \to p,\ \neg r \to s,\ s \to t$$
$$\text{Conclusion: } t$$

Valid Arguments

Hypotheses: $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$
Conclusion: t

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Remember you can also use truth table to show this albeit with $32 = 2^5$ rows!

How do we use quantifiers with rules of inference?

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all students and Sachin is a student.

“All students are smart”

“Therefore, Sachin is smart.”

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

"There is someone who got an A in COMPSCI 230."
"Let's call her Amelie and say that Amelie got an A"

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

"Amelie got an A in the class."
"Therefore, someone got an A in the class."

Old Example Re-Revisited

Our Old Example:

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  "All human beings are mortal."

  "Sachin is a human being."

- Does it follow that "Sachin is mortal?"

Solution:

- Let H(x): "x is a human being."
- Let M(x): "x is mortal."
- The domain of discourse U is all human beings.
- "All human beings are mortal." translates to $\forall x\ H(x) \rightarrow M(x)$

  "Sachin is a human being." translates to H(Sachin)

To show:

$$\begin{array}{l} \forall x\ (H(x) \rightarrow M(x)) \\ \quad H(Sachin) \\ \hline \therefore M(Sachin) \end{array}$$

Old Example Re-Revisited

To show:

$$\frac{\begin{array}{c}\forall x\,(H(x) \rightarrow M(x)) \\ H(Sachin)\end{array}}{\therefore M(Sachin)}$$

Step	**Valid Argument**	**Reason**
(1)	$\forall x\,(H(x) \rightarrow M(x))$	Premise
(2)	$H(Sachin) \rightarrow M(Sachin)$	Universal instantiation from (1)
(3)	$H(Sachin)$	Premise
(4)	$M(Sachin)$	Modus ponens from (2) and (3)

Universal Modus Ponens

Universal modus ponens combines universal instantiation and modus ponens into one rule.

$\forall x\,(P(x) \rightarrow Q(x))$
P(a), where a is a particular element in the domain

$\therefore Q(a)$

This is what our previous example used!

The Lewis Carroll Example Revisited

- Premises:
  1. "All lions are fierce."
  2. "Some lions do not drink coffee."

  Conclusion: Can we conclude the following?

  3. "Some fierce creatures do not drink coffee."
- Let L(x): "x is a lion." F(x): "x is fierce." and C(x): "x drinks coffee." Then the above three propositions can be written as:
  1. $\forall x \, (L(x) \rightarrow F(x))$
  2. $\exists x \, (L(x) \land \neg C(x))$
  3. $\exists x \, (F(x) \land \neg C(x))$
- **How to conclude 3 from 1 and 2?**

The Lewis Carroll Example Revisited

1. $\forall x\,(L(x) \rightarrow F(x))$
2. $\exists x\,(L(x) \land \neg C(x))$
3. $\exists x\,(F(x) \land \neg C(x))$

How to conclude 3 from 1 and 2?

1.	$\exists x\,(L(x) \land \neg C(x))$	Premise
2.	$L(Foo) \land \neg C(Foo)$	Existential Instantiation from (1)
3.	$L(Foo)$	Simplification from (2)
4.	$\neg C(Foo)$	Simplification from (2)
5.	$\forall x\,(L(x) \rightarrow F(x))$	Premise
6.	$L(Foo) \rightarrow F(Foo)$	Universal instantiation from (5)
7.	$F(Foo)$	Modus ponens from (3) and (6)
8.	$F(Foo) \land \neg C(Foo)$	Conjunction from (4) and (7)
9.	$\exists x\,(F(x) \land \neg C(x))$	Existential generalization from (8)

Example: Is Moo Carnivorous?

- Premises:

1. “If x is a lion, then x is carnivorous.”
2. “Moo is not carnivorous.”

Conclusion: Can we conclude the following?

3. “Moo is not a lion.”

- Let L(x): “x is a lion.” C(x): “x is carnivorous.”
- Then the above three propositions can be written as:

1. $\forall x\ (L(x) \rightarrow C(x))$
2. $\neg C(Moo)$
3. $\neg L(Moo)$

- **How to conclude 3 from 1 and 2?**

Example: Is Moo Carnivorous?

1. $\forall x\,(L(x) \rightarrow C(x))$
2. $\neg C(Moo)$
3. $\neg L(Moo)$

How to conclude 3 from 1 and 2?

1. $\forall x\,(L(x) \rightarrow C(x))$	Premise
2. $L(Moo) \rightarrow C(Moo)$	Universal instantiation from (1)
3. $\neg C(Moo)$	Premise
4. $\neg L(Moo)$	Modus tollens from (1) and (2)

Universal Modus Tollens

Universal modus tollens combines universal instantiation and modus ponens into one rule.

$\forall x\ (P(x) \rightarrow Q(x))$

$\neg Q(a)$, where a is a particular element in the domain

$\therefore\ \neg P(a)$

This is what our previous example used!