Casino Math

Second Edition

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DEDICATIONS

For Bobby and Andy, the best sons any man could ever hope for;
And for Terry, my soulmate without whom I would be forever lost.

Bob Hannum

To my wife, Linda, for the love and understanding to allow me the chance to pursue my goals. To my children, Trace and Dani, who are blessed with all the talent in world, my hope and pledge is that they will have the same opportunity.

Tony Cabot

ACKNOWLEDGEMENTS

This second edition of *Casino Math* is, we feel, much improved over the first. As with any project of this magnitude, there are many who deserve thanks, but special mention must be made of Stew Ethier, who helped correct several errors and whose well-considered comments contributed significantly to the enhancement of numerous sections. Bill Eadington and Judy Cornelius of the Institute for the Study of Gambling and Commercial Gaming were kind enough to provide us a forum for the publication of this book. Howard Schwartz from the Gambler's Book Store has provided suggestions for this edition. We also thank the many University of Denver students, current and past, who drew our attention to errors and made helpful suggestions for revisions.

FOREWORD

> **The desire to risk is an essential part of human nature.**
>
> David Johnston
> *Temples of Chance* (1992)

This book presents the mathematics underlying casino games, provides a precise account of the odds associated with these games, and describes the role of mathematics in casino gaming management. The reader will gain an understanding of how and why casino games produce the expected revenues. Topics discussed in this text range from the basic principles of probability, odds, expectation, house advantage, and the law of averages, to price setting using game odds, gaming and economic regulations, standards for fairness, player worth, and rebate programs. Common casino measurements such as drop, handle, win, hold, and theoretical win are explained and illustrated for each of the major games. No formal background in mathematics is necessary, although comfort with the notion of percentages and quantitative reasoning is helpful. The primary intended audience is casino managers, or potential managers, but this book will serve as a valuable reference for anyone interested in the mathematical foundations of casino games and the use of mathematics in gaming management practice.

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PREFACE

> **Can one even as much as touch a gambling table without becoming immediately infected with superstition?**
>
> Fyodor Dostoevsky
> *The Gambler* (1867)

Critics of the gaming industry have long accused it of creating the name "gaming" as more politically correct than calling itself the "gambling industry." The term "gaming," however, has been around for centuries and more accurately describes the operators' view of the industry because most often casino operators are not gambling. Instead, they rely on mathematical principles to assure that their establishment generates positive gross gaming revenues.[1] The operator, however, must assure the gaming revenues are sufficient to cover deductions like bad debts, expenses, employees, taxes and interest.

Despite the obvious, many casino professionals limit their advancements by failing to understand the basic mathematics of the games and their relationships to casino profitability. One casino owner would often test his pit bosses by asking how a casino could make money on blackjack if the outcome is determined simply by whether the player or the dealer came closest to 21. The answer, typically, was because the casino maintained "a house advantage." This was fair enough, but many could not identify the amount of that advantage or what aspect of the game created the advantage.

As in any industry, managers must know a great deal about many things. Given that products offered by casinos are games, managers must understand why the games provide the expected revenues. In the gaming industry, nothing plays a more important role than mathematics.

Mathematics also should overcome the dangers of superstitions. An owner of a major strip casino once experienced a streak of losing substantial amounts of money to a few "high rollers." He did not attribute this losing streak to normal volatility in the games, but to bad luck. His solution was simple. He spent the evening spreading salt throughout the casino to ward off the bad spirits.

Before attributing this example to the idiosyncrasies of one owner, his are atypical only in their extreme. Superstition has long been a part of gambling – from

1 Gross gaming revenues are usually expressed as the sum of all amounts received as winning less all amounts paid out as losses.

both sides of the table. Superstitions can lead to irrational decisions that may hurt casino profits. For example, believing that a particular dealer is unlucky against a particular (winning) player may lead to a decision to change dealers. As many, if not most, players are superstitious. At best, he may resent that the casino is trying to change his luck. At worst, the player may feel the new dealer is skilled in methods to "cool" the game. Perhaps he is even familiar with stories of old where casinos employed dealers to cheat "lucky" players.

Understanding the mathematics of a game also is important for the casino operator to ensure that the reasonable expectations of the players are met. For most persons, gambling is entertainment. It provides an outlet for adult play.[2] As such, persons have the opportunity for a pleasant diversion from ordinary life and from societal and personal pressures.

As an entertainment alternative, however, players may consider the value of the gambling experience. For example, some people may have the option of either spending a hundred dollars during an evening by going to a professional basketball game or at a licensed casino. If the house advantage is too strong and the person loses his money too quickly, he may not value that casino entertainment experience. On the other hand, if a casino can entertain him for an evening, and he enjoys a "complimentary" meal or drinks, he may want to repeat the experience, even over a professional basketball game.

Likewise, new casino games themselves may succeed or fail based on player expectations. In recent years, casinos have debuted a variety of new games that attempt to garner player interest and keep their attention. Regardless of whether a game is fun or interesting to play, most often a player will not want to play games where his money is lost too quickly or where he has an exceptionally remote chance of returning home with winnings.

Mathematics also plays an important part in meeting players' expectations as to the possible consequences of his gambling activities. If gambling involves rational decision-making, it would appear irrational to wager money where your opponent has a better chance of winning than you do. Adam Smith suggested that all gambling, where the operator has an advantage, is irrational. He wrote "There is not, however, a more certain proposition in mathematics than that the more tickets [in a lottery] you advertise upon, the more likely you are a loser. Adventure upon all the tickets in the lottery, and you lose for certain; and the greater the number of your tickets, the nearer you approach to this certainty."[3]

Even where the house has an advantage, however, a gambler may be justified if the amount lost means little to him, but the potential gain would elevate him to a higher standing of living.[4] For example, a person with an annual income of $30,000 may have $5 in disposable weekly income. He could save or gamble this money. By saving it, at the end of a year, he would have $260. Even if he did this for years, the savings would not elevate his economic status to another level. As an alternative, he could use the $5 to gamble for the chance to win $1 million. While the odds of

2 V. Abt, et. al., Business of Risk (1985).

3 Quoted in David Weinstein and Lillian Deitsh, The Impact of Legalized Gambling: The Socioeconomic Consequences of Lotteries and Off-Track Betting, 147.

4 Id. at 131.

winning are remote, it may provide the opportunity to move to a higher economic class.

Since the casino industry is heavily regulated and some of the standards set forth by regulatory bodies involve mathematically related issues, casino managers also should understand the mathematical aspects relating to gaming regulation. Gaming regulation is principally dedicated to assuring that the games offered in the casino are fair, honest, and that players get paid if they win. Fairness is often expressed in the regulations as either requiring a minimum payback to the player or, in more extreme cases, as dictating the actual rules of the games offered. Casino executives should understand the impact that rules changes have on the payback to players to assure they meet regulatory standards. Equally important, casino executives should understand how government mandated rules would impact their gaming revenues.

BRIEF HISTORY OF GAMBLING

1

Consumers spend more on legal gambling in the United States than movie tickets, recorded music, theme parks, spectator sports, and video games combined. In 2003, gross gaming revenues totaled $73 billion, with gamblers placing about $860 billion in legal wagers.[5]

> Gambling … the revolutionary idea that defines the boundary between modern times and the past is a mastery of risk: the notion that the future is more than the whim of the gods and that men and women are not passive before nature.
>
> - Peter L. Bernstein, *Against the Gods: The Remarkable Story of Risk*, p. 1

EARLY HISTORY OF GAMING

Gambling has held human beings in thrall for millennia. In has been engaged in everywhere, from the dregs of society to the most respectable circles. Pontius Pilate's soldiers cast lots for Christ's robe as He suffered on the cross. The Roman Emperor Marcus Aurelius was regularly accompanied by his personal croupier. The Earl of Sandwich invented the snack that bears his name so that he could avoid leaving the gaming table in order to eat. George Washington hosted games in his tent during the American Revolution. Gambling is synonymous with the Wild West. And "Luck Be a Lady Tonight" is one of the most memorable numbers in Guys and Dolls, a musical about a compulsive gambler and his floating crap game. The earliest known form of gambling was a kind of dice game played with what was known as an astragalus, or knuckle-bone. This early ancestor of today's dice was a squarish bone taken from the ankles of sheep or deer, solid and without marrow, and so hard as to be virtually indestructible. Astragali have surfaced in archeological digs in many parts of the world. Egyptian tomb paintings picture games played with

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5 American Gaming Association; Christiansen Capital Advisors.

astragali dating from 3500 BC, and Greek vases show young men tossing the bones into a circle.[6]

While some may think that gambling is a recent phenomenon, it actually dates back to antiquity. Dice have been recovered from Egyptian tombs, while the Chinese, Japanese, Greeks and Romans all were known to play games of skill and chance for amusement as early as 2300 B.C.

Both Native Americans and European colonists brought a history of gambling from their own cultures that helped shape America's views and practices. Native Americans developed games and language describing gambling and believed that their gods determined fate and chance. In 1643, explorer Roger Williams wrote about the games of chance developed by the Narragansett Indians of Rhode Island.

British colonization of America was partly financed through lottery proceeds, beginning in the early 17th century. Because lotteries were viewed as a popular form of voluntary taxation in England during the Georgian era, they also became popular in America as European settlers arrived here. Prominent individuals such as Ben Franklin, John Hancock and George Washington sponsored a half-dozen lotteries that operated in each of the 13 colonies to raise funds for building projects. Between 1765 and 1806, Massachusetts authorized lotteries to help build dormitories and supply equipment for Harvard College; other institutions of higher learning, including Dartmouth, Yale and Columbia, also were financed through lotteries. A lottery even was approved to finance the American Revolution.[7]

An intuitive sense of the notions underlying probability has probably characterized winning gamblers since gambling was invented. The Greeks had a word for probability, *eikos*, with the modern meaning of "to be expected with some degree of certainty," and Aristotle came close to putting quantities to it when he wrote in *De Caelo* that "… to repeat the same throw ten thousand times with the dice would be impossible, whereas to make it once or twice is comparatively easy."

> We are all gamblers. And, as long as chance and uncertainty persist as features of human existence, we will continue to be so. In the modern world, however, as never before, chance has become an irreducible aspect of daily life: risk, speculation, indeterminism and flux are our constant companions in social, economic and personal affairs: we have entered the Age of Chance.
>
> - Gerda Reith, *The Age of Chance* (1999), p. 1

The intuitions of gamblers began to find their way into mathematics in 1494, when a Franciscan monk named Luca Paccioli posed what came to be known as the "problem of the points," drawing from a gambling game called *balla*. "A and B are playing a fair game of *balla*," he stipulated. "They agree to continue until one has won six rounds. The game actually stops when A has won five and B three. How should the stakes be divided?" The first approach to answering the question was given

6 This paragraph taken from Peter L. Bernstein, Against the Gods: The Remarkable Story of Risk, p. 12.

7 The preceding paragraphs taken from Mike Roberts, "The National Gambling Debate: Two Defining Issues," Whittier Law Review, Vol. 8, no. 3, 1997; Scarne's New Complete Guide to Gambling; Encarta Online Encyclopedia.

about fifty years later by Girolamo Cardano, a Renaissance polymath and self-confessed chronic gambler, but was not published until 1663.

The credit for inventing probability theory goes to Blaise Pascal and Pierre de Fermat, who in the course of a correspondence in the 1650s solved the problem of the points by means of what has become known as Pascal's Triangle, a way of laying out the number of ways in which a particular event can occur. Armed with Pascal's Triangle, it is possible to determine the proportion that any one, or any combination, of those events represents of the total.[8]

GAMING REVENUE: 10-YEAR TRENDS[9]

As the gaming industry has expanded throughout the United States, the gross annual revenue has steadily increased. Gross gambling revenue (GGR) is the amount wagered minus the winnings returned to players, a true measure of the economic value of gambling. GGR is the figure used to determine what a casino, racetrack, lottery or other gaming operation earns before taxes, salaries and other expenses are paid - the equivalent of "sales," not "profit." In 2003 for example, the commercial casino industry had GGR of more than $27 billion (not including deepwater cruise ships, cruises-to-nowhere or non-casino devices), but paid $11.8 billion in wages and benefits and more than $4.3 billion in taxes, plus other expenses.

The following tables show the growth from 1993 to 2003 in both the commercial casino industry and gaming as a whole, which includes pari-mutuel wagering, lotteries, casinos, legal bookmaking, charitable gaming and bingo, Indian reservations and card rooms.

YEAR	TOTAL COMMERCIAL CASINO	TOTAL GAMING
1993	$11.2 billion	$34.7 billion
1994	$13.8	$39.8
1995	$16.0	$45.1
1996	$17.1	$47.9
1997	$18.2	$50.9
1998	$19.7	$54.9
1999	$22.2	$58.2
2000	$24.3*	$61.4
2001	$25.7*	$63.3
2002	$26.5*	$68.7
2003	$27.0*	$72.9

*Amount does not include deepwater cruise ships, cruises-to-nowhere or non-casino devices.
Sources: American Gaming Association, Christiansen Capital Advisors LLC.

STATE STATISTICS

The chart below includes statistics for the 11 commercial casino states in the categories of current number of operating casinos, gross casino gaming revenue,

8 The preceding paragraphs taken from Charles Murray, Human Accomplishment (2003), p 232.
9 American Gaming Association

gaming tax revenue, casino employees and casino employee wages. Combined totals of all the states in each category also are included.

2003 STATE STATISTICS

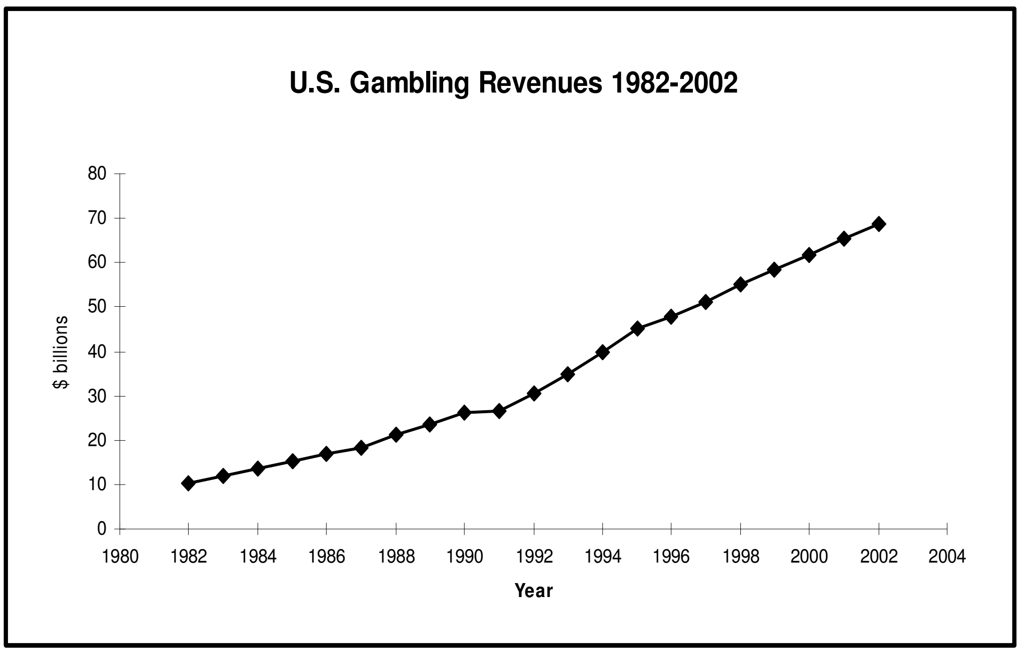
State	Current # of Operating Casinos	Gross Casino Gaming Revenue	Gaming Tax Revenue	Casino Employees	Casino Employee Wages
Colorado	44	$698.2 m	$95.6 m	7,364	$207.1 m
Illinois	9	$1.7 b	$719.9 m	9,101	$376.4 m
Indiana	10	$2.2 b	$702.7 m	16,555	$589.5 m
Iowa	13	$1.0 b	$209.7 m	8,764	$278.5 m
Louisiana	18	$2.0 b	$448.9 m	20,755	$452.7 m
Michigan	3	$1.1 b	$250.2 m	8,087	$368.0 m
Mississippi	29	$2.7 b	$325.0 m	30,377	$1.0 b
Missouri	11	$1.3 b	$369.0 m	10,700	$310.0 m
Nevada*	256	$9.6 b	$776.5 m	192,812	$7.0 b
New Jersey	12	$4.5 b	$414.5 m	46,159	$1.2 b
South Dakota**	38	$70.4 m	$5.5 m	1,734	$34.9 m
TOTAL	**443**	**$27.0 b**	**$4.3 b**	**352,428**	**$11.8 b**

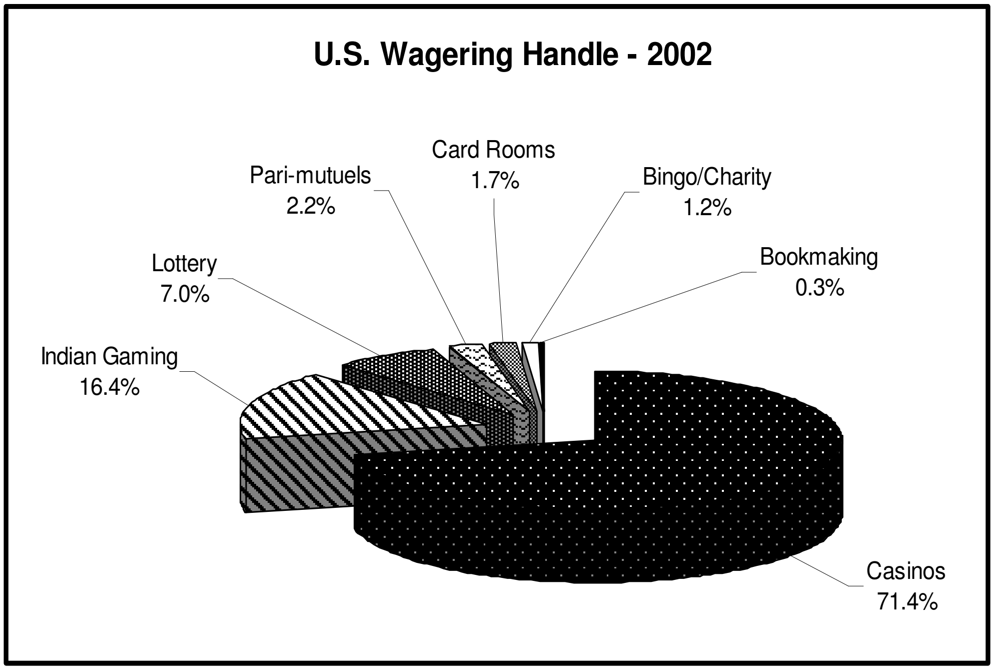
*Includes locations with gross gaming revenue of at least $1 million.

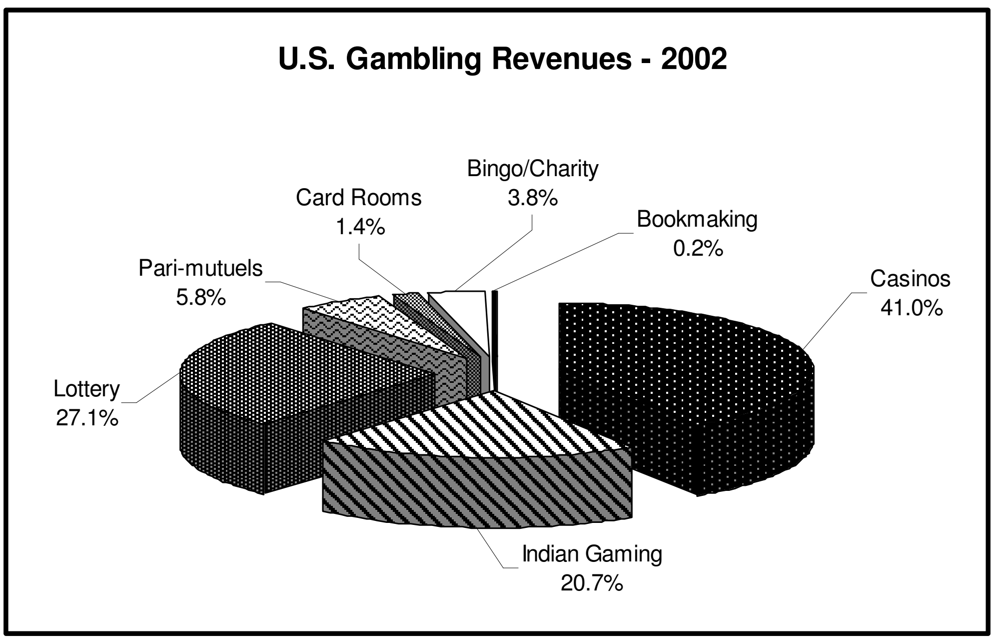
**Employees and employee wages for 2001.

Sources: Casino operators, and state gaming regulatory agencies and associations.


U.S. Wagering Handle 1982-2002
$ billions
900
800
700
600
500
400
300
200
100
0
1980
1982
1984
1986
1988
1990
1992
1994
1996
1998
2000
2002
2004
Year


U.S. Gambling Revenues 1982-2002
$ billions
80
70
60
50
40
30
20
10
0
1980
1982
1984
1986
1988
1990
1992
1994
1996
1998
2000
2002
2004
Year


U.S. Wagering Handle - 2002
Card Rooms
1.7%
Pari-mutuels
2.2%
Bingo/Charity
1.2%
Lottery
7.0%
Bookmaking
0.3%
Indian Gaming
16.4%
Casinos
71.4%


U.S. Gambling Revenues - 2002
Bingo/Charity
3.8%
Card Rooms
1.4%
Bookmaking
0.2%
Pari-mutuels
5.8%
Casinos
41.0%
Lottery
27.1%
Indian Gaming
20.7%

FEDERAL STUDY COMMISSION: ECONOMIC IMPACT*

After a two-year study of legalized gambling in the United States, the congressionally mandated National Gambling Impact Study Commission found numerous benefits of casino gaming. The following are conclusions reached in both the 1999 NGISC final report and commission-funded research:

- 'As it has grown, it [gambling] has become more than simply an entertainment past-time: the gambling industry has emerged as an economic mainstay in many communities and plays an increasingly prominent role in state and even regional economies. ...'
- 'Research conducted on behalf of the commission confirms the testimony of ... casino workers and government officials that casino gaming creates jobs and reduces the level of unemployment and government assistance in communities that have legalized it.'
- 'Gambling appears to have net economic benefits for economically depressed communities.'
- '... a new casino of even limited attractiveness, placed in a market that is not already saturated, will yield positive economic benefits on net to its host economy.'
- 'Those communities closest to casinos experienced a 12% to 17% drop in welfare payments, unemployment rates and unemployment insurance.'
- 'The Commission also heard from a number of local officials in jurisdictions where casinos are located. Among those who informed the commissioners with their testimony were Elgin, Illinois Mayor Kevin Kelly, Mayor Scott King from Gary Indiana, as well as mayors from Bettendorf, Iowa and Alton, Illinois. The Commission also heard from Mayors A.J. Holloway, Bobby Williams, Bob Short and Eddy Favre of Biloxi, Tunica, Gulfport and Bay St. Louis, Mississippi respectively. Without exception these elected officials expressed support for gambling and recited instances of increased revenues for their cities. They also discussed community improvements made possible since the advent of gambling in their communities and reviewed the general betterment of life for the citizenry in their cities and towns.'

The commission also made numerous recommendations supporting the conclusion that casino gaming can have a positive impact on the economy:

- 'The Commission recommends to State, local and Tribal governments that (when considering the legalization of gambling or the repeal of gambling that is already legal) they should recognize that, especially in economically depressed communities, casino gambling has demonstrated the ability to generate economic development through the creation of quality jobs.'
- 'The Commission recommends to State, local and Tribal governments that (when considering the legalization of gambling) casino development should be targeted for locations where the attendant jobs and economic development will benefit communities with high levels of unemployment and underemployment and a scarcity of jobs for which the residents of such communities are qualified.'
- 'The Commission recommends to State, local and Tribal governments that when planning for gambling-related economic development, communities with legal gambling or that are considering the legalization of gambling should recognize that destination resorts create more and better quality jobs than casinos catering to a local clientele.'

**American Gaming Association* (http://www.americangaming.org/); *National Gambling Impact Study Commission* Final Report; *National Research Council* (NRC) of the *National Academy of Sciences*, Pathological Gambling; Adam Rose and Associates, The Regional Economic Impacts of Casino Gambling: Assessment of the Literature and Establishment of a Research Agenda; *National Opinion Research Center*, Gambling Impact and Behavior Study, Report to the NGISC.

One of the most striking features to emerge from a cursory glance at the phenomenon [gambling] is its almost universal prevalence throughout history and across cultures: it unites peoples as diverse as the ancient Greeks and the North American Indians with the inhabitants of modern Western societies. Despite its apparent insignificance as 'mere' games then, it can be said to constitute a fundamental feature, in some form or another, of human life.

- Gerda Reith, *The Age of Chance* (1999), p. 1-2

CASINO ADVERTISING*

In June 1999, the U.S. Supreme Court struck down the advertising restriction on the commercial casino industry in its decision in the case Greater New Orleans Broadcast Association v. United States. Until then, the Communications Act of 1934 had prohibited all television ads that showed gambling activity by commercial casinos.

Before the ruling, commercial casinos had been allowed to promote only their non-gaming resort features. The 9th U.S. Circuit Court had affirmed the decision of a Nevada District Court overturning that portion of the federal law in 1997. The 5th Circuit Court of Appeals, however, had upheld the ban. The Supreme Court's decision overturned the ban nationwide, and cleared up any inconsistency between the two lower court decisions.

Exceptions to the Communications Act of 1934 had been made over the years, allowing lotteries, horse and dog racing and Indian casinos to advertise freely. Only commercial casinos were still banned from showing gambling activity. As a result, the high court found in favor of the Greater New Orleans Broadcast Association.

The issue of casino advertising was one followed not only by the gaming industry itself, but also by the advertising industry and the media. Both groups wanted the ban lifted, saying it violated First Amendment rights.

The American Gaming Association also voiced its opinion on the issue, filing an amicus brief with the Supreme Court to address some of the social cost arguments made by the U.S. Justice Department in support of the ban.

**American Gaming Association* (http://www.americangaming.org/).

MATHEMATICAL THEORY FOR CASINO GAMES

2

This chapter presents the mathematical foundations necessary for understanding how casino games work. Percentages, odds, and probability are the foundation of the gaming industry. The mathematics underlying the games assures the casinos will make money. Since games are the products sold by the gaming industry, managers must understand how the mathematics behind the games produces the expected revenues.

PROBABILITY

> Probability is the very guide of life.
>
> Cicero (106-43 B.C.)

Outcomes in games of chance are, by definition, uncertain. The branch of mathematics used to describe and calculate uncertainty is called *probability*. Not surprisingly, the roots of modern probability theory came from gambling problems posed by seventeenth-century gamblers to mathematicians. The problems involved the chances of winning or losing certain wagers. Casinos face the same concerns today. One Las Vegas casino owner retained a mathematician simply to answer questions about the probable outcome of events that were occurring or might occur in his casino. This could include players suggesting unusual bets or requesting deviations from normal house limits or rules.

Most people are familiar with the basic notion of probability. When tossing a fair coin, the probability of heads is ½ since you would expect heads to occur about ½ the time in a very large number of tosses. In the formal study of the laws of chance, probability quantifies uncertainty by expressing the likelihood that an event will occur. It is a measure of the uncertainty associated with the outcome of a random experiment. The experiment may be the toss of a pair of dice, the draw of a playing card, the selection of a certain combination of symbols on a slot machine, or the number resulting from the spin of a roulette wheel.

The Origins of Probability Theory Part 1

Although evidence exists that games of chance have been around for several thousand years, the birth of the formal study of probability is generally ascribed to the correct analysis of two gambling propositions in the mid-17th century.

The Chevalier de Mere, a French nobleman and gentleman gambler, had made considerable money betting even money that a six would be rolled at least once in four throws of a single die.

Incorrectly reasoning that the probability of winning this wager was 4/6, he proposed a second wager he believed would result in the same rate of success: that a double six would be rolled at least once in twenty-four rolls of a pair of dice.

When he lost at this second wager, the Chevalier consulted mathematician Blaise Pascal to explain. Pascal in turn corresponded with another mathematician, Pierre de Fermat. In their series of exchanges, Pascal and Fermat solved the two dice propositions and laid the foundations for a new branch of mathematics now known as probability.

Probability reflects how often we expect things to occur and can be interpreted to be the proportion of times an event will occur in the long run. That is, the probability of an event is the approximate proportion of times the event will occur in many repetitions of the random experiment. Probability can be expressed as a fraction, decimal, or percentage. The probability of heads is ½, or .50, or 50%. Sometimes probability is referred to as *chance*. A person may say that there is a 50% chance of getting heads when tossing a coin.

EXAMPLE: CARDS

A standard deck of 52 playing cards has four Aces. If a card is selected randomly from the deck, the probability of it being an Ace is 4/52, or 1/13. It also could be expressed as a 7.7% chance of randomly selecting an ace. The probability that it is a diamond is 13/52 (or 25%), the probability of it being red is 26/52 (or 50%), and the probability of it being a face card is 12/52 (or 23.1%).

EXAMPLE: DICE

A fair die has six sides with values 1 through 6, each of which is equally likely to appear face-up if the die is thrown. The probability of getting the number 4 is 1/6, and the probability of getting a 6 is also 1/6. When tossing a pair of dice, there are 36 (6 x 6) possible outcomes. To see this, it may be helpful to think of the two dice as being different colors, say red and green. One possible outcome is that the red die is a 4 and the green die is a 3. Another outcome is that the red die is a 3 and the green die is a 4. Each of these outcomes, (R4,G3) and (R3,G4), has probability 1/36, but the probability of throwing a total of seven is 6/36 because there are 6 combinations (outcomes) that produce a total of seven: (R1,G6), (R2,G5), (R3,G4), (R4,G3), (R5,G2), and (R6,G1). Similarly, the probability of throwing totals of 2, 3, 4, 5, 6, 8, 9, 10, 11, and 12, are 1/36, 2/36, 3/36, 4/36, 5/36, 5/36, 4/36, 3/36, 2/36, and 1/36, respectively.

EXAMPLE: ROULETTE

A double-zero roulette wheel has 38 numbers – 18 red, 18 black and 2 green. In a single spin of this wheel, the probability the ball lands on the number 8 is 1/38, the probability it lands on the number 00 is 1/38, the probability it lands on a red number is 18/38, and the probability it lands on a number in the first dozen (1 through 12) is 12/38. The double-zero wheel is favored in the United States. A single-zero roulette wheel, favored in Europe, has no 00 but is otherwise the same as the double-zero wheel. In a single spin of the single-zero wheel, the probability of getting the number 8 is 1/37, the probability of red is 18/37, and the probability of the first dozen is 12/37.

BASIC PROBABILITY RULES

For each of the "experiments" in the simple examples above, each possible outcome has the same chance of occurring. For example, when a pair of dice is tossed, there are 36 possible outcomes and each has probability equal to 1/36. When a card is selected randomly from a single deck of 52 playing cards, each card has a 1/52 chance of being selected. In roulette, each number is equally likely to occur. Although many basic experiments can be analyzed using this "equally likely" outcomes property, for other more complicated events, a few basic rules are essential to calculating the relevant probabilities.

Familiarity with these rules provides the foundation for understanding the mathematics of casino games. These rules are as follows:

PROBABILITY RULE ONE

> ***Rule 1***
> *The probability of an event will always be between 0 and 1.*

The probability of an event represents the likelihood that the event will occur. If an event has a probability of zero, it means that event is impossible. If an event has a probability of one, then that event must occur. The closer a probability is to one, the more likely that event is to occur, and the closer the probability is to zero the less likely it is to occur.

PROBABILITY RULE TWO

> ***Rule 2***
> ***(Complement Rule)***
>
> *The probability of an event occurring plus the probability of that event not occurring equals 1.*

The opposite of an event is referred to as its complement. The complement of heads is tails, the complement of odd is even, and the complement of Spade is non-Spade. The Complement Rule is true because the total probability of all possible outcomes must equal 1. For example, when selecting a card from a deck, either a Spade is selected (13/52 or 25%) or something other than a Spade is selected (1 – 13/52 = 39/52 or 75%). Likewise, the probability that either a Spade or a non-Spade is selected is 13/52 + 39/52 or 52/52, or expressed in percentage as 25% + 75%, or 100%.

RULE THREE

> ***Rule 3***
> ***(Addition Rule)***
>
> *For mutually exclusive events, the probability of at least one of these events occurring equals the sum of their individual probabilities.*

The Addition Rule comes in two flavors: one for mutually exclusive events and one for events that are not mutually exclusive. The rule given here is for mutually exclusive events. Events are mutually exclusive if the occurrence of one precludes the occurrence of the other. As an example, consider the probability that a card randomly selected from a standard deck is either a Heart or a Spade. Because these events are mutually exclusive, each with probability 13/52, the probability of getting either a Heart or Spade is 13/52 + 13/52 = 26/52, or ½ (or 50%).

When rolling a pair of dice, the probability of rolling a 7 or 11 is (6/36 + 2/36) = 8/36, and the probability of rolling a 2, 3, or 12 is (1/36 + 2/36 + 1/36) = 4/36.[10]

RULE FOUR (A)

> ***Rule 4A***
> ***(Multiplication Rule – independent events)***
>
> *For independent events, the probability of all of them occurring equals the product of their individual probabilities.*

While the Addition Rule finds the probability of at least one of several events occurring, the Multiplication Rule is used to find the probability of them all occurring. This rule also comes in two flavors – a general version and a version for independent events. Rule 4A applies to independent events.

Events are independent if the occurrence of one has no effect on the probability of the occurrence of the others. If you flip a coin two times, whether you flip heads or tails on the first flip has no influence on whether you will flip heads or tails on the second flip. The two flips of the coin are independent events. Thus, the probability of two heads in two tosses is ½ x ½, or ¼. Similarly, the probability of getting five heads in 5 tosses of a fair coin can be calculated to be (½ × ½ × ½ × ½ × ½) = $(½)^5$ = 1/32, or about .031. The probability of getting five consecutive twelve's in five tosses of a pair of dice is $(1/36)^5$ = .000000017, or about 1 in 60,466,176. The probability of ten consecutive red numbers in ten spins of a roulette wheel is $(18/38)^{10}$ = .0005687, or about 1 in 1,758. The calculations in these examples are valid because

10 Individual dice probabilities are covered later in the section on craps.

the events involved are independent trials. The outcome on any given trial has no effect on the outcome of any other trial.

Until they were stolen in 1951, the late Desert Inn Casino proudly displayed a pair of dice with a plaque that read "These dice made 28 consecutive passes."[11] This was among the most successful recorded runs on the craps table in Las Vegas history. The reason these events occur so infrequently is a function of the multiplication rule for independent events. The probability of making a single pass at the craps table is .493 or 49.3%. The odds of making 28 straight passes is $(.493)^{28}$ or about 1 out of 400 million.

RULE FOUR (B)

Rule 4B is similar to Rule 4A, but is used when the events in question are not independent. For example, try to calculate the probability of randomly selecting three cards from a deck and all three of those cards being Aces. The probability that the first card will be an Ace is 4/52. If you drew the Ace, there would be one less Ace and one less card overall in the deck. In fact, there would be three Aces among the 51 remaining cards. Therefore, the probability the second card you draw would be an Ace given the first is an Ace is 3/51. Now, if you also drew an Ace as your second card, the probability that you would draw an Ace as the third card given the first two are Aces is 2/50 (two fewer Aces and two fewer cards). So the probability of drawing three cards and all being Aces is $4/52 \times 3/51 \times 2/50$, or .000181 (1 in 5,525).

Notice that the trials here are not independent since the outcome of the first draw will affect the outcome on the second, and the outcome of the second will affect the outcome on the third. If you were to select the three cards with

Rule 4B
(Multiplication Rule – dependent events)

For non-independent events, the probability of all of them occurring equals the product of their conditional probabilities, where the conditional probability of one event is affected by the event(s) that came before it.

The Origins of Probability Theory Part 2

The solutions to the two dice problems solved by Pascal and Fermat in 1654, and generally regarded as the birth of probability, are based on the multiplication rule (Rule 4) and the complement rule (Rule 2). The probability of rolling at least one six in four throws of a single die is most easily calculated by first finding the probability this does not occur:

$$P(\text{no sixes}) = (5/6)x(5/6)x(5/6)x(5/6) = .482.$$

The probability of getting at least one six, then, is 1 − .482 = .518. Similarly, the probability of rolling at least one double six in twenty-four throws of a pair of dice is given by

$$1 - (35/36)^{24} = 1 - .509 = .491.$$

11 Barney Vinson, Las Vegas Behind the Tables, Part 2, (1988).

replacement (replace each card into the deck after selecting, so you are drawing from the complete deck of 52 each time), the trials would now be independent and the probability of getting three Aces would be $4/52 \times 4/52 \times 4/52$, or .000455 (1 in 2,197).

These examples highlight a key difference between games like roulette and craps and the game of blackjack. Roulette and craps involve independent trials whereas blackjack involves dependent trials. In roulette and craps, the probabilities remain the same from trial to trial (it doesn't matter if there are ten consecutive red numbers; the probability of red is still 18/38 on the eleventh trial).

In blackjack, the trials are dependent since the composition of the cards remaining to be played depends on which cards have already been played. It is this feature, dependent trials, which causes the advantage in blackjack to flow back and forth between the house and the player and is why "card-counting" can be effective in blackjack.[12]

ODDS

Odds are another way to express probability. Whereas probability represents the relationship between the desired event and all possible events, odds describe the relationship between a desired event and all non-desired events. As a general rule, odds are stated against an event in casinos. For example, the probability of selecting a Spade from a shuffled deck of 52 cards is 13/52 or 1/4. This is because there are 13 favorable events (Spade) out of the total of 52 possible outcomes. Expressed as odds, the odds against getting a Spade are 3 to 1, because there are 3 non-spades for every 1 Spade. As another example, in double-zero roulette, the probability of any single number is 1/38 and so the true odds against any given number are 37 to 1. To generalize, if the probability of an event is X/Y, then the odds against that event are Y–X to X.

To convert from odds to probability, if the odds against are X to Y, then the probability in favor is Y/(X+Y). For example, when throwing a pair of dice, the probability of rolling a seven is 1/6, for odds against of (6–1) to 1, or 5 to 1. The odds against rolling a twelve are 35 to 1, for a probability of 1/(35+1), or 1/36.

True odds are different than payoff odds. For the roulette example above, the true odds against a single number are 37 to 1, but the casino's payoff odds are 35 to 1. This, of course, is how the casino makes money.[13]

Sometimes odds are expressed as X *for* Y rather than X *to* Y. This means that the amount X to be awarded to a winning player will include the original bet. A player winning 4 for 1 on a $5 wager will be given $20, an amount that includes the $5 wagered. Thus, 4 for 1 odds are equivalent to 3 to 1 odds. In both cases, a winning wager will be paid a profit of 3 units, in addition to the player keeping the original 1 unit bet. *Even money* means the payoff is 1 to 1, for a profit of 1 unit.

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12 Hands dealt from the same shoe in baccarat are also dependent trials, but it has been shown that card-counting is not effective in baccarat.

13 There is one bet in the casino, the free odds bet in craps, in which the payoff odds are equal to the true odds.

PERMUTATIONS

Permutations refer to the number of different ways in which objects can be arranged in order. In a permutation, each item can appear only once and each order of the items' arrangement constitutes a separate permutation.

EXAMPLE: HORSE RACING

Consider a race with five horses, labeled A, B, C, D, and E. In horse racing, an exacta (or perfecta) is a wager that wins if the race's first two finishers are picked in exact order. The number of possible exacta tickets is just the number of ways of picking and ordering two of the five horses. With this five-horse race, it is easy to just list the twenty possible exacta tickets (the first letter designates the 1st place finisher and the second letter designates the 2nd place finisher): AB, AC, AD, AE, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EA, EB, EC, ED. Although a bit more tedious, it is also possible to list the 60 possible trifecta tickets, where the race's first three finishers are selected in exact order.

Obviously if the set of objects is large, enumerating all the possible permutations is not feasible. The following formula shows how to calculate the number of permutations without having to list them out.

PERMUTATIONS FORMULA

The number of permutations of *n* objects taken *r* at a time (each object can appear only once and order is important) is given by:

$$P_{n,r} = \frac{n!}{(n-r)!} = n \times (n-1) \times \ldots \times (n-r+1)\,,$$

where $$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1\,.$$

The exclamation point in the above permutations formula is called the factorial operation, and the expression $n!$ is read "n factorial." As examples, $4! = 24$, $5! = 120$, and $10! = 3{,}628{,}800$. By definition, $0! = 1$ (this is needed in the formula for combinations).

Applying the permutations formula with n=5 and r=2 yields $P_{5,2} = 20$, and n=5 and r=3 gives $P_{5,3} = 60$, confirming the exacta and trifecta values in the horse racing example. As illustrations using larger values of n, the formula shows there to be 90 different exacta and 720 different trifecta tickets in a ten-horse race, and 240 exacta and 3,360 trifecta possibilities when there are sixteen horses.

Note that when r=n, the permutations formula reduces to $n!$, which is the number of ways of ordering the entire set of n items. The set of three objects, A, B, and C, for example, can be ordered in $3! = 6$ ways: ABC, ACB, BAC, BCA, CAB, and CBA.

The next example shows selection processes in which order is important but where an object may be repeatedly selected, and so the permutations formula does not apply.

EXAMPLE: NUMBERS GAMES

Arizona's *Pick 3* and Texas' *Pick 3* numbers games pay 500 for 1 if the three digit number selected by a player matches, in the exact order, the three-digit number selected by a random draw. Since a digit may appear on a given ticket more than once (4-4-8, for example), the permutations formula does not apply when determining the number of possible tickets. The value obtained by applying the permutation formula, 720, counts only those tickets in which all three digits are different. There are, however, 1,000 ($10 \times 10 \times 10$) possible *Pick 3* tickets, and the probability of winning the 500 for 1 payoff is 0.001. A similar numbers game in which four digits are selected is played in several states – Delaware's *Play 4*, Pennsylvania's *Big 4*, Ohio's *Pick 4*, and Florida's *Play 4* are examples. These pay a top prize of 5,000 for 1 for matching all four randomly selected digits in exact order. In these four-digit numbers games, there are 10,000 possible tickets and the probability of winning the top prize is 1 in 10,000.

COMBINATIONS

Unlike permutations, combinations refer to the number of ways that a set of items can be selected from a group, without regard to the order in which the members of the set are arranged. To introduce the idea of combinations, consider a horse racing wager where the bettor picks the top two finishers but does not need to specify the order of finish for the selected horses.

EXAMPLE: HORSE RACING

A quinella is a racing wager that wins if the bettor selects the race's first two finishers in any order. In a race with five horses, A, B, C, D, and E, there are ten possible quinella combinations: AB, AC, AD, AE, BC, BD, BE, CD, CE, and DE. Since a quinella only needs to identify the two top finishers but not the order, there are fewer possible quinella tickets than exacta tickets, where the order of finish must also be specified.

As with permutations, there is a formula to compute the number of combinations. This formula, given below, shows how to calculate the number of ways to select x objects from a collection of n objects when order is not relevant.

COMBINATIONS FORMULA

The number of combinations of n objects taken x at a time (each object can appear only once and order is not relevant) is given by:

$$\binom{n}{x} = \frac{n!}{x!(n-x)!},$$

where $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$.

As examples, $\binom{4}{2} = 6$, $\binom{5}{3} = 10$, $\binom{20}{3} = 1{,}140$, and $\binom{10}{10} = 1$.

The last of these calculations required the use of the fact that 0! = 1. The number of combinations of n objects taken x at a time, $\binom{n}{x}$, is usually read "n choose x."

A key difference between combinations and permutations is that the order in which the selected objects are arranged is important in permutations but is not relevant in combinations. Many gaming applications exist in which items are selected but order is not important. These include keno, lotteries, and poker. Before moving on to the details of keno, two relatively easy illustrations for lotteries and poker are given.

EXAMPLE: LOTTERY

The Colorado Lotto drawing selects 6 numbers from the field 1 to 42. To win the jackpot, the player must match all 6 selected numbers, and order is irrelevant. Since there are $\binom{42}{6} = 5{,}245{,}786$ possible combinations of 6 numbers selected from 42, the odds of winning the Colorado Lotto jackpot are about 1 in 5.2 million.

This calculation can be viewed as taking the number of ways to select 6 numbers from 42 in a specific order, $P_{42,6} = 42 \times 41 \times 40 \times 39 \times 38 \times 37 = 3{,}776{,}965{,}920$, divided by the number of possible rearrangements or orderings, 6! = 720.

EXAMPLE: POKER

The combinations formula can be used to compute the probability of being dealt a royal flush. The number of 5-card hands that can be dealt from a standard deck of 52 cards is $\binom{52}{5} = 2{,}598{,}960$. Since 4 of these are royal flushes, the probability of being dealt a royal flush is 4 divided by 2,598,960, or 1 in 649,740. Like the Lotto example above, it may be helpful to realize that the calculation of the number of 5-card hands can be completed by taking the number of possible 5-card ordered hands, $P_{52,5} = 52 \times 51 \times 50 \times 49 \times 48 = 311{,}875{,}200$, and dividing by the number of possible orderings of these 5 cards, 5! = 120. The combinations formula is helpful in computing other poker probabilities as well.

EXPECTATION (EXPECTED VALUE)

Expectation, or expected value, represents how much money a player can expect to win or lose in the long run on a particular wager. If the player's expectation is negative, as is typical for most bets made in a casino, the player can expect to lose money over the long haul.

An emphasis should be placed on "in the long run". Once a player was playing a dollar slot machine that displayed a sign that said the game had a 98% payback. The player became angry and filed a complaint with the Nevada State Gaming Control Board after playing $100 and receiving absolutely no payout. She felt she should have received $98 in return. She was never satisfied with the explanation, even from the regulators, that the 98% was simply an average return in the long run.

EXPECTED VALUE OF A WAGER

Expected value is a function of both the probabilities of winning and losing the bet, as well as the amount won or lost. The following formula can be used to calculate the expected value of a wager.

$$EV = \sum (Net\ Pay_i \times P_i),$$

where *Net Pay*$_i$ equals the net payoff and P_i equals the probability of the payoff.

Here's a simple example. Suppose you randomly select one card from a shuffled deck of 52 cards. You wager $1 and win even money if the card is a Spade. Under the rules of this game, you'll win $1 with probability 1/4 and you'll lose $1 with probability 3/4. The mathematical expectation, or expected value, for this wager is:

EV = [(+$1) × (1/4)] + [(−$1) × (3/4)] = −$0.50.

This means that, in the long run, you'll lose 50 cents for every dollar wagered. This should make sense since on the average you'll lose $1 three out of every four times you bet, and win $1 once, for an average loss of $0.50 per dollar wagered.

To take another example, consider double-zero roulette. A winning wager on a single number is paid 35 to 1. If a $5 bet on the number 17 is made, the net payoffs are +$175 (if it wins) and −$5 (if it loses). These payoffs have probabilities 1/38 and 37/38, respectively. The expected value is:

EV = (+$175)(1/38) + (−$5)(37/38) = −$0.263.

Demystifying Some Mathematical Symbols

Do not be intimidated by formulas. The symbol ∑ is the mathematical notation for the sum of all the calculations inside the parentheses marks. For example, if there are three values designated x_1, x_2, and x_3, then $\sum x_i$ simply means the sum $x_1+x_2+x_3$. Thus if $x_1=2$, $x_2=4$, and $x_3=1$, then $\sum x_i$ equals 7. In the EV formula, each possible net payoff (*Net Pay*$_i$) should be multiplied by its probability (P_i) and the results should be added together.

This means a player can expect to lose a little over a quarter for each $5 wagered on the number 17, in the long run. Put another way, the casino can expect to win $0.263 for every $5 wagered on the number 17. Clearly, the particular number on which the wager is placed makes no difference and the expected value is −$0.263 for a $5 wager on any single number. The expectation calculation for a $25 bet on red, which pays even money, is:

$$EV = (+\$25)(18/38) + (-\$25)(20/38) = -\$1.316.$$

The casino can expect to win about $1.32 of every $25 wager on red.

HOUSE ADVANTAGE

House advantage, or house edge, is the opposite or negative of the player expectation, expressed as a percentage of the wager. Sometimes house advantage is referred to as PC, for percentage, although care must be taken when discussing PC since this term is sometimes used for hold percentage. Hold is not the same as house advantage and there is much confusion among some casino personnel regarding the meaning of “PC.” Hold and hold percentage are discussed in Chapter 2 when common casino measurements are examined in more detail.

> "Amid the action and reaction of so dense a swarm of humanity, every possible combination of events may be expected to take place, and many a little problem will be presented which may be striking and bizarre... "
>
> Sherlock Homes[14]

House advantage (HA) is the most commonly used indicator of the value of a particular game or bet. From the casino’s perspective, the house advantage represents how much, in terms of percentage of the money wagered, the casino can expect to retain in the long run.[15] Double-zero roulette, for example, has a house advantage of 5.3%. Thus, for example, if many wagers totaling $1,000,000 are made on the roulette tables in a given year, the casino will make (win) about $53,000.

> ***A CAUTION***
>
> Two issues can cause confusion regarding the value of the house advantage for a particular game or wager. The first is whether to include ties (in Baccarat, for example, or for the don’t pass line bet in Craps) and the second is whether to express the house advantage in terms of the “base” bet or average bet amount (examples include Caribbean Stud Poker, Let It Ride, and Three Card Poker). Both apparent discrepancies boil down to how the bet handle, or total amount wagered, is figured. The important thing, however, is that the expected player loss in dollars and cents is the same regardless of which of these ways the house advantage is reported. It is what the expected player loss is compared to that results in different percentage edges. These issues are discussed further in Chapter 2.

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14 Conan Doyle, A. “The Adventure of th New York: Doubleday, pp. 244-257, 1988.

15 This interpretation assumes the house a amount wagered. Although this method of expressing the advantage is common, some games or wagers admit other methods. In several games, for example, the final amount put at risk is not always equal to the original bet and the house advantage is often expressed per hand, or relative to the base bet, rather than per unit wagered. See Different Ways To Express Win Rate in Chapter 2.

Remember that expectation is expressed in monetary units while house advantage is expressed in percentage units. For the roulette examples above, the casino's expectations on a $5 bet on a single number and a $25 bet on red are $0.263 and $1.32, respectively, but the house advantage on both wagers is 5.26% ($0.263/$5 and $1.32/$25).

Computing House Advantage Directly From Odds

Although house advantage is usually calculated using the expected value formula, it can be computed directly from the payoff and true odds. If the payoff is x to 1 and the true odds (against) are y to 1, the house advantage is $(y-x)/(y+1)$. The following summarizes this result.

Payoff: x to 1
True Odds: y to 1
House Advantage: $\dfrac{y-x}{y+1}$

For example, the true odds against winning a single number bet at roulette are 37 to 1 and the payoff is 35 to 1. The house advantage is (37-35)/(37+1) = 2/38 = 0.0526, or 5.26%.

More generally, if the payoff is x to z and the true odds against are y to w, the house advantage is equal to $(yz-xw)/[z(y+w)]$.

Payoff: x to z
True Odds: y to w
House Advantage: $\dfrac{yz-xw}{z(y+w)}$

For example, in craps a winning place bet on the number five pays 7 to 5, and the true odds against this bet are 3 to 2. The house advantage on this place bet is equal to [(3)(5)–(7)(2)]/[5(3+2)] = 1/25 = 0.04, or 4.00%. Similarly, a place bet on the number four, with true odds of 2 to 1 and payoff of 9 to 5, carries a casino advantage of [(2)(5)–(9)(1)]/[5(2+1)] = 1/15 = 0.0667, or 6.67%.

The Law of Large Numbers (Law of "Averages")

Expectation and house advantage are long run averages. That is, for a large number of trials (wagers) these values represent what can be expected to occur. Actual play will deviate from these values, but when the number of trials is large enough, the percentage of the total amount wagered that is retained by the casino should be close to the house advantage.[16] This idea is the law of large numbers, sometimes known as the *law of averages*.

Put another way, by making enough trials, you can essentially ensure that the ratio of successes to total trials will be about equal to the theoretical probability. Thus, as the number of bets increases, the total casino win divided by the total action will tend to get closer and closer to the expected casino win divided by the total action.

A simple coin-tossing example may help shed light on the law of large numbers. Suppose you threw 10 coins on the floor. You may get 50% heads – the expected value – but it would not be surprising if you got 60% or more. The actual probability of getting 60% or more heads when randomly tossing 10 coins is 0.377.[18] Now imagine throwing 100 coins on the floor. What are the chances of getting 60% or more heads? It's not very likely – the probability is 0.028, or about 1 in 35. If you threw 1,000 coins on the floor it's virtually impossible to get 60% or more heads. The probability is about 0.000000000136 (less than 1 in 7 billion). This is the effect of the law of large numbers. Although you probably will not get exactly 50% heads, the larger the number of tosses, the more likely you are to observe an actual percentage close to 50%.

> **LAW OF LARGE NUMBERS**
> **(LAW OF AVERAGES)**
>
> In repeated independent trials of the same experiment, the actual proportion of occurrences of an event eventually approaches its theoretical probability.
>
> Another consequence of a large number of wagers is an increased likelihood of the truly unusual, like 28 passes in a row at the craps table. But, of course, the chances of the player failing 28 times in a row are even more likely.[17]

The same principle applies to a series of independent wagers. Instead of coin tosses, consider a series of wagers on the color red in the game of roulette. A player could be ahead after only 10 wagers (the probability is $p = 0.314$ for ten bets on red in double-zero roulette[19]), but it is less likely he would be ahead after 100 wagers ($p \approx 0.265$), even less so after 1,000 wagers ($p \approx 0.045$), and it's virtually impossible after 10,000 wagers ($p \approx 0.000000065$). The larger the number of spins (wagers), the more likely it is that the actual percentage won by the casino will be close to the expected

16 But see note 9 on page 10, infra.

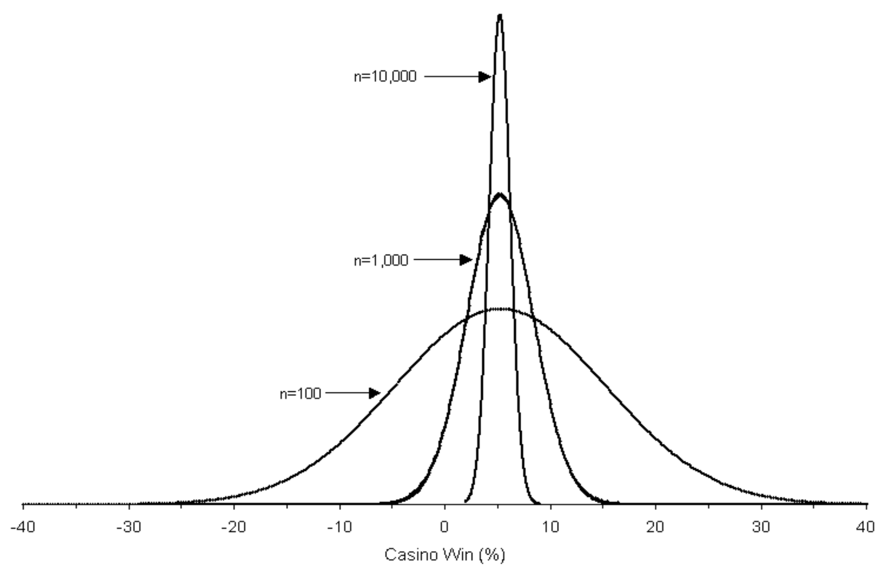
17 Diaconis, P. and Mosteller, F. "Methods of Studying Coincidences." Journal of the American Statistical Association, 84, pp. 853-861 (1989).

18 The probability of observing x heads in n tosses of a coin can be computed using the binomial probability distribution. The binomial formula is given by P(x) = n!/[x!(n–x)!]px(1–p)n–x, where p=.5 for a fair coin toss, and n! = n·(n–1)·(n–2)··· 2·1. For large n the binomial distribution can be well approximated by the normal distribution.

19 The probabilities associated with the possible outcomes in a series of spins of the roulette wheel can be calculated using the binomial probability distribution (or, for large n, approximated by the normal probability distribution).

value of 5.26%. Figure 1 shows the distributions of probability for the percentage won by the casino for a series of 100, 1,000, and 10,000 wagers on red.

FIGURE 1



Central Limit Theorem

> ***Theorem* defined ...**
> A statement that can be demonstrated to be true by accepted mathematical operations and arguments.

Among the most powerful theorems in probability and statistics is the central limit theorem. The central limit theorem states that for a large number of independent trials (wagers), the distribution of the total, or percentage of the total, can be described by the normal curve. This theorem is responsible for each of the preceding figures resulting in the familiar bell curve, or more formally, the normal probability distribution.

For the series of 100, 1,000, and 10,000 wagers on red, the curves depicting the distribution of possible outcomes center on an expected value of 5.26%. This is the casino advantage for a bet on red in double-zero roulette.

Again, the larger the number of spins (wagers), the more likely it is that the actual percentage won by the casino is close to the expected value of 5.26%. The curve for 100 wagers is tighter around the expected percentage of 5.26% than the curve for 10 wagers, and the curve for 1,000 wagers is even tighter around the expected percentage than the curve for 100 wagers. With 10 wagers, there is more leeway in what the actual percentage outcome will be than for 100 wagers, and even less latitude for 1,000 wagers. This is the law of large numbers operating in the gambling environment.

Since the normal curve has been well studied, assessing probabilities for any phenomenon that can be described with this curve is possible. In particular, it can gauge how close, in terms of probability, the actual percentage casino win will be to the theoretical or expected win.

VOLATILITY

Casino managers should love the central limit theorem. It dictates that if you get enough play in your casino, you should get pretty close to your expected results. Although some deviation will occur between the observed win and theoretical win, this deviation should be small when the number of trials is large. If the number of trials is small, however, a substantial difference may exist between the actual and expected win.

Volatility analysis is an important tool for the casino manager. If the deviation starts to fall outside the norm to the casino's disadvantage, the manager should look for answers. A casino manager needs to understand what constitutes a normal deviation from the theoretical win and what should raise the specter of further inquiry. These issues are discussed in Chapters 3 and 4. Likewise, if the deviation starts to fall outside the norm to the casino's advantage, the astute regulator should ask questions.

STANDARD DEVIATION

To gain an understanding about how much deviation from the theoretical house advantage can be expected, and what fluctuations can be considered normal (due to chance variation), a good place to start is a common statistical measure called the *standard deviation*. The standard deviation tells how much variation can be expected when observing repeated occurrences of a random variable.

For example, the total amount won by the casino in a series of one thousand $5 bets on red in roulette is a random variable. For one person who makes a series of one thousand $5 bets on red, the casino may win a total of $300. For the next person making a series of one thousand $5 bets on red, the win may be $250. If many people make a series of one thousand $5 bets on red, there will many different outcomes. Some variations will naturally occur. The standard deviation for the total amount won will show whether these different outcomes will tend to be about the same or will be wildly different.

A small standard deviation means the outcomes tend to be similar. A standard deviation of zero, the smallest possible value, would indicate the outcomes are always exactly the same. A large standard deviation indicates the outcomes are very different or highly variable. The standard deviation is a crucial tool in volatility analysis.[20]

The next section shows how to calculate the standard deviation and use it to predict the likely maximum and minimum win that will occur in a series of wagers. It also shows how to determine whether an observed win or loss in a series of wagers is merely due to normal statistical fluctuations or is more likely explained by other reasons.

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20 Basic concepts of volatility analysis are covered in this chapter, volatility in slots and baccarat are treated in Chapter 3, and the importance of volatility in casino operations is discussed in Chapter 6.

WAGER STANDARD DEVIATION

To assess the likely deviation between actual win and house advantage in a series of wagers, computing the standard deviation for a single wager is first required. Since the standard deviation is the square root of a measure known as the *variance*, defining the variance is a prerequisite to determining the standard deviation. The variance for a single wager can be calculated from the following formula:

$$VAR_{wager} = \sum [(Net\ Pay_i - EV)^2 \times P_i].$$

Standard deviation is the square root of the variance:

$$SD_{wager} = \sqrt{VAR_{wager}}\ .$$

As an example, consider a $5 wager on red at double-zero roulette. From the casino's perspective, the possible outcomes are +$5 and −$5 with probabilities 20/38 and 18/38, respectively. The calculations for the expected value to the casino and the standard deviation for this wager are shown in the following table.

Standard Deviation – $5 Even Money Bet at Roulette

X	P(X)	X∗P(X)	X−EV	$(X-EV)^2$	$(X-EV)^2*P(X)$
+5	0.526316	+2.631579	4.736842	22.437673	11.809302
−5	0.473684	−2.368421	−5.263158	27.700831	13.121446
	EV = +0.263158			*VAR* = 24.930748	
				SD = 4.993070	

The wager's expected value is $0.263 and, as usual, represents the average casino win per $5 wagered on red. The standard deviation for this wager is $4.993.

Alternative Formula for Variance and Standard Deviation

Some math books show the following alternative formula for calculating the variance and standard deviation:

$$VAR = \sum [(Net\ Pay_i)^2 \times P_i] - (EV)^2$$

For the roulette example, this formula would be implemented as follows:

$$VAR = [(+5)^2(.526316)+(-5)^2(.473684)] - (.263158)^2$$
$$= 25 - .069252$$
$$= 24.930748.$$

The standard deviation can be obtained by taking the square root of the variance.

Standard deviation is the key to assessing the likely fluctuations that will occur in a long series of wagers. It is easiest to perform the necessary analysis using the expectation and standard deviation expressed on a per unit basis.

PER-UNIT EV AND STANDARD DEVIATION

The per-unit expected value and standard deviation can be obtained by computing the wager expected value and standard deviation, as above, assuming a $1 bet, or from the wager expected value and standard deviation as follows:

$$EV_{per\ unit} = \frac{EV_{wager}}{wager\ amount},$$

$$SD_{per\ unit} = \frac{SD_{wager}}{wager\ amount}.$$

For the roulette example, the per-unit expected value and standard deviation are:

$$EV_{per\ unit} = \frac{\$0.263158}{\$5} = 0.052632,$$

$$SD_{per\ unit} = \frac{\$4.993070}{\$5} = 0.998614.$$

The per-unit expected value of 0.052632 means the casino will, in the long run, retain 0.0526 unit of every unit wagered. On the average, the casino will win $0.263 of every $5 wager, $5.26 of every $100 wager, and $26.32 of every $500 wager.

CONFIDENCE LIMITS – ANALYZING FLUCTUATIONS IN CASINO WIN

What statistical fluctuations should the casino expect in a series of wagers and what are extraordinary? As will be shown, expected fluctuations can be examined in three ways: (1) win percentage, (2) number of units won, or (3) the total win amount. The following table gives the expected value and standard deviation of the win percentage, the number of units won, and the total dollars won in a series of *n* independent identical wagers.

Expected Values and Standard Deviations for a Series of Wagers		
Win Percent	$EV_{win\%} = EV_{per\ unit}$	$SD_{win\%} = \frac{SD_{per\ unit}}{\sqrt{n}}$
Number of Units	$EV_{units} = n \times EV_{per\ unit}$	$SD_{units} = \sqrt{n} \times SD_{per\ unit}$
Total Win	$EV_{win} = unit\ wager \times n \times EV_{per\ unit}$	$SD_{win} = unit\ wager \times \sqrt{n} \times SD_{per\ unit}$

To illustrate these calculations, consider a series of 1,000 wagers of $5 each on red in roulette. From the formulas in the above table, we have:

$$EV_{win\%} = 0.052632 .$$

$$SD_{win\%} = \frac{0.998614}{\sqrt{1,000}} = 0.031579 .$$

$$EV_{units} = 1,000 \times 0.052632 = 52.632 .$$

$$SD_{units} = \sqrt{1,000} \times 0.998614 = 31.579 .$$

$$EV_{win} = \$5 \times 1,000 \times 0.052632 = \$263.16 .$$

$$SD_{win} = \$5 \times \sqrt{1,000} \times 0.998614 = \$157.89 .$$

As previously discussed, for a large series of independent wagers, the actual win will follow a normal distribution – imagine repeating the series of wagers a very large number of times and recording the win each time. Using a table of standard normal probability values,[21] the probability associated with any observed win in a series of wagers can be found from the theoretical win percentage and standard deviation.

Some Practical Casino Uses For Confidence Limits

The following series of statements, sometimes collectively referred to as the "empirical rule," can be used to find approximate confidence limits for the actual win percentage:[22]

- ♣ About 68% of the values will be within 1 standard deviation of the expected value,
- ♣ About 95% of the values will be within 2 standard deviations of the expected value,
- ♣ Virtually all the values (99.7%) will be within 3 standard deviations of the expected value.

Applying these statements to a large series of independent wagers give some guidelines for win volatility in a casino:

- ♠ About 2/3 of the time the actual win will be within 1 standard deviation of expected win,
- ♠ About 95% of the time actual win will be within 2 standard deviations of expected win,

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21 See Table 1, Appendix. You can obtain normal distribution probability values from a computer software package such as Excel.

22 The percentages given in these statements are derived from the standard normal probability table, but are rounded for convenience and ease of recollection. See Table 1, Appendix. More precisely, the percentage of values within one, two and three standard deviations of the mean of a normal distribution are 68.27%, 95.45%, and 99.73% respectively.

♠ The actual win will almost never be the actual win be more than 3 standard deviations from expected win.

You can use these results to assess whether an observed win in a series of wagers can be attributed to chance variation. For our 1,000 even money roulette wagers, the actual casino win percentage will be in the interval 0.052632 ± 0.031579, or between 2.11% and 8.42%, about 2/3 of the time. The casino win should be outside the interval 0.052632 ± 2(0.031579), or −1.05% (i.e., a player win of 1.05%) to 11.58%, only 5% of the time. The casino win for this series should almost never be less than 0.052632 − 3(0.031579) = −4.21% or greater than 0.052632 + 3(0.031579) = 14.74%.

In terms of number of units won, 2/3 of the time the casino will win 52.632 ± 31.579, or between 21.05 units and 84.21 units. Only 5% of the time should the casino win be outside the interval 52.632 ± 2(31.579), or −10.53 units to 115.79 units. The casino win for this series should almost never be less than 52.632 − 3(31.579) = −42.11 units or greater than 52.632 + 3(31.579) = 147.37 units.

In terms of total amount won, about 2/3 of the time the casino's total win for this series will be in the interval $263.16 ± $157.89, or between $105.26 and $421.05. Only about 5% of the time will the total casino win for this series will be outside the interval $263.16 ± 2($157.89), or −$52.63 to $578.95 and almost never will it be less than −$210.53 or greater than $736.84.[23]

Testing For Deviations

The casino should regularly test for deviations. One slot operator put a control test into place for each gaming device that would signal deviations beyond the confidence bands, but only tested on a monthly basis. Dishonest employees soon learned the extent to which they could steal money from the slot machines and still stay within the accepted confidence bands. Had the slot operator tested on a monthly, quarterly and annual basis, the confidence limits would have shrunk closer together (on a percentage basis) as the number of wagers (coin-in) increased. Therefore, the employee theft that was within the confidence limits on a short (monthly) trial would have been outside the confidence limits on a quarterly basis and been substantially outside the confidence limits on an annual basis. Moreover, the problem may have been caught earlier had the slot operator tested on a casino-wide basis as opposed to each individual machine.

Some Miscellaneous Observations

♦ The confidence limits for any one of the three variables – win percentage, number of units won, and total win amount – can be derived from those of either of the other two.

♦ The unit amount wagered ($1, $5, or $500) does not affect the expected value, standard deviation or confidence limits for the win percentage or number of units won.

23 Minor discrepancies in these calculations appear due to rounding of the final displayed values.

- ♦ The number of wagers has a direct effect on how close the actual win percentage will be to the expected win percentage (house advantage).

This last point is extremely important and speaks to where we see the law of large numbers at work in gambling. To further illustrate, the following table provides upper and lower confidence limit values for win percentage for various numbers of even-money wagers (e.g., red) in roulette. Note how the confidence limits shrink closer together as the number of wagers increases.

Effect of Number of Bets on Actual Win Percentage

	Number of Bets (on red in roulette)				
	100	*1,000*	*10,000*	*100,000*	*1,000,000*
$EV_{win\%}$	5.26%	5.26%	5.26%	5.26%	5.26%
$SD_{win\%}$	9.99%	3.16%	1.00%	0.32%	0.10%
95% Lower Conf. Limit	–14.71%	–1.05%	3.27%	4.63%	5.06%
95% Upper Conf. Limit	25.24%	11.58%	7.26%	5.89%	5.46%

CONFIDENCE LIMITS AS CONTROLS

Confidence limits giving upper and lower bounds for the actual casino win are similar to control limits used to monitor a statistical process. Control limits are typically plotted in a control chart, a common tool used in quality management to analyze variation, to track an ongoing process and signal when the process is "out of control."

Game volatility can be examined in a similar way by plotting upper and lower confidence limits for the casino win as a function of the number of plays. Figure 2, for example, shows the 95% confidence bands for casino win percentage in a series of even money roulette wagers.

FIGURE 2
95% Confidence Bands for Win % (Roulette Even Money Wager)

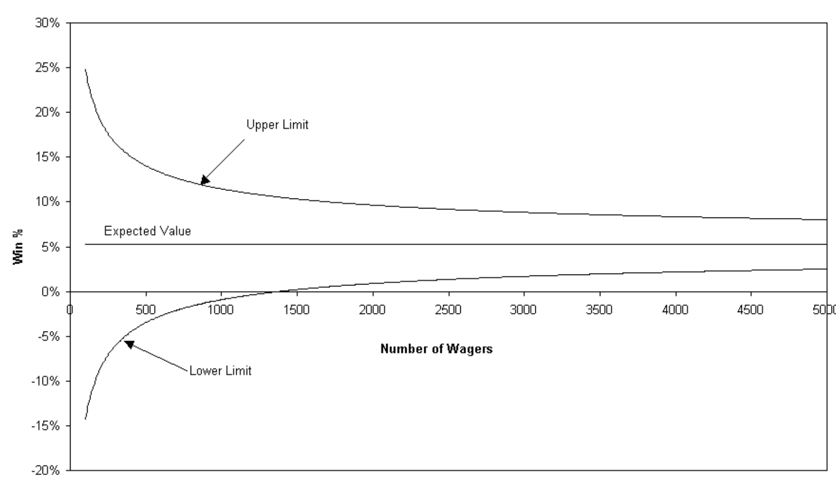
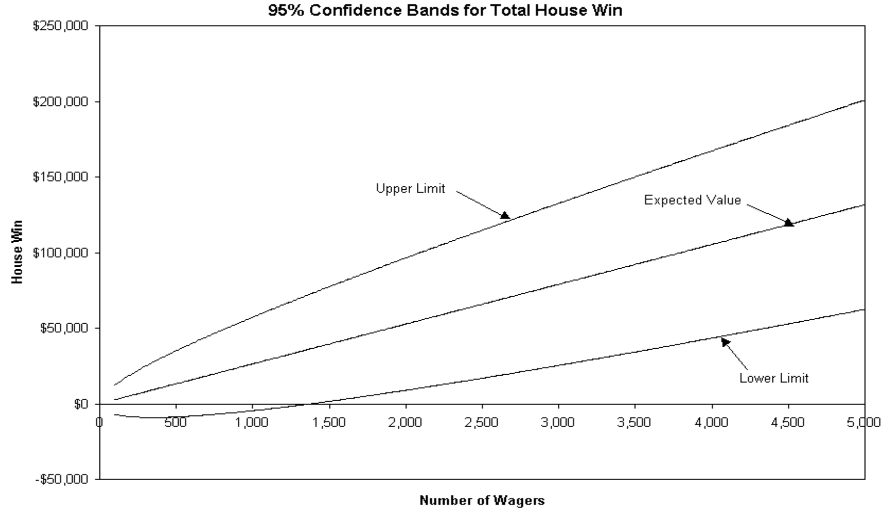


Figure 2 can be used as a control chart.[24] The confidence bands show the boundaries of normal statistical fluctuations for the actual casino win percentage for a given number of plays. An observed win that falls between the upper and lower confidence bands can be attributed to normal statistical fluctuation, while an observed win outside the confidence bands is likely due to other than normal statistical fluctuations and requires further scrutiny.

The shape of the confidence bands highlights that as the number of wagers increases, the closer the actual win will be to the theoretical win, in percentage terms. In absolute terms, however (total dollars or total units), the deviation between actual and expected win will increase as the number of wagers increases. Figure 3 shows the 95% confidence bands for total casino win in a series of $500 even money wagers in double-zero roulette.

FIGURE 3
ROULETTE - $500 per Wager Red
95% Confidence Bands for Total House Win



Note that the confidence limits for percentage win (Figure 2) shrink together as the number of wages increases, whereas they grow apart for the total win (Figure 3). Charts such as the ones above for roulette can be constructed for any wager or any game. The confidence level need not be set at 95%, but can be any desired level.[25]

VOLATILITY INDEX – A MEASURE USED FOR SLOT MACHINES

Slot machine volatility is measured by its volatility index. If the fluctuations in actual win percentage tend to be larger for machine A than machine B, the volatility

24 In the usual statistical process control setting, a random sample of the same size is taken periodically from the process and the relevant statistic from each sample plotted over time on a control chart with fixed control limits. Thus, it is common for a control chart to show flat confidence bands. Since the confidence limits in this chart are shown for different numbers of wagers (different sample sizes), the resulting confidence bands are not flat.

25 A common form of control chart in statistical process monitoring will set control limits at ±3 standard deviations resulting in a 99.7% confidence level.

index for A will be greater than that for B. The volatility index is a function of the number and size of the payouts, and is defined to be:

$$V.I. = z \times SD_{per\ unit}$$

where z represents the standard normal probability value corresponding to a desired confidence level. The volatility index typically reported by slot machine manufacturers is based on a 90% confidence level, for which z = 1.65. Thus the typical volatility index is as follows:

Volatility Index – 90% Confidence
$V.I. = 1.65 \times SD_{per\ unit}$

If a 95% confidence level were used, the appropriate z value is 1.96 and the volatility index would be $1.96 \times SD_{per\ unit}$.[26]

Upper and lower limits for actual payback percentage[27] for a series of n plays on the machine are given by the formula:

Upper & Lower Limits for Payback Percentage
$Theoretical\ Payback \pm \frac{V.I.}{\sqrt{n}}$

Calculation of the limits on actual slot payback percentage is similar to that for the confidence limits on roulette win percentage.

The plus or minus part of the slot limits calculation, $\pm \frac{V.I.}{\sqrt{n}}$, represents the "margin of error" (see the boxed sidebar at the end of this chapter). Slot volatility and the calculation of the volatility index are examined further in Chapter 3.[28]

A Caveat About Confidence Levels

Although the illustrated calculations for confidence limits and margin of error were based on 95% confidence and the slot volatility index is typically based on a 90% confidence level, any level of confidence can be used. The 95% level is

26 Previous calculations based on rough guidelines such as the "empirical rule" used the approximate z-value of 2, rather than a more precise 1.96, for 95% confidence.

27 Slot analysis is typically done in terms of payback percentage rather than win percentage.

28 It makes sense to define the "volatility index" for other games too. The volatility index for any game would then be the basis for the calculation of the margin of error and confidence limits, just as it is for slots.

somewhat standard across a wide variety of applications but if a higher or lower degree of confidence is desired, it requires only a different z value. A normal probability table[29] or a computer package can be used to find the appropriate number of standard deviations, or z value, associated with the desired confidence. For 95% confidence this z value is 1.96 (sometimes rounded to 2 for rough calculations) and for 90% the z value is 1.65. Put another way, it is possible to find the probability that the actual win percentage will be more or less than any amount. To illustrate, the following table shows the probabilities of casino win outcomes in various ranges after 1,000 bets on red in double-zero roulette.

Casino Win after 1,000 Bets on Red in Double-Zero Roulette (in units)	
Range	*Probability*
Less than –20	1.07%
–20 to 0	3.71%
0 to 20	10.29%
20 to 40	19.39%
40 to 60	24.77%
60 to 80	21.47%
80 to 100	12.63%
100 to 120	5.04%
120 to 140	1.36%
More than 140	0.28%

ANOTHER EXAMPLE – BACCARAT

A further example of fluctuation analysis is provided for the game of baccarat. The following table summarizes the expected value (from the casino's perspective) and standard deviation calculations for a $1 bet on the banker.

Expected Value and Standard Deviation – $1 Banker Bet in Baccarat					
X	P(X)	X∗P(X)	X−EV	$(X-EV)^2$	$(X-EV)^2*P(X)$
+1	0.4462466	+0.4462466	0.989421	0.978954	0.436855
−0.95	0.4585974	−0.4356675	−0.960579	0.922712	0.423153
0	0.0951560	0	−0.010579	0.000112	0.000011
EV = +0.0105791				VAR = 0.860019	
				SD = 0.927372	

29 See Table 1, Appendix.

Since the calculations in the above table were based on a $1 wager, the resulting expected value and standard deviation can be considered to be on a per unit basis. The next table shows 95% margins of error for the win percent, number of units won, and total win amount for varying length series of $500 wagers.

Volatility Analysis – $500 Banker Bet in Baccarat				
	Number of Bets			
	1,000	*10,000*	*100,000*	*1,000,000*
Win %				
EV	1.06%	1.06%	1.06%	1.06%
95% Margin of Error	±5.87%	±1.85%	±0.59%	±0.19%
Units Won				
EV	10.6	105.8	1,057.9	10,579.1
95% Margin of Error	±58.7	±185.5	±586.5	±1,854.7
Win Amount				
EV	$5,290	$52,895	$528,954	$5,289,535
95% Margin of Error	±$29,326	±$92,737	±$293,261	±$927,372

Note that as the number of bets increases, the margin of error for win percentage decreases, but the margin of error for units won and for win amount increase.

Volatility associated with baccarat is further explored in Chapter 3.

Factors Affecting Volatility

When analyzing volatility in a series of wagers, as the discussion above shows, the number of wagers affects the fluctuations in win percentage and win amount, either units or dollars. The larger the number of wagers, the closer the actual win will be to the expected win in percentage terms and the larger the deviation will be between actual win and expected win in absolute terms – total amount won, units or dollars. The amount of a wager also impacts the volatility of the dollar amount won (but not units or percentage won).

Another factor is the size of the payoff. Other things being equal, wagers with a larger payoff will have a larger standard deviation.[30] Since the size of the payoff has a direct effect on the per unit standard deviation, this factor plays a role in all forms of volatility analysis – win percent, number of units won, and total amount won.

To take a simple example, consider two wagers with equal expected values. For wager A you win $1 with probability 0.5 and lose $1 with probability 0.5. For wager B you win $999,999 with probability 0.000001 and lose $1 with probability 0.999999. It is easy to verify that both wagers have zero expected value (fair games!), but wager

30 For games like slot machines with multiple possible payoffs, the largest (jackpot) payoff generally has the greatest effect on volatility, although all possible payoffs contribute.

A has a smaller standard deviation, and over a long series will be less volatile than wager B.

Questions to Ask if the Win Percentage Falls Outside Normal Confidence Levels

- Is the data correct?
- Is there an extraordinary event that caused the deviation?
- Is there a mechanical or personnel error?
- Are players cheating you?
- Are your employees stealing from your casino?

In general, for single payoff wagers with the same house advantage, a wager having a low probability of winning a large amount is more volatile than a wager with a larger probability of winning a smaller amount. In roulette, for example, a straight up bet on a single number paying 35 to 1 with probability 1/38 is a more volatile wager than an even money wager on red paying 1 to 1 with probability 18/38. For a given number of wagers, the win percentage for the single number bet is more likely to be further from the theoretical win percentage (house advantage) than the even money wager. The following table shows this effect by giving the margin of error for the basic single payoff wagers in roulette and baccarat. Note the wagers with a smaller probability of a larger payoff have larger margins of error, reflecting greater volatility. This principle extends to wagers with more than one payoff as well. Slot machines with larger jackpot payoffs, for example, generally have a greater volatility index (see Chapter 3 for more on slot machines).

Effect of Payoff on Volatility

				95% Margin of Error	
	Payoff	*Prob*	*Expected Win %*	*10,000 Bets*	*100,000 Bets*
ROULETTE					
Even Money	1	.4737	5.26%	±2.00%	±0.63%
Column/Dozens	2	.3158	5.26%	±2.79%	±0.88%
Double Street	5	.1579	5.26%	±4.38%	±1.38%
5-Number Bet	6	.1316	7.89%	±4.73%	±1.50%
Corner Bet	8	.1053	5.26%	±5.52%	±1.75%
Street	11	.0789	5.26%	±6.47%	±2.05%
Split	17	.0526	5.26%	±8.04%	±2.54%
Straight-up	35	.0263	5.26%	±11.53%	±3.64%
BACCARAT					
Banker	0.95	.4586	1.06%	±1.85%	±0.59%
Player	1	.4462	1.24%	±1.90%	±0.60%
Tie	8	.0952	14.36%	±2.64%	±0.84%

GAMBLING FALLACIES AND SYSTEMS

The Gambler's Fallacy

A common gamblers' fallacy called 'the doctrine of the maturity of the chances' (or 'Monte Carlo fallacy') falsely assumes that each play in a game of chance is not independent of the others and that a series of outcomes of one sort should be balanced in the short run by other possibilities. A number of 'systems' have been invented by gamblers based largely on this fallacy; casino operators are happy to encourage the use of such systems and to exploit any gambler's neglect of the strict rules of probability and independent plays.

Encyclopedia Britannica

Myths, misconceptions, and fallacies about gambling abound, and run the gamut from merely believing that slot machines that haven't hit in a while are "due" to full-blown gambling systems that purportedly can beat negative expectation games. Players who fall prey to these misunderstandings usually end up losing more money than they would otherwise. Casino managers are also vulnerable to losing money – the casino's money – and therefore would be well served to be aware of such fallacious thinking. In all aspects of gaming, decisions and actions based on superstition, myths, and "bad math" invite disaster. This section discusses some of the most common gambling fallacies and betting systems.

THE LAW OF AVERAGES AND COMMON FALLACIES

Serious misunderstandings about independent events and the law of averages are at the heart of many gambling fallacies. One of the most common fallacies, for example, is that the plays in a game of chance are not independent and therefore a series of outcomes of one sort should be balanced in the short run by outcomes of another sort. This particular myth is so common it has been dubbed "the gambler's fallacy."

Another popular misconception is that gambling systems can be devised to win in the long run against the casino's negative expectation games of chance. The existence of such a system is mathematically impossible, of course, thanks to the law of large numbers.[31] As mentioned, this mathematical law is sometimes referred to as the law of averages and correctly interpreted means that the gambler's total loss divided by his total action will approach his expected loss divided by his total action. A key here is the notion of ratio or percentage. The law does not state that the amount of money lost will approach the expected amount lost. The law of averages is about the percentage deviation between actual and expected results – the greater the number of trials, the smaller the *percentage* deviation. The actual (or absolute) deviation, however, should grow as the number of trials increases. The following simple example using tosses of a fair coin illustrates this point.

31 But see notes 10-12, and accompanying text, supra.

Comparing Absolute and Percentage Deviation – Tosses of a Fair Coin				
Number of Tosses	*Number of Heads*	*Percent Heads*	*Deviation – Number of Heads*	*Deviation – Percent Heads*
100	53	0.530	3	0.030
1,000	520	0.520	20	0.020
10,000	5,100	0.510	100	0.010
100,000	50,500	0.505	500	0.005
1,000,000	501,000	0.501	1,000	0.001

This table shows how the percentage of heads approaches the expected 50% as the number of trials increases – the deviation between the percentage of heads and the expected percent decreases as per the law of averages – but the deviation between the actual number of heads and the expected number increases. For gamblers, while the percentage of money lost should approach the theoretical percentage loss, the difference between the amount of money lost and the expected loss will likely grow.

When reading the fallacies that follow, keep in mind the correct interpretations of independent events and the law of averages, as these are the two most commonly misunderstood mathematical concepts related to gambling fallacies.

Hot and Cold Streaks

A fair coin is flipped ten times and results in eight heads and two tails. Does this mean that more tails will show up in the next few flips? Those who subscribe to this fallacy believe that the "law of averages" will rescue them in the midst of a losing streak by turning things around.[32] After all, in the long run there should be 50% heads and 50% tails, so more tails should be coming up now to "even out" things. Others might expect more heads to show up because heads are "hot." Both views are incorrect. Because the tosses are independent trials, the probability of heads is 50% on each and every toss regardless of what happened on previous tosses.

Situations similar to this coin example occur in the casino. Suppose, for example, that eight spins of the roulette wheel result in eight consecutive red numbers. Is black now more likely on the ninth spin because it is "due," or is red more likely now because it is hot? Many people believe that if one color has come up several times in a row in roulette then the other color is now either more or less likely. The truth, of course, is that each color has the same probability of occurring on every spin, regardless of the outcomes of the past spins. Viewing it from the long term, for every sequence of eight consecutive reds that is followed by a black on the ninth spin, there will another sequence of eight reds followed by a ninth red. The bottom line is that in a game like roulette, every trial is independent and past history is irrelevant.

This principle applies to other casino games involving independent events. The dice may appear to be "hot" at a craps table, but this does not change the odds of the shooter's throwing the various numbers. Similarly, keno numbers do not turn hot or cold. For independent events, each outcome is entirely unrelated to past outcomes and

32 This fallacy is the previously mentioned "gambler's fallacy."

so "hot" and "cold" streaks do not really exist in the sense that particular outcomes become more or less likely to occur than normal. Streaks are just statistical fluctuations and no one can predict whether a streak will continue or when it will end.

Some betting systems use the fallacious logic of riding streaks. Others use the equally faulty logic of betting against the streak. The former, of course, is based on the hot streak fallacy; the latter on the mistaken notion that in the short run things must even out according to the "law of averages." Considered thought about the nature of independent events and the correct application of the law of averages will reveal these systems to be worthless.

Debunking the Notion of Betting on a Winning Streak

In July 2000 column on baccarat in a major gaming newspaper, the columnist was describing "excellent player opportunities" in baccarat. He stated:

"Let's say that the player has won the last three decisions. What would you bet on the player or the bank? If you said the bank, you would be wrong! Bet on the player, because the player is having the winning streak . . . Winning streaks offer some of the best opportunities a player can get . . . Betting on streaks will ensure that you will never be on the wrong side of any winning streak. There will be times when the expected trend or pattern does not materialize but there will also be times, probably many more times when the expected trend or pattern continues or repeats, and this is a huge advantage for the player."

Unfortunately, this is bad advice. The probability that the player will win at baccarat is 44.62%. If the player wins three times in a row, the probability that it will win the fourth time will be almost exactly the same.

Luck

Closely related to the hot/cold streak fallacy described above is the notion of "luck." Players experience, or attribute to themselves, good luck when they are winning and bad luck when they are losing. As with hot and cold streaks, however, luck is merely a natural statistical fluctuation in one direction or another – good luck is a fluctuation of successful bets and bad luck is a fluctuation of unsuccessful bets. It is a fallacy to think that luck exists as a tangible quantity.

Betting Systems

A betting system is a method of placing and varying bets in a predetermined way to try and beat a game that is favorable to the house and unfavorable to the player. It is a mathematical fact, however, that betting systems do not work for negative expectation games.[33] It is not possible to change or manipulate the laws of probability and mathematics to turn a negative expectation game into a positive one. Every such proposed gambling system has at least one fundamental flaw that prevents the player from gaining the advantage.[34]

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33 See, for example, Probability and Measure, 3rd edition, by Patrick Billingsley (John Wiley & Sons, 1995).

34 Games of both chance and skill where the player has some special knowledge or skill that allow him to make bets with a positive expectation (blackjack, for example) are the exception.

MARTINGALE OR DOUBLING-UP PROGRESSION

Probably the most common and oldest gambling system is the Martingale, or doubling-up system. Commonly used for bets with even-money propositions in games such as roulette, the idea behind this system is simple. The player keeps doubling the stake with each consecutive losing bet so that when he eventually wins, he covers all the losing bets and makes a small profit. Assuming an initial bet of one unit, if you lose the first bet, then you bet two units. If you lose that one, you bet four units, and if you lose that one you bet eight units. This progression continues until you finally win a bet at which time your winning amount will equal one unit more than the sum of all previous losing bets. The net profit is one unit.

The problem with the doubling-up system is that occasionally the player will either hit the maximum bet limit imposed by the casino or run out of money. A string of consecutive losses will eventually necessitate a bet size prohibited by either house imposed bet limits or finite capital. Starting with a $5 bet, for example, and incurring ten consecutive losses would require a bet of $5,120. Twenty consecutive losses would require a bet of $5,242,880. Players who think the chance of such disastrous losing streaks is so small it can be ignored are guilty of fallacious reasoning. While the probability is relatively small, these losing streaks will sometimes occur and when they do, the money lost is more than enough to compensate for the one-unit profits gained when the system "works." The intelligent gambler knows that when this system is played for some time, the devastating cycles will restore his overall return to the negative theoretical expectation. In short, the player will win a small amount frequently and lose a large amount occasionally, but the percentage return remains the same.

A variation on the Martingale is the Great Martingale system, whereby the player doubles his previous losing bet and adds one unit. Thus the bet progression would be 1 unit, 3 units, 7 units, 15 units, and so on. With each bet, the potential win increases by one unit. The fundamental flaw with this system is the same as with the Martingale – the Great Martingale is just a quicker way of going broke.

D'ALEMBERT OR PYRAMID SYSTEM

The D'Alembert system calls for the player to increase his bet size after a loss and decrease the bet size after a win. Assume one-unit increments. If the player wins his first bet, the sequence is over and the player begins a new one. If the first bet loses, the player increases the second bet by one unit. Thereafter, the bet is increased one unit after each loss and decreased by one unit after each win. The end result is that if the number of wins eventually balances the number of losses, the player will have profited one unit for each win in the sequence.

The D'Alembert, or pyramid, system is based on the so-called "law of equilibrium," which falsely asserts that any two opposite events, such as red and black in roulette, must eventually win an equal number of times. This so-called "law" is the same "gambler's fallacy" encountered above in the discussion of hot and cold streaks where a gambler will bet on black after a series of consecutive reds, believing black must be due. It is also referred to as the doctrine of the maturity of chances and the Monte Carlo fallacy. Those who succumb to this fallacy are mistaking the short run for the long run. The ratio of blacks to reds in roulette will approach 50% in the long run, but there is no guarantee this will be the case in the short term.

Pyramid systems will break even in the long run against a 50/50 (fair) random process and will lose at the house percentage edge against a house favorable game.[35] As with the Martingale System, casino betting limits and the player's finite capital are the downfall of the pyramid system.

Cancellation System

The typical cancellation system proceeds as follows. The player writes down a series of positive numbers whose sum is equal to the amount he desires to win. At any stage, the next bet is the sum of the first and last numbers currently in the series. If the bet wins, the two numbers just used are crossed off the list. If the bet loses, the player writes the amount lost at the end of the list. Betting continues in this fashion, crossing out numbers when a bet is won and writing down a new number (an amount of money) when a bet is lost, until all the numbers (old and new) have been canceled out. At this point the player will have won an amount equal to the sum of the original numbers in the list. To illustrate the cancellation system, suppose the player seeks to win 25 units and uses as an initial list 2, 3, 1, 6, 5, 8. The table below shows what might happen.

Cancellation System

Current List	*Bet Size*	*Outcome*	*Current Overall Profit*
2, 3, 1, 6, 5, 8	10	Lose	−10
2, 3, 1, 6, 5, 8, 10	12	Lose	−22
2, 3, 1, 6, 5, 8, 10, 12	14	Win	−8
3, 1, 6, 5, 8, 10	13	Lose	−21
3, 1, 6, 5, 8, 10, 13	16	Lose	−37
3, 1, 6, 5, 8, 10, 13, 16	19	Win	−18
1, 6, 5, 8, 10, 13	14	Lose	−32
1, 6, 5, 8, 10, 13, 14	15	Lose	−47
1, 6, 5, 8, 10, 13, 14, 15	16	Win	−31
6, 5, 8, 10, 13, 14	20	Win	−11
5, 8, 10, 13	18	Win	+7
8, 10	18	Lose	−11
8, 10, 18	26	Win	+15
10	10	Lose	+5
10, 10	20	Win	+25
Total	215	7 Wins, 8 Losses	25 unit profit achieved

35 See, for example, The Casino Gambler's Guide, by Allan Wilson (1970).

Note that a flat bet would lose with this sequence since there are eight losses and only seven wins. With the cancellation system used here, a 25-unit profit is achieved as advertised. The goal is not always realized, however, as bet sizes may escalate so that the next required bet exceeds the player's or table's limit. Again, the familiar flaw as with the Martingale and D'Alembert systems – house limits and player resources.

Margin of Error

Another way to represent the 95% confidence limits is to report the "plus or minus" part of the result. For example, the 95% confidence limits for the series of 100,000 wagers on red in roulette, 4.63% and 5.89%, were derived from the calculation 5.26% ± 0.63%. This means there is a 95% probability that the actual win percentage will be within 0.63% of the expected win percentage. That is, 0.63% represents the maximum deviation from the theoretical win percentage that can be expected (95% of the time) for 100,000 plays. The maximum deviation is similar to what statisticians call the *margin of error*. In statistical sampling, the margin of error represents the maximum likely error attached to a statistic. Ordinarily the statistic is being used to estimate an unknown population parameter. In our present context, the 100,000 wagers constitute a sample from the population of all such wagers, and the observed win percentage would serve as an estimate of the theoretical (population) win percentage. Here, however, the objective is not to estimate an unknown parameter since the relevant parameter, theoretical win percentage, is known. Rather, we are interested in knowing how unusual an observed win percentage is, given the known theoretical win percentage. For the 100,000 roulette wagers on red, an observed win percentage that is not within ±0.63% of the theoretical win percentage is unusual since the probability of that happening is only 0.05. As long as the observed win percentage is within ±0.63% of the theoretical win percentage, the difference can be attributed to normal statistical fluctuations. In this sense, 0.63% serves as a 95% confidence level "margin of error" for the win percent associated with a series of 100,000 wagers on red. Similarly, the margin of error for 1,000 wagers is 6.32%, for 10,000 wagers it is 2.00%, and for 1,000,000 wagers it is 0.20%.

SUMMARY OF FORMULAS & RULES

Probability Rules

- ♦ Rule 1. The probability of an event will always be between 0 and 1.
- ♦ Rule 2. (Complement Rule). The probability of an event occurring plus the probability that event not occurring equals 1.
- ♦ Rule 3. (Addition Rule). For mutually exclusive events, the probability of at least one of these events occurring equals the sum of their individual probabilities.
- ♦ Rule 4A. (Multiplication Rule – independent events). For independent events, the probability of all of them occurring equals the product of their individual probabilities.
- ♦ Rule 4B. (Multiplication Rule – dependent events). For events that are not independent, the probability of all of them occurring equals the product of their conditional probabilities, where the conditional probability of one event is affected by the event(s) that came before it.

Odds

- ♣ If probability in favor is X/Y, the odds against are Y–X to X
- ♣ If the odds against are X to Y, the probability in favor is Y/(X+Y)

Wager Expected Value

$$EV = \sum (Net\ Pay_i \times P_i)$$

House Advantage

$$HA = \frac{EV}{Wager} \times 100\%$$

If payoff = x to 1 & true odds = y to 1, then $HA = \frac{y - x}{y + 1}$

If payoff = x to z & true odds = y to w, then $HA = \frac{yz - xw}{z(y + w)}$

Wager Variance

$$VAR = \sum [(Net\ Pay_i - EV)^2 \times P_i]$$

Wager Variance (Alternative)

$$VAR = \sum [(Net\ Pay_i)^2 \times P_i] \ - \ (EV)^2$$

Wager Standard Deviation

$$SD = \sqrt{VAR}$$

Wager Expected Value, Per Unit (House Advantage, as decimal)

$$EV_{per\ unit} = \frac{EV_{wager}}{wager\ amount}$$

Wager Standard Deviation, Per Unit

$$SD_{per\ unit} = \frac{SD_{wager}}{wager\ amount}$$

Series Win Percentage

$$EV_{win\%} = EV_{per\ unit}$$

$$SD_{win\%} = \frac{SD_{per\ unit}}{\sqrt{n}}$$

Series Number of Units

$$EV_{units} = n \times EV_{per\ unit}$$

$$SD_{units} = \sqrt{n} \times SD_{per\ unit}$$

Series Win Amount

$$EV_{win} = unit\ wager \times n \times EV_{per\ unit}$$

$$SD_{win} = unit\ wager \times \sqrt{n} \times SD_{per\ unit}$$

Volatility Principle

For a large series of independent wagers:

♠ 2/3 of the time actual win will be within 1 SD of expected win,

♠ Only ≈ 5% of the time will actual win be more than 2 SD's from expected win,

♠ Virtually never (0.3%) will the actual win be more than 3 SD's from expected win.

COMMON CASINO MEASUREMENTS

3

Since casino games are built on probability, odds, percentages, and related statistical information, the measurements necessary to describe and monitor transactions and other business associated with these games are somewhat unique. Terms such as handle, drop, and hold are standard fare with casino operations personnel. This chapter explains the common measurements used in connection with casino game operations.

BASIC TABLE GAME AND SLOT OPERATIONS

This brief introductory section presents an overview of the basic terminology peculiar to table game and slot operations. Familiarity with these preliminary terms is necessary before proceeding to the primary measurements used by casinos and regulators to analyze play at a casino.

TABLE GAMES

Casino managers must understand the very different nature of table games and slot machines when using mathematics to assess performance. The critical difference is in the amount of information a casino can accurately record on the play of table games versus the play of slot machines. Someday, casinos will have the ability to track every transaction that occurs in the casino, whether it is the amount of each bet at a blackjack table or the quality of a video blackjack player's play. But until technology meets these needs in a cost-efficient manner, the casino must rely on limited or less-than-perfect information to do its analyses. This is particularly true with table games where the casino does not generally have the advantage of the advanced player tracking systems.

When a casino player approaches a table game to play, he usually exchanges cash for chips. After giving the player the chips, the dealer will drop the cash into the drop box attached to the underside of a table game. Some players are not cash players, but will gamble using credit instruments or "markers." A marker is an instrument used to document the extension of credit to a player. In this case, the player will sign a credit instrument in the amount of the chips received at the table. Like the cash received, the credit instruments also are placed in the drop box.

Besides cash and credit instruments, the drop box contains foreign chips, and credit and fill slips. Foreign chips are chips from other casinos played or exchanged by players. A credit slip (not related to credit extended to a player) is an instrument used to document a transfer of (usually excess) chips from a table game to the cashier – a game credit. A fill slip is an instrument used to document a transfer of chips from the cashier to a table game – a fill.

SLOT MACHINES

Slot machines earned their name because they have a slot that accepts coins or tokens for play from players. Most slot machines today are equipped with a currency acceptor box, or bill validator, that allows the slot machine to accept currency. A player will insert coins or currency into the machine to begin play. Most slots have two types of meters: hard meters and soft meters. The two hard meters, which resemble the odometers in cars before the 1990s, track "coin-in" and "coin-out," respectively. Coin-in is the money inserted by the player into the slot machine. Coin-out is the money that is paid by the slot machine to the player. Coin-in typically goes to the slot machine hopper, a coin storage bank inside the machine through which all payouts are made. If the hopper is full, the machine will divert additional coins to the drop bucket stored in a locked cabinet beneath a slot machine. If the hopper becomes empty, a casino employee will undertake a "fill" by adding coins or tokens.

Slot meters record (1) total credits by coin-in and coin-out, (2) total credit through the bill validator, (3) total credits, total credits played, total credits won, and total credits paid, and (4) total games played and total games won.

HANDLE

Handle is the total amount of money wagered or put at risk by a player. A player's handle is sometimes referred to as his *action*. In principle, handle may be calculated using the following formula:

$$Handle = Pace \times Average\ Bet \times Duration$$

Pace refers to the speed of the game in decisions per hour in terms of hands dealt, wheel spins, dice rolls, etc. Average bet refers to the average amount of money wagered per decision. Duration is the length of time a player plays, in hours (a unit of time other than hours can be used as long as the pace is also expressed in this unit).

Slot machine meters and tracking systems allow for easy determination of an individual player's slot handle. For table games, however, it is impractical to monitor every bet a player makes, or how long he plays, and the speed of the particular game he is playing. Although technology may change this is the future, player table game handle is currently estimated using reasonable values established through occasional observation of the player's average bet and duration of play. Although the speed of a table game is affected by several factors, including the particular dealer and the number of players at the table, an average value is often used for game pace.

To illustrate the use of the formula, suppose through observation a manager estimates that a roulette player averages $50 a bet on each spin for a two-hour period at a double-zero table completing 60 spins per hour. The estimated total amount wagered is:

Handle = 60 spins/hr x $50/spin x 2 hrs = $6,000.

Another player averaging $100 per bet for two-hours at the same table would produce an estimated handle of $12,000.

In most retail businesses, gross revenue is determined by multiplying the number of each product sold by its selling price. In the casino environment, the game is the product, the house advantage serves as the price, and the handle represents quantity. This perspective yields the gaming equivalent of the familiar "revenue equals price times quantity" equation:

Revenue = House Advantage x Handle.

Of course in the short term, the house advantage may not apply throughout the duration of play – the house advantage is the *long-run* percentage of wagered money retained by the casino; the actual win percentage realized during a given period may deviate from this theoretical win percentage. From this perspective the above equation gives the *expected* revenue.

DROP

Drop is the total amount of the contents in a game's drop box (or drop bucket and currency acceptor box in the case of slots).

TABLE GAMES

For table games, drop consists of the currency and foreign chips in the drop box plus the value of credit instruments (markers) issued or redeemed at the table. That is, drop = (cash + markers + chips) contained in the drop box.

SLOTS

For slots, the drop is the total amount of currency and coin in the slot machine currency acceptor box and drop bucket. That is, drop = (coins + currency) contained in the drop bucket and currency acceptor box. The slot drop does not include coins or tokens in the slot machine's hopper.

DROP VS. HANDLE

For table games, drop and handle are not equivalent. Handle, a theoretically precise notion, is difficult to measure for table games. Drop, while empirically precise, is a conceptually elusive term. Many factors can affect a table's drop and it is not clear exactly what it is that drop measures in terms of gambling volume. To give an example of the difficulties associated with table game drop, consider the following two players. Player A exchanges $1,000 cash for chips at a blackjack table and then, betting his winnings, makes 800 bets at $50 each for a total of $40,000 in wagers before walking away broke. For Player A, the drop is $1,000 and the handle is $40,000. Player B also buys-in for $1,000 but then makes only two bets at $50 each and (taking his remaining chips) walks away. For Player B, the drop is also $1,000 but the handle is $100. This player may take his remaining chips and play them at another table (at which his drop will be zero) or may cash them out at the cashier cage.

For slots, other gaming devices, keno, and sports books, where it can be accurately measured, drop is (in principle) equal to handle. This is not 100% accurate for slots and gaming devices because the casino usually does not take into account the change in the hopper level. For example, suppose the hopper holds and contains 1,000 coins. If a player is winning, he will be paid from the hopper so that at the conclusion of play, the hopper contains only 500 coins. If the drop is conducted at this time, it may ignore that the hopper is less 500 coins. Moreover, it requires players to lose 500 coins in the next drop cycle before it contributes any coins to the next drop.

Factors that Impact the Accuracy of the Drop

- ♥ Credit policy converting cash for chips or money plays
- ♥ Promotional items such as lucky bucks or 7 for 5
- ♥ Checks retained in float
- ♥ False drop
- ♥ Rim credit

WIN

Win refers to the total amount won by the casino at a table or device.

TABLE GAMES

For table games, win is the amount of the drop minus the change in the table's chip inventory, including chips issued during fills and chips removed during credits. That is,

$$\begin{aligned} Win &= drop - missing\ chips \\ &= drop - (begin\ chips + fills - credits - end\ chips). \end{aligned}$$

SLOTS

For slots, win is the amount in the drop bucket and bill validator less any jackpots paid and accounting for any change in the hopper inventory, including fills.

$$Win = drop - jackpots - change\ in\ hopper\ inventory.$$

THEORETICAL WIN

Note that "win" as described above refers to actual win. Theoretical win is the amount the casino expects to win and can be determined as follows:

$$\text{Theoretical Win} = \text{H.A. x Handle}$$

Since handle can be found by multiplying the average bet times the game pace times the duration of play, theoretical win can be represented as:

$$\text{Theoretical Win} = \text{H.A. x Pace x Avg. Bet x Duration}$$

Theoretical win is also called expected win.

WIN PERCENTAGE

Win percentage is the win divided by handle:

$$\text{Win Percentage} = \frac{\text{Win}}{\text{Handle}}$$

As with the above description of win, this refers to the actual win percentage, and not theoretical or expected win percentage.

Actual win percentage represents the fraction of money wagered that is retained by the casino over a given period of time or number of trials. The theoretical win percentage is the house advantage, or the long-run average percentage of money wagered that is retained by the casino. For a large number of trials, the actual win percentage should approximate the theoretical win percentage.

For slots and other gaming devices, the actual win percentage is generally known. For table games, however, win percentage is difficult to measure since handle is usually unknown. For further discussion on these and other related issues, see the section later in this chapter on different ways to express win rate.

THEORETICAL WIN PERCENTAGE

The percentage of money wagered that the casino can expect to win in the long run is the theoretical, or expected, win percentage . This value is just the house advantage:

$$\text{Theoretical Win \%} = \frac{\text{Theoretical Win}}{\text{Handle}} = \text{House Advantage}$$

For a large number of trials, the actual win percentage should be close to the theoretical win percentage.

HOLD

Hold, or more specifically hold percentage, is the percentage of the drop that is won by the casino. That is,

$$Hold\ \% = \frac{win}{drop}$$

Note that for slot machines and other gaming devices, the hold percentage is equivalent to the win percentage since drop is the same (in principle) as handle. Slot hold represents the percentage of all money wagered by players that the machine retains.

For table games, however, where handle and drop are not generally equal, hold percentage is not the same as win percentage. The actual fraction of money wagered at a table game that the casino retains is not the hold percentage, but the win percentage – win divided by handle. As mentioned above, table game handle is difficult to measure and therefore the win percentage is generally unknown. Hold percentage is a known measure of the "win rate" for table games and is used as a rough gauge of how these games are performing.

Hold percentage for table games is represented by game revenue, i.e. the drop plus the change in chip inventory, divided by drop.

$$Hold\ \% = \frac{Drop + Change\ in\ chip\ inventory}{Drop}$$

Change in chip inventory can be calculated by taking current inventory, adding any credits, and subtracting fills and closing inventory.

EXPECTED HOLD PERCENTAGE

Using the expected win rather than actual win in the hold percentage formula will yield expected, or theoretical, hold:

$$Expected\ Hold\ \% = \frac{Expected\ Win}{Drop},$$

or

$$Expected\ Hold\ \% = \frac{H.A. \times Pace \times Avg.\ Bet \times Duration}{Drop}.$$

Many factors impact hold percentage by including the theoretical win percentage, the speed of the game, the average bet size, number of bets and the number of players at the table. These and other factors that can affect hold percentage are discussed below.

HOLD PERCENTAGE AS A MANAGEMENT TOOL

Casinos use hold percentage as a management tool for determining the performance of games. They use it because other measurements of performance are not readily available. For example, a comparison of theoretical and actual win percentages could assist casino management in learning if casino personnel or cheaters at a particular table are stealing. Substantial unexplainable deviations (either small but prolonged deviations or large and short deviations) may be warnings of potential problems. Unfortunately, with table games, most casinos do not have the technology to track actual win percentage because it is a function of the amount wagered on each hand divided by the amount paid out on each hand. A casino cannot determine either measurement without sophisticated and costly tracking systems. Therefore, casino management must rely on more crude measurements to determine if circumstances exist that require special management intervention.

TYPICAL HOLD PERCENTAGES – NEVADA TABLE GAMES

The following table shows recent hold percentages for selected games in Nevada (the Nevada Gaming Revenue Report labels the hold percentage for each game as "win percent," even for table games).

Hold Percentages
Selected Nevada Table Games, year ending September 30, 2000.

	# of Units	*Win Amount (thousands)*	*Win (Hold) Percent*
Blackjack	3,734	1,182,359	12.64%
Craps	493	505,761	13.80%
Roulette	469	298,126	24.24%
Baccarat	108	541,600	17.36%
Mini-Baccarat	122	141,597	12.57%
Keno	174	89,768	27.14%
Caribbean Stud	113	59,242	27.71%
Let It Ride	174	75,228	21.69%
Pai Gow	48	35,032	20.18%
Pai Gow Poker	267	111,970	21.87%

Source: Gaming Revenue Report, Nevada State Gaming Control Board

FACTORS THAT IMPACT HOLD PERCENTAGE

House advantage is probably the biggest factor that impacts hold percentage. The above table shows that hold percentage is not the same as house advantage. With the exception of keno, the hold percentages in the above table are significantly larger than the overall game house advantages. For example, a player at the baccarat table can bet on player (with a house advantage of 1.24%), banker (with a house advantage of 1.06%) or a tie (with a house advantage of 14.36%). All three bets have a substantially less house advantage than the experienced hold percentage. The house advantage represents the percentage of money wagered that the casino will theoretically win in the long run. The hold percentage can be viewed as the percentage of the buy-in that is won by the casino. Since for many table games, players typically wager more than the buy-in (e.g., by re-betting winnings, or even using chips bought at another game), the hold percentage tends to be larger than the house advantage.

Factors That Influence Average Number of Bets per Session

- ♠ Number of Bets Per Session
- ♠ Number of players per table.
- ♠ Use of automatic shufflers.
- ♠ Dealer experience.
- ♠ Game procedures that slow or speed up the game.
- ♠ Player satisfaction.
- ♠ Location.
- ♠ Marketing

Other game related factors that also impact the hold percentage include the average bet size, the average number of bets, the game pace, the number of players at the table, and the duration of play. Some of these factors are related to each other, and each is, in turn, affected by a variety of other considerations.

Average bet size can be influenced by table minimums and marketing. The latter concerns whether the casino tends to attract grind (low bet) or high limit players.

Average number of bets per session can be influenced in two ways: by increasing the number of bets per hour such that a player plays more hands in a set amount of time or by increasing the length of the player's session. Many things impact how long the player decides to play. An example is the player's general satisfaction with the game or the overall casino ambiance. Ambiance includes all of the factors that make players want to keep playing for as long as possible. Each player group is different, but all respond to casino environments that meet their desires and expectations – comfort, friendliness, service, cleanliness and ambiance.

Game pace refers to the number of decisions per hour. This can be influenced by the use of automatic shuffler, the skill level or dealers and the number of players at the table.

FACTORS THAT IMPACT AVERAGE BET SIZE

- ♦ Table limits
- ♦ Player mix

The following simple example illustrates the effect of number of players and game pace on hold percentage. Suppose two tables, A and B, offer the same game with a 2% house advantage. One player sits at Table A and buys in for $500, bets $25 per hand and plays for one hour. Six players sit at Table B and buy in for $500 each, bet $25 per hand and play for an hour. With a nearly full table, only 60 hands per hour are dealt at Table B compared to 200 hands per hour at Table A. Assume the players' loss is consistent with the house advantage, so that at the end of the hour, the player at Table A has lost $100, while those at Table B have lost a combined $180. With these win figures, the hold percentages for the two tables would be:

Table A: Hold % = $100/$500, or 20%.

Table B: Hold % = $180/$3,000, or 6%.

The results are summarized in the following table:

Effect of High Table Occupancy on Hold %		
Number of players	*1*	*6*
Drop/Player	$500	$500
Total Drop	$500	$3,000
Average Bet per player	$25	$25
Total Bets per Hand	$25	$150
House Advantage	2%	2%
Hands per Hour	200	60
Win/Hour	$100	$180
Hold % per Hour	20%	6%

Note that although higher table occupancy generally leads to a greater win, the hold percentage tends to be lower.

FALSE DROP

False drops are caused by either a small average bet size (compared to the buy-in) or a short period of play, or both. The following examples will illustrate false drops and how they are created.

Suppose a person approaches a table and "buys in" for $100. The game is rigged so that the casino wins every hand, and so has a theoretical win or house advantage of 100%. The $100 goes into the drop box and is the drop. Suppose further that the player plays 5 hands at a $5 per hand and loses all 5 hands. He then leaves the table with $75 in chips. For this player, the handle is $25, the casino win percentage is 100%, and the hold percent is $25/$100 = 25%.

Now suppose the game is fairer and the house advantage is 4%. Starting with the same $100 and betting $5 per hand, another player plays 500 hands, winning 240 hands and losing 260 hands before running out of chips. The handle for this player is $2500, the casino win is $100, the win percentage is 4%, and the hold percentage is $100/$100 = 100%.

The hold percentage is significantly less for the first player than the second even though the theoretical advantage is much greater in the first case (100%) than the second (4%). This is because the first player only played a few hands and so the casino could only win a small fraction of what would normally be expected for this player's buy-in. This phenomenon – players making only a few small bets relative to their buy-in – creates a *false drop* and can considerably reduce the hold percentage in the short term.[36]

Effects of false drops created by a small average bet size (relative to the buy-in) and playing for a short period are further illustrated in the two tables that follow. The first table shows the hold percentages per hour for two players playing the same

36 Short-term hold percentage is also affected by statistical fluctuations, as actual win percentage will deviate from theoretical win percentage. See Chapter 1 for more on statistical fluctuations and volatility.

amount of time with the same house advantage, but with different average bet sizes. The second table compares the hold percentages for two players betting the same amount per hand with the same house advantage, but playing for different amounts of time.

FALSE DROP CREATED BY SMALL AVERAGE BET		
	$20 Player	*$100 Player*
Buy-in (Drop)	$1,000	$1,000
House Advantage	2%	2%
Hands per Hour	100	100
Win/Hour	$40	$200
Hold % per Hour	4%	20%

FALSE DROP CREATED BY SHORT DURATION OF PLAY		
	$100 Player	*$100 Player*
Buy-in (Drop)	$1,000	$1,000
House Advantage	2%	2%
Hands per Hour	100	100
Win/Hour	$200	$200
Hours Played	0.1	2
Hands Played	10	200
Total Win	$20	$400
Hold %	2%	40%

FLAWS IN MEASUREMENTS THAT IMPACT DROP AND HOLD

When analyzing the impact of game related factors on hold percentage, the casino manager also should understand that the method of calculating drop and, thus, hold, is flawed. Drop is supposed to be the sum of all chips purchased. But, even this number cannot be accurately determined for a particular portion of the casino. For example, a player may buy chips at the craps table, play for a short while and then leave the craps pit for a blackjack table. By playing the chips purchased in the craps pit at the blackjack table, the player impacts the hold percentage in both the blackjack pit and the craps pit. No attempt is made to reconcile these circumstances, and drop is calculated on the contents of the drop box, which includes cash, markers, and foreign chips.

Use of rim credit in the casino also impacts drop. Rim credit is when the casino allows a player to place a bet on credit without executing a credit instrument. At the conclusion of play, the player will usually sign a marker for the exact amount of his losses. This results in a 100% hold. Likewise, if he walks before signing the marker, the casino must use a dummy marker for calculating the drop or else the drop is reduced by the amount owed by the player.

A casino may have either lax or rigid policies toward redemption of credit instruments at the table. The casino may insist that players do not "walk" from the table with chips purchased with credit instruments, but must redeem outstanding credit instruments. Where this policy is in place, the hold percentage will increase. For example, suppose a player draws 10 credit instruments of $10 each. He then plays one hand and loses $10. If he is allowed to walk with $90, the hold percent is ($100-$90)/$100, or 10%. If he is forced to redeem $90 in markers, the hold percentage is ($10-$0)/$10, or 100%.

DIFFERENT WAYS TO EXPRESS WIN RATE

Rate of win can be described in several ways including hold percentage, win percentage, and theoretical win. Even experienced casino personnel can be confused about the differences. It is common to find two casino managers discussing "win percentage," the first manager understanding it to be the house advantage while the second believes the discussion is about hold percentage (still a third may interpret it to be the actual win percentage, win divided by handle).

With respect to house advantage, other issues can cause misunderstandings. For games where ties occur and no money changes hands, the advantage can be expressed either including or excluding ties. For example, one manager may quote the house advantage on the player bet in baccarat as 1.24% while another may insist it is 1.37%. For other games where the final amount put at risk may be more or less than the original wager (e.g., Caribbean Stud, Let It Ride), should the house advantage be reported relative to the base bet or the average bet?

This section focuses on some of the most common misunderstandings and issues surrounding win rate. The first three segments elaborate further on the differences among theoretical win percentage, actual win percentage, and hold percentage. The last three parts address the meaning of the term "PC," the difference between house advantage relative to a base bet or an average bet, and the effect of ignoring ties.

HOLD PERCENT VS. HOUSE ADVANTAGE

Even those familiar with the gaming industry and otherwise knowledgeable gaming professionals sometimes confuse hold percentage and house advantage. In one instance, a regulatory board questioned whether a game that had a very conservative house advantage should be approved because the hold percentage on a field trial was high.

A glance again at the last column in the Hold Percentages Table on page 44 underscores the difference between hold percentage and house advantage, a difference that is often a source of confusion. While a few games (e.g., keno, the big six wheel) and a few isolated wagers in other games (e.g., craps any seven, baccarat tie bet, sic bo two of a kind) have a huge house advantage, if casino games generally carried overall house advantages anywhere near those in the last column of the Hold

Percentages Table, players would soon disappear from the gaming tables. The overall house advantage for all games except keno in the table is less than 6%, some significantly so.

The hold percent figures are much greater than the house advantage because these figures are not the win divided by the handle, which should approximate the house advantage for a large number of wagers, but win divided by drop. Since drop is typically far less than handle for table games (bettors tend to re-bet their winnings – the "churn"), the hold percent tends to be greater than the house advantage.

For keno, the hold is about what you'd expect for the house advantage since drop is equivalent to handle for this game. This is also true for slot machines and sports betting. For table games, however, hold percent is a rough measure of the performance of the game and cannot be expected to approximate the house advantage even for large numbers of bets.

THEORETICAL WIN PERCENTAGE VS. ACTUAL WIN PERCENTAGE

Theoretical win percentage is just the house advantage. The actual win percentage, as measured by win divided by handle, approaches the theoretical house advantage as the number of trials increases. This mathematical fact is expressed as

$$\lim_{n\to\infty}\left(\frac{Win}{Handle}\right) = House\ Advantage$$

> ***Lim*** *is a math term that is short for limit. It simply means that as the number of trials increases (all the way to infinity), the term in the parentheses on the left side of the equal sign will become closer and closer to the term on the right side of the equal sign.*

For slots, where handle is easily measured, this means the slot machine's actual payback percentage will be close to the theoretical payback percentage. The law of large numbers can be observed by measuring the deviation between actual payback percentages and theoretical payback percentages over time. It is possible, in fact, using mathematical theory (see section in Chapter 3 on Slot Machines) to assess the likelihood that this deviation will be larger or smaller than any specified amount. This is how control limits can be determined to allow the slot manager to decide whether there is cause for concern (possible tampering, etc.) regarding a machine's performance.

Since handle is more difficult to measure for table games, however, and actual win percentage is not obtained, observing the law of large numbers is not easy in this sense. Rather, for practical reasons, hold percentage is the measure of a table game's performance (see below).

ACTUAL WIN PERCENTAGE VS. HOLD PERCENTAGE

Win percentage refers to the actual win percentage and is equal to win divided by handle. Hold percentage is equal to win divided by drop. Given that win can be expressed relative to either drop or handle, the term "win percentage" is not very descriptive – percentage of what? Expressing win as a percentage of drop yields hold percentage while win as a percentage of handle gives win percentage. Part of the confusion surrounding this distinction may arise because for slots, actual win percentage and hold percentage are, in principle, the same things.

For table games, however, win percentage and hold percentage are not identical. As mentioned above, win percentage is not readily calculable because, absent sophisticated electronic tables, casinos cannot measure handle on the gaming tables, as it would involve registering every bet made. Instead, hold percentage is only a rough management tool that can be used to assess table performance.

Factors That Impact Actual Win Percentage

- House advantage.
- Employee errors in making change or payouts.
- Game volatility.
- Player cheating/scams.
- Employee cheating/stealing.
- Player skill levels.
- Player disputes settled in favor of player for relations purposes.
- Equipment maintenance.

An important feature of win percentage is that for a large number of trials, the actual win percentage will closely approximate the theoretical win percentage, or house advantage, a meaningful and mathematically clear-cut concept. It is less clear, however, what the long term hold percentage represents. Since hold is affected by factors such as the pace of the game, game interest, dealer conduct, and theft, the best that can be derived from differences in long term hold percentages between time periods or across properties is that one or more of many factors is different for these time periods or properties. This can still be helpful to casino management though because some (but not all) of these factors are troublesome, such as cheating or theft. For the same reason that casino management can use statistics to help detect problems in the casino, so can regulators. Commonly, gaming regulators require casinos to provide a variety of statistical reports on casino and game performance.[37]

WHAT IS PC?

Casino personnel often refer to a game's "PC." For example, the roulette's PC this week was 24%. In this context PC refers to the hold percentage. Others may use PC to denote the house advantage. Unfortunately, no generally accepted definitions exist for these two terms, so as with a reference to a generic "win percentage," it is important to clarify exactly what is meant when using the term.

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37 For example, Nevada requires that any fluctuations of plus or minus 3% from the base level – which is defined as the statistical win to statistical drop percentage for the previous business year – must be investigated and the results of the investigation must be documented and retained. Nevada regulations also require each casino to determine statistical drop to maintain a more accurate hold percentage. Statistical drop results from adding currency and chips in the drop box to markers transferred to the cage and markers repaid in the pit with chips.

Nevada gaming regulations also require that statistics be maintained for each table and type of game, by shift, by day, cumulative month-to-date and cumulative year-to-date.

Reports also must be generated monthly indicating month-to-date and year-to-date hold percentages for each slot machine and a comparison of the actual hold percentage to the theoretical hold percentage for each machine.

For keno, reports are prepared which indicate win, write and the win-to-write hold percentage for each shift, day, month-to-date and year-to-date.

BASE BET OR AVERAGE BET TO DETERMINE HOUSE ADVANTAGE?

Several games offer the opportunity for the player to add money to or subtract money from the original bet. Caribbean Stud Poker and Let It Ride are two such games. In these cases, the final amount of money put at risk is not always equal to the original bet and so the house advantage can be expressed relative to the base bet or the average bet size.

Consider the game of Let It Ride, where the player may have one, two, or three units in play on a given hand. Although play starts with three units in the betting circles, the player may withdraw two of these and only one unit must be put at risk. Thus the base bet is 1 unit. Most of the time (84.6%), the first two units will be withdrawn and the bet size will be one unit. About once every 12 hands (8.5%), the player will risk two units and about once every 14 hands (6.9%), he will risk three units. This gives an average bet size of 1.224 units.[38]

Let It Ride's house advantage is usually reported to be 3.5%. This means the player will lose 3.5% of every $1 base bet, or the casino will win 3.5 cents per hand. The player, however, is betting an average of $1.224 per $1 base bet, and so is losing 2.86% of the average amount put at risk (.0286 x $1.224 = 3.5 cents). Thus, the casino advantage is 3.5% relative to the base bet, or 3.5% per hand, but 2.86% relative to the average bet, or 2.86% per unit wagered. Both ways of expressing this house advantage are correct, and both lead to the same expected casino win: 3.5 cents per hand assuming a $1 base bet.

Although the house advantage per unit wagered represents the expected percentage of money wagered that the casino retains (because the actual win percentage – i.e., actual win divided by handle, if it could be measured – will approach the house advantage as the number of trials increases), the per hand advantage might be a preferred way to express the casino advantage for the following reasons: (1) the house advantage per hand is the more realistic measure of how much the casino will earn and what the player can expect to lose, as the result of a single hand, (2) it can be misleading to quote the advantage per unit wagered without also reporting the average bet size, and (3) it makes it easier to rate players for purposes of comps.

Let It Ride is not the only game where the average bet size is greater than the single hand base bet. The following table shows the house advantage relative to the base bet (House Advantage per Hand), the house advantage relative to the average bet (House Advantage per Unit Wagered), and the average bet size for the most common games in which the final amount bet may differ from the original amount bet.

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38 These frequencies and resulting average bet size assume proper strategy.

Win Rate Comparison per Hand vs. Per Unit Wagered			
GAME	*House Advantage per Hand*	*House Advantage per Unit Wagered*	*Avg. Bet Size*
Let It Ride	3.51%	2.86%	1.22
Caribbean Stud Poker	5.22%	2.56%	2.04
Three Card Poker (Ante/Play)	3.37%	2.01%	1.67
Three Card Poker (Pair Plus)	2.32%	2.32%	1.00
Casino War*	2.88%	2.68%	1.07
Red Dog	2.80%	2.37%	1.18
Blackjack**	0.54%	0.49%	1.10

*Assumes player always doubles his bet to go to war when there's a tie.
**Six decks, Las Vegas Strip rules, basic strategy.

IGNORING TIES

In some casino games the outcome of certain wagers may be a tie, so that no money is won or lost. The player and banker bets in baccarat and the don't pass bet in craps are examples. For these wagers, the house advantage can be expressed either including or excluding tie hands, or in other words, relative to the total number of hands or relative to the number of resolved bets.

The player bet in baccarat, for example, will win about 44.62%, lose 45.86%, and push about 9.52% of the hands. Thus for every 10,000 units wagered on the player, on average the outcome will be 4,462 units won and 4,586 units lost, for a net loss of 124 units. Relative to the 10,000 units wagered (or total number of hands), this represents a house advantage of 1.24%. Relative to the 9,048 hands that resulted in a resolution, however, the 124-unit house win is 1.37% (more precise win, loss and tie percentages result in a value of 1.364965% – see below). These advantages can also be determined using the usual formula for expected value. With ties included,

$$EV = (+1)(.4462466) + (-1)(.4585974) + (0)(.0951560) = -0.01235.$$

Ignoring ties, the player bet will win 49.3175% and lose 50.6825% of the resolved bets, yielding an expectation of

$$EV = (+1)(.493175) + (-1)(.506825) = -0.01364965.$$

Regardless of which way the advantage is expressed – 1.24% if ties are included or 1.37% if ties are ignored – the casino's expected win is the same. If, for example, 5,000 bets of $100 each are made on player, the expected win is $6,200 and can be computed using either method of expressing the casino advantage (except for round-off error):

1.24% x $100 x 5,000 hands = 1.37% x $100 x 4,524 decisions.

For the baccarat banker bet, the house advantage is 1.06% if ties are included and 1.17% ignoring ties:

$$EV = (+0.95)(.4585974) + (-1)(.4462466) + (0)(.0951560) = -0.01058,$$

and

$$EV = (+0.95)(.506825) + (-1)(.493175) = -0.0116916.$$

A similar situation exists for the don't pass bet in craps, which is barred from winning (and results in a tie) when a twelve is rolled. The necessary mathematics shows the house advantage to be 1.36% if the ties are included and 1.40% if ties are ignored.

SUMMARY OF COMMON CASINO MEASUREMENTS

HANDLE = total amount of money wagered

$$HANDLE = Pace \times Average\ Bet \times Duration$$

DROP = amount in drop box (or bucket plus bill validator)

DROP = amount of cash or cash equivalents exchanged by players for chips (cheques)

WIN = total amount won by the casino

$$THEORETICAL\ WIN = H.A. \times Handle$$

$$THEORETICAL\ WIN = H.A. \times Pace \times Avg.\ Bet \times Duration$$

$$WIN\ \% = \frac{Win}{Handle}$$

$$THEORETICAL\ (EXPECTED)\ WIN\ \% = \frac{Theoretical\ Win}{Handle}$$

$$THEORETICAL\ (EXPECTED)\ WIN\ \% = House\ Advantage$$

$$\lim_{n \to \infty} \left(\frac{Win}{Handle} \right) = House\ Advantage = Theoretical\ Win\ \%$$

(i.e., actual Win% should be close to the House Advantage for a large number of trials)

$$HOLD\ \% = \frac{Win}{Drop}$$

$$EXPECTED\ HOLD\ \% = \frac{Expected\ Win}{Drop}$$

$$EXPECTED\ HOLD\ \% = \frac{H.A. \times Pace \times Avg.\ Bet \times Duration}{Drop}$$

GAMES OF PURE CHANCE

4

> **He nourished a conviction that there must be some logic lurking somewhere in the results of chance.**
>
> Joseph Conrad
> *The End of the Tether* (1899)

Casino games can be roughly divided into four categories: house banked games of pure chance, house banked games involving skill, non-banked pooled games of pure chance, and non-banked pooled games involving skill. Games within each of these categories have similar mathematical principles. For example, because the mathematical analysis of roulette, craps, and slot machines does not require attention to strategy and computing odds, probabilities and house advantages for these games of pure chance tends to be straightforward and relatively easy. Only mathematically insignificant differences in how the random numbers and outcomes are generated differentiate these games. Games involving skill, such as blackjack and variants of poker, require attention to strategy and are not as easy to analyze with direct computations, often necessitating more complex mathematics and computer analysis.

This chapter covers the mathematics behind casino games that do not involve an element of skill. The banked games of pure chance include several of the largest revenue producers in the casino: slot machines, keno, roulette, craps, and baccarat as well as casino war, sic bo, and the big six wheel. The final section touches on pooled games of pure chance, including progressive slots and bingo.

BANKED GAMES

House-banked games are simply those where the casino is staking its money against the players. Without doubt the most popular table games, and the ones that generate the most revenue, are the all house-banked, such as roulette, craps, blackjack and baccarat. In Nevada in 2000, these four games accounted for about $2.6 billion, or more than three-quarters of all table games revenue. Slot machines, however, have become the undisputed king of the casino floor, generating over $6 billion, or nearly two-thirds of all gaming revenue in Nevada in 2000.[39]

SLOT MACHINES

> ***House Advantage:***
> 5% (average, Nevada)
> 8% (average, others)
>
> ***Typical Hold:***
> 5% – 8%

Slot machines come in an incredible variety of models. The term "slots" is used today to refer to both traditional, but now computer controlled, reel type machines, and the plethora of new video games now on the market.

In American casinos, slot machines are more popular than table games. On average they produce 70%-75% of the gaming revenues and fill at least 80% of the floor space.[40] In 2000, the installed base of all slot machines in the United States was about 487,227.[41]

TYPES OF SLOTS

Four types of slot machines are most popular.[42] The traditional spinning reel machines occupy 57% of the slot market, compared to video poker with 25%, followed by multi-line, multi-play video devices with 14%, and wide area progressives with 4%. Most video games are either poker games or electronic versions of the traditional spinning reel machines. Other less popular video games are electronic versions of other casino games such as keno and blackjack. Video "reel" machines often are themed and can be based on board games (Monopoly, Battleship), television shows (Wheel of Fortune), and rock and roll icons (Elvis).

The differences between traditional spinning reel machines and video "reel" machines are only in the way that the results are displayed to the player. Since traditional spinning reel machines are no longer mechanical but controlled by computer microchips like video "reel" machines, they also can be programmed for virtually any number of "stops," any hit frequency, and any house advantage within the limits of applicable regulations.

In other words, the symbols on each reel do not have an equal chance of appearing on the pay line. Instead, the computer in the slot machine produces a random number, each of which corresponds to a particular combination of symbols on the pay line of the slot machine. Once the random number is generated, the computer instructs the slot machine to stop the reels so that they display the combination of symbols that corresponds to the random number. For example, suppose a three line slot machine has 20 symbols on each reel and only one "7" appears on each reel. If each symbol had an equal chance of appearing, the probability of obtaining three "7"s would be $1/20 \times 1/20 \times 1/20$, or 1/8,000. In computer-controlled slot machines,

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39 Nevada State Gaming Control Board Revenue Reports.

40 Bear Stearns, An Investor's Guide to Slot Machines, Bear Stearns, p.10 (Fall 1999).

41 Id. The average daily win per slot machine varies considerably between states from a high of $353.73 in Connecticut to $81.80 in Nevada. Other states include Ill. ($279.30), Ind. ($224.15), N.J. ($217.86), La. ($207.75), Ia. ($202.78), Mo. ($127.77), Miss. ($114.98) and Co. ($94.60) (1998).

42 Id.

however, the probability of obtaining three “7”s can be set by the computer to be substantially more remote. For example, the probability that this combination appears could be only one out of 3,000,000. This allows the casino to profitably offer higher top jackpots.

Multi-line, multi-play video devices are clearly becoming more popular, taking shares from the traditional spinning reel. These machines come in a wide variety of types depending on how the games pay winning players. Line games allow the player to activate additional pay lines by playing more coins. In a typical reel slot, for example, three horizontal lines and the diagonals might be activated to pay back with winning combinations. With today’s video slots, many more pay lines can be activated. Multipliers pay for winning combinations on a single line only – usually the center horizontal line. Payback for additional coins is simply a multiple of the payback for a single coin. Multi-line, multi-play video games combine both multipliers and line games.

Buy-a-Pay games allow the player to “buy” additional winning symbols with extra coins. For example, one coin inserted may only pay out for three cherry symbols, two coins will pay for three cherries or three bars, and three coins will pay for three cherries, three bars, or three 7’s.

Casino Advantage

House advantage for slot machines is oftentimes referred to as the theoretical win percentage or theoretical hold. This value represents the long-term percentage of money wagered that the casino retains (wins). The flip side of the house advantage, the proportion of wagered money returned to the player, is the payback percentage. For example, if a machine holds 5% (about average for Nevada slot machines) the payback percentage is 95%.

Hold percentage is a theoretical amount that can be set to virtually any value with the computer program that controls the machine. Whatever this theoretical hold, the actual percentage will vary somewhat but will be close to the theoretical amount after a large number of plays. Assessing the likelihood of a certain difference between actual and theoretical win is possible because of the random selection of outcomes. The volatility section that follows later in this chapter addresses how much disparity can be expected between actual and theoretical win for slot machines.

Randomness and Gaming Devices

Today’s computer controlled gaming devices depend on random number generators (RNG’s) to select a “random” symbol combination and ensure the game’s fairness. In jurisdictions like Nevada, regulations require machines to be tested to ensure they are sufficiently random.[43] One property of randomness that is particularly important for casino games is unpredictability. Random does not mean totally

43 Since the computer must be programmed to select the combination, the RNG produces pseudo-random numbers that act like random numbers. To determine whether a series of numbers possesses random properties, mathematicians apply numerous statistical tests for randomness Some of these tests include the runs, goodness-of-fit, and various other chi-squared tests. See, for example, Knuth The Art of Computer Programming, Vol. 2, (1998) for further details.

unpredictable, however, only unpredictable in the short term. Because of the law of large numbers, it is possible to make predictions about what will happen in the long run in a completely random game – this is, in fact, an appealing feature of certain types of random phenomenon. For example, if a balanced coin is tossed 1,000 times, although you probably won't get exactly 500 heads, you know you will get about 500 heads and 500 tails. Calculations will show the chances of getting between 490 and 510 heads are about 49%, the chances of getting between 470 and 530 heads are about 95%, and the chances of getting more than 550 heads are about 0.07%. Similarly, if a fair six-sided die is rolled 6,000 times, we cannot know what number will roll on any given throw, but we can make predictions about how often each number will roll. Random sequences exhibit short run variability but long run stability. Because of this long run stability it is possible to assess how much deviation from the theoretical win percentage can be expected (see Slot Volatility that follows later in this chapter).

Variation Across Slot Machines

The theoretical hold percentage varies across slot machines quite widely, as much as between 0.5% and 30% (some video poker machines if played with expert strategy will even pay back over 100% – a negative house advantage), and depends on the type of machine, denomination, and even the jurisdiction. Generally the higher denomination machines have lower house advantages and those in Nevada tend to have lower advantages than those in other jurisdictions. To get an idea of this variation, examine the following tables showing actual win percentages for the major types[44] of slots in Nevada, Colorado, and Mississippi.

Nevada Slots 2000

Denomination		**Total**	**Total**	**Amount**	**House Win/**	**House**
5 cents	53,946	18,281.1	1,321.1	338,879	24,489	7.23
10 cents	968	417.8	25.7	431,706	26,579	6.16
25 cents	88,920	46,227.1	2,516.1	519,873	28,296	5.44
50 cents	2,768	2,469.5	114.8	892,171	41,480	4.65
1 dollar	34,181	33,870.9	1,508.1	990,928	44,122	4.45
5 dollars	3,619	7,280.8	268.2	2,011,849	74,128	3.68
25 dollars	552	1,721.9	53.1	3,119,493	96,234	3.08
100 dollar	299	1,248.6	42.9	4,247,028	145,958	3.44
Total	185,248	111,518	5,850	601,993	31,580	5.24

Source: Nevada State Gaming Control Board. Excluding Megabucks and Others categories. Figures

44 These tables exclude a few types that represent a tiny fraction of the slot action in each state (for example, 1 cent, 2 dollar, and 10 dollar), as well as progressives.

Colorado Slots 2000						
Denomination	**# of Machines**	**Total Amount Wagered ($millions)**	**Total House Win ($millions)**	**Amount Wagered/ Machine (in dollars)**	**House Win/ Machine (in dollars)**	**House Win %**
5 cents	4,224	2,231.8	149.2	528,403	35,333	6.69
10 cents	102	134.2	7.6	1,322,280	74,415	5.63
25 cents	5,961	3,623.3	207.4	607,843	34,796	5.72
50 cents	200	152.6	9.0	764,425	45,018	5.89
1 dollar	3,078	3,509.9	167.4	1,140,512	54,409	4.77
5 dollars	289	644.2	26.7	2,227,761	92,349	4.15
Total	13,854	10,296.0	567.3	743,266	40,956	5.51
Source: Colorado Division of Gaming. Figures subject to rounding.						

Mississippi Slots 2000						
Denomination	**# of Machines**	**Total Amount Wagered ($millions)**	**Total House Win ($millions)**	**Amount Wagered/ Machine (in dollars)**	**House Win/ Machine (in dollars)**	**House Win %**
5 cents	10,123	456.2	40.4	45,061	3,992	8.86
10 cents	197	8.1	0.6	41,072	3,100	7.55
25 cents	14,845	653.1	50.2	43,993	3,384	7.69
50 cents	2,424	189.3	11.9	78,112	4,925	6.31
1 dollar	9,141	905.4	47.0	99,046	5,140	5.19
5 dollars	1,278	296.0	14.6	231,630	11,444	4.94
25 dollars	157	63.1	4.0	401,974	25,691	6.39
100 dollars	82	25.4	1.2	310,360	14,457	4.66
Total	38,247	2,596.6	170.0	67,891	4,445	6.55
Source: Mississippi Gaming Commission. Figures subject to rounding.						

With the exception of the 5-cent and 10-cent slots, machines in Colorado have slightly larger win percentages than comparable machines in Nevada. Slots in Mississippi are consistently more house favorable than both Colorado and Nevada. Overall, Nevada slots win 5.2%, Colorado 5.5%, and Mississippi 6.5%. For quarter and dollar slots, the two most popular machines in terms of both amount wagered and revenue produced (about 70% of both totals in Nevada and Colorado, and 60% in Mississippi), win rates are 5.0% in Nevada, 5.3% in Colorado, and 6.2% in Mississippi. Higher win percentages can be found in other jurisdictions in the United States (Midwest riverboats, Atlantic City, Foxwoods) where an average win of 8% to 10% is common.

The preceding tables show that higher denomination machines generally have lower win percentages. This is not only true for Nevada, Colorado and Mississippi, but in other jurisdictions as well. The most popular machines are the 25-cent and 1-dollar slots, followed by the 5-cent machines. In terms of house win per machine, generally the larger denomination machines bring in the most revenue.[45]

Hit Frequency

Hit frequency is the percentage of time the machine pays something to the player. Hit frequency is important in assessing whether a particular slot machine will appeal to a particular audience. There is, however, a tradeoff between hit frequency and the amount of the payout. For a given payback percentage, hit frequency and average payout amount per hit are inversely related. One machine can pay out small amounts quite frequently while another with the same theoretical payback percentage may pay out large amounts very infrequently.

From the casino's perspective, these two machines will generate the same amount of revenue in the long run, but to the player they will behave quite differently. Casino management must decide what balance they need to strike between hit frequency and payout amounts, and the mix of types of machines to offer to the playing public.

Par (PC) Sheets

A par sheet is a document prepared by a slot machine manufacturer that shows the possible outcomes from the play of a slot machine, the probability of occurrence of each, and the contribution of each winning outcome to the payback percentage. It also includes other statistical information such as the hit frequency, volatility index, and confidence limits for actual payback percentage. Par sheets are sometimes referred to as PC sheets, game sheets, specification sheets, or theoretical hold worksheets.

The following tables show portions of par sheets for two hypothetical slot machines, including distribution summaries of hit and payout listings, and a summary of much of the information given in a typical par sheet. The hold percentage represents the expected house win, and the payback percentage indicates the expected return to the player, after a large number of plays.

45 The handful of 10-cent machines in Colorado is an exception.

PAR SHEET SUMMARY DATA – SLOT MACHINE A				
Reel Strip Number 1A HOLD % 5.053				
MODEL # : HYPO-A PAYTABLE XYZ-A				
90% Confidence Value, 10,000,000 pulls - LOW %: 94.60 HIGH %: 95.29				
COIN #	PERCENT PAY BACK	HIT FREQ	TOTAL HITS	TOTAL PAYS
1	94.947%	36.799%	325,570	840,034
Reels: 3 Stops: 96 Reel Combos: 884,736.				

PAYOUT	HITS
0	559,166
2	295,827
5	28,256
10	400
25	300
40	240
50	180
100	150
150	100
200	50
250	20
300	15
500	12
750	10
1,250	6
1,500	3
2,000	1
TOTAL	884,736

VOLATILITY INDEX = 10.990

90% CONFIDENCE VALUES		
Handle Pulls	Lower %	Upper %
1,000	60.19	129.70
10,000	83.96	105.94
100,000	91.47	98.42
1,000,000	93.85	96.05
10,000,000	94.60	95.29

PAR SHEET SUMMARY DATA – SLOT MACHINE B				
Reel Strip Number 1B HOLD % 5.168				
MODEL # : HYPO-B PAYTABLE XYZ-B				
90% Confidence Value, 10,000,000 pulls LOW %: 94.25 HIGH %: 95.42				
COIN #	PERCENT PAY BACK	HIT FREQ	TOTAL HITS	TOTAL PAYS
1	94.832%	14.840%	108,183	691,328
Reels: 3 Stops: 90 Reel Combos: 729,000.				

PAYOUT	HITS
0	620,817
2	74,324
5	14,256
10	13,200
25	4,200
40	960
50	500
100	240
150	180
200	100
250	120
300	40
500	16
750	24
1,250	12
1,500	10
2,000	1
TOTAL	729,000

VOLATILITY INDEX = 18.438

90% CONFIDENCE VALUES		
Handle Pulls	Lower %	Upper %
1,000	36.53	153.14
10,000	76.39	113.27
100,000	89.00	100.66
1,000,000	92.99	96.68
10,000,000	94.25	95.42

For slot machine A, the casino will retain about 5.053% of the total amount wagered and 94.947% will be returned to the player in the long run. Ranges of normal fluctuation around these theoretical percentages are given through 90% confidence limits for payback percentage for 1,000, 10,000, 100,000, 1,000,000, and 10,000,000 pulls. After 100,000 plays on slot machine A, for example, the probability is 0.90 that the actual payback percentage will be between 91.47% and 98.42%. An actual win of

3% for this device should not be alarming after 100,000 plays, but would be reason for concern and further investigation if it resulted after one million or more pulls. The actual win percentage should, with 90% probability, be between 93.85% and 96.05% after 1 million pulls, and within less than 0.4% of the theoretical value after 10 million plays. These confidence limits are based on the volatility index, a base measure of the variation in actual win percentage associated with a slot machine. With slot machine B's larger volatility index, the confidence limits are somewhat wider (farther from the theoretical payback percentage) for a given number of plays.[46]

Other information given in a par sheet includes the number of reel combinations in a complete cycle, the number of times each possible (non-zero) payout will occur in the cycle (hits), the hit frequency, total hits, and total pays in the cycle. With 3 reels and 96 stops for each reel, slot machine A has 96 x 96 x 96 = 884,736 possible reel combinations. In a complete cycle, 559,166 combinations will return nothing to the player and 325,570 combinations will pay some amount, for a hit frequency of 325,570/884,736 = 36.8%. The hit frequency for slot machine A is more than double that for slot machine B even though both machines hold about 5%. This is because machine A pays out small amounts more frequently while machine B compensates for the less frequent payouts with higher amounts. Finally, total pays can be found by multiplying each payout amount by its number of hits in a cycle and summing.

Slot Volatility

The preceding par sheet discussion refers to how the actual hold for a slot machine may vary from its overall theoretical advantage by an amount that depends on the payout distribution – in particular the volatility index associated with the distribution – and the number of plays. This difference can be relatively large for a small number of games played. As the number of plays increases, the actual hold will more closely approximate the machine's overall edge. The volatility index is the basis for calculating how much variation can be expected and for determining whether an observed hold percentage on a given machine is normal or unusual for a given number of plays. In general, the larger the volatility index, the greater fluctuations will be in actual hold percentage. This section examines the derivation and use of this index.

Volatility index can be calculated by multiplying the slot wager standard deviation times the appropriate z score, where the z score is the standard normal probability value corresponding to the desired confidence level.[47] Formally,

$$\textit{Volatility Index:}\ \ V.I. = z\sigma,$$

where z equals the z score for the desired confidence level, and σ is the slot machine's per coin standard deviation. The calculation of the standard deviation was discussed in detail in Chapter 1. For a slot machine, the per-coin standard deviation can be computed by:

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46 The volatility index and the derivation of the confidence limits are covered in the section on Slot Volatility.

47 A 90% confidence level is commonly used for the slot volatility index, with z score equal to 1.65.

$$\sigma = \sqrt{\sum (Net\ Pay_i - EV)^2 \times P_i}\ ,$$

where

$Net\ Pay_i$ = the net payoff per coin wagered

= (payout minus number of coins wagered) divided by coins wagered,

EV = house expected value (theoretical edge) for number of coins wagered,

P_i = probability of each Net Pay.

As an example, the table following shows the standard deviation and volatility index calculations for slot machine A from the aforementioned par sheet discussion.

Volatility Index – Calculations For Slot Machine A						
Net Pay	*Hits*	*Prob*	*Pay x Prob*	*Pay-EV*	*(Pay-EV)²*	*(Pay-EV)² x Prob*
1	559,166	0.632015	0.632015	0.95	0.90	0.569762
-1	295,827	0.334368	-0.334368	-1.05	1.10	0.369010
-4	28,256	0.031937	-0.127749	-4.05	16.41	0.523986
-9	400	0.000452	-0.004069	-9.05	81.91	0.037033
-24	300	0.000339	-0.008138	-24.05	578.43	0.196136
-39	240	0.000271	-0.010579	-39.05	1524.94	0.413667
-49	180	0.000203	-0.009969	-49.05	2405.95	0.489493
-99	150	0.000170	-0.016785	-99.05	9811.01	1.663379
-149	100	0.000113	-0.016841	-149.05	22216.06	2.511038
-199	50	0.000057	-0.011246	-199.05	39621.11	2.239149
-249	20	0.000023	-0.005629	-249.05	62026.16	1.402139
-299	15	0.000017	-0.005069	-299.05	89431.22	1.516236
-499	12	0.000014	-0.006768	-499.05	249051.43	3.377976
-749	10	0.000011	-0.008466	-749.05	561076.69	6.341741
-1249	6	0.000007	-0.008470	-1249.05	1560127.22	10.580290
-1499	3	0.000003	-0.005083	-1499.05	2247152.48	7.619739
-1999	1	0.000001	-0.002259	-1999.05	3996203.00	4.516831
TOTAL	884,736	EV = 0.050526		*VAR* = 44.367605		
SD = 6.660901						
Volatility Index = 10.990487						

In the above table, the *Net Pay* column gives the possible casino profits for a coin played. The second column gives the number of hits in each cycle for each possible payout. The probabilities in column three are obtained by dividing the number of hits for each payout by the total cycle combinations (884,736).[48] The fourth column gives the *Net Pay* times its probability for each payout, the sum of which equals the theoretical machine advantage, or *EV*=0.050526. The last three columns show the deviation between *Net Pay* and the *EV*, the square of this deviation, and the square of this deviation multiplied by the respective probability for each payout. The sum of the values in the last column, 44.3676, equals the wager variance. The square root of the variance is the wager standard deviation:

$$\sigma = \sqrt{\sum (Net\ Pay_i - EV)^2 \times P_i} = \sqrt{44.3676} = 6.6609.$$

At the 90% confidence level the volatility index is 1.65 times the standard deviation:

$$V.I. = z\sigma = 1.65(6.660901) = 10.990487.$$

This, of course, corresponds to the volatility index given in the par sheet (10.990). Although the slot volatility index is typically based on 90% confidence, other confidence levels could be achieved using an appropriate z value in the final calculation. A 95% confidence level would call for z = 1.96; a 99% confidence level would require z = 2.58. Appropriate z values for other confidence levels can be obtained from a standard normal probability table.[49]

The volatility index can be used to derive confidence limits for actual payback percentage for a given large number of plays.[50] These limits are computed using the formula:

$$Payback\ Percent \pm \left(\frac{V.I.}{\sqrt{n}} \times 100 \right),$$

where n is the number of plays.[51] Using slot machine A from above, the theoretical payback percentage is equal to 94.947% and the confidence limits for the actual payback percentage after one million plays are given by (rounding):

$$94.947\% \pm \frac{10.990}{\sqrt{1{,}000{,}000}} = 94.947\% \pm 1.0990\% = 93.85\%\ to\ 96.05\%\ .$$

As n becomes large, the interval from lower to upper confidence limit shrinks. This is another example of how the Law of Large Numbers works to ensure the actual win percentage will, in the long run, be close to the theoretical win percentage. Figure

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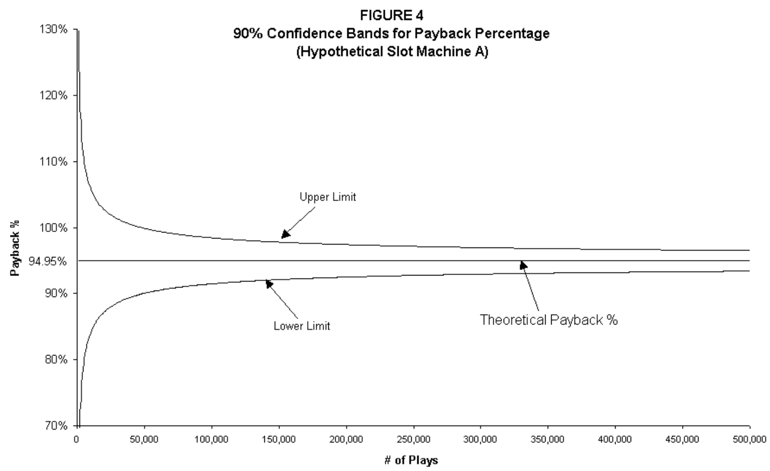
48 Although this value is at the foot of the hit column, it is not the total number of hits used in computing the hit frequency since the former includes “hits” with zero payout.

49 See Table 1, Appendix.

50 These limits are valid for a large number of plays due to the Central Limit Theorem, which ensures that the distribution of actual payback percentages in repeated samples of a given large size will be about normal.

51 The multiplication by 100 in the plus/minus part of the formula is necessary because the calculation is expressed in terms of payback percent rather than payback proportion.

4 shows this effect by displaying the 90% confidence bands for the realized payback percentage as a function of the number of plays for hypothetical slot machine A.



As mentioned earlier, to calculate confidence limits using a confidence level other than 90%, use the appropriate *z* value found from a standard normal table.

Effect of Payout Structure

Previous discussions on hit frequency and volatility touched on the effect of the payout structure of a slot machine on its long run behavior. To further illustrate these ideas, consider the table below showing the payout distributions for three hypothetical machines.

All three machines in the above table return roughly the same percentage to the player, but their behaviors over a long series of plays are dramatically different. Slot machine 1 will provide some type of payback over sixty times as often as slot machine 2 and twice as frequently as slot machine 3.

Because the overall payback percentages are about the same, the average payout per hit is much less for machine 1 (2.63) compared to machines 2 (170.55) and 3 (5.30). Slot machine 1 will appeal to players who want more frequent payouts at the expense of smaller amounts, while machine 2 offers relatively large payouts infrequently. In comparing the first two slot machines it would not be surprising if more players consider machine 1 to be “loose” and machine 2 “tight” even though their overall payback percentages are about the same. In terms of frequency of hits and size of payouts per hit, machine 3 falls between the first two. Machine 1 has the largest top payout (100,000 compared to 10,000 and 2,000 for machines 2 and 3) and the largest hit frequency. Hit frequency and payout percentage determine the entire payout distribution.

Payout Distributions For 3 Slot Machines					
SLOT MACHINE 1		SLOT MACHINE 2		SLOT MACHINE 3	
Payout	*Hits*	*Payout*	*Hits*	*Payout*	*Hits*
0	566,966	0	879,816	0	726,817
2	296,827	2	300	2	128,924
5	20,624	5	300	5	14,256
10	50	10	300	10	9,840
25	50	25	400	25	2,400
40	50	40	400	40	800
50	40	50	500	50	500
100	30	100	800	100	320
150	25	150	600	150	240
180	20	200	400	200	180
200	15	300	400	250	240
300	12	500	300	300	100
400	10	1,000	200	500	50
500	8	2,000	10	750	32
1,000	5	3,000	6	1,250	24
1,200	3	5,000	3	1,500	12
100,000	1	10,000	1	2,000	1
Cycle	884,736	Cycle	884,736	Cycle	884,736
Hits	317,770	Hits	4,920	Hits	157,919
Payback %	94.50%	Payback %	94.84%	Payback %	94.66%
Hit Freq.	35.92%	Hit Freq.	0.56%	Hit Freq.	17.85%
V.I.	175.57	V.I.	43.64	V.I.	21.04

In the example above, the volatilities of the three machines reflect the relative size of the top payouts. Slot machine 1 offers a top payout of 100,000 compared to 10,000 for machine 2, and only 2,000 for machine 3. Because of this, confidence limits for a given number of plays will be wider for slot machine 1 than those for machines 2 and 3. For machine 1 it will take more plays to achieve a given probability that the actual win percentage will be within a specified amount of the theoretical win percentage. Slot volatility is most greatly affected by the size of the largest payout(s). All else equal, machines with larger top payouts will be more volatile than those with smaller top payouts. Progressive slot machines, with their huge jackpot payouts, are particularly volatile – when the jackpot hits, the actual payback percentage spikes.

Jackpot Frequency and the Poisson Distribution

A jackpot on a certain slot machine has been hit three times in the past month. On another machine, the jackpot has not been hit in over three years. A player has collected on two royal flushes on the same video poker machine within one hour. Another player has played 200,000 hands with no royal flush. How unusual are these occurrences? Should there be cause for concern that something is amiss with these machines?

It is possible to calculate the probabilities of such events using the Poisson probability distribution. This distribution, named after the 19^{th} century French mathematician Simeon Poisson, is commonly used to calculate the probability of rare events. Applied to gaming devices, the relevant formula for the probability of X jackpot (or royal flush) hits in *n* plays is:

$$P(X) = \frac{e^{-\mu}\mu^{X}}{X!}$$

where μ is the average number of jackpot hits over the course of the *n* plays (or the number of cycles in the *n* plays), and *e* is the mathematical constant approximately equal to 2.71828. The value of μ will depend on the particular game and the number of plays, and can be computed by dividing the number of plays in question by the number of plays in a cycle. For example, a 3-reel, 90 stop per reel slot machine with a cycle of 729,000 plays and one jackpot combination will average one jackpot for every 729,000 plays, 1.372 for every one-million plays, 6.859 for every 5-million plays, and 13.717 for every 10-million plays. Optimal play on a Jacks-or-Better video poker machine will produce a royal flush about once every 40,000 hands, 5 times every 200,000 hands, and 25 times every one-million hands.

To illustrate the probability calculation, consider the slot machine described above. The chance that the jackpot will be not be hit over the course of one cycle (μ = 1) is:

$$P(0) = \frac{e^{-1} \times 1^{0}}{0!} = 0.3679.$$

This implies that 63.21% of the time there will be at least one jackpot over the course of a cycle. This same machine will yield an average of 2.7435 jackpots over the course of 2 million plays, so the probability the jackpot is not hit during this many plays is:

$$P(0) = \frac{e^{-2.7435} \times (2.7435)^{0}}{0!} = 0.0643\,.$$

Applying the Poisson formula using other values of *X* yields the following table for this slot machine.

Jackpot Probabilities: Slot Machine with 729,000–Play Cycle			
Number of Plays	*Hits*	*Probability*	*Cumulative*
1 million	0	25.37%	25.37%
1 million	1	34.80%	60.16%
1 million	2	23.87%	84.03%
1 million	3	10.91%	94.94%
1 million	4	3.74%	98.68%
1 million	5	1.03%	99.71%
1 million	6 or more	0.29%	100.00%
3 million	0	1.63%	1.63%
3 million	1	6.72%	8.35%
3 million	2	13.82%	22.17%
3 million	3	18.96%	41.13%
3 million	4	19.50%	60.63%
3 million	5	16.05%	76.69%
3 million	6	11.01%	87.70%
3 million	7	6.47%	94.17%
3 million	8	3.33%	97.50%
3 million	9	1.52%	99.02%
3 million	10 or more	0.98%	100.00%

Similar calculations can be done for the likelihood of royal flushes on video poker machines. The next table shows the probabilities of getting various numbers of royal flushes on a Jacks-or-Better machine assuming optimal play.

Royal Flush Probabilities: Jacks-or-Better*			
Number of Hands	*Hits*	*Probability*	*Cumulative*
100,000	0	8.21%	8.21%
100,000	1	20.52%	28.73%
100,000	2	25.65%	54.38%
100,000	3	21.38%	75.76%
100,000	4	13.36%	89.12%
100,000	5	6.68%	95.80%
100,000	6	2.78%	98.58%
100,000	7	0.99%	99.58%
100,000	8	0.31%	99.89%
100,000	9	0.09%	99.97%
100,000	10 or more	0.03%	100.00%
200,000	0	0.67%	0.67%
200,000	1	3.37%	4.04%
200,000	2	8.42%	12.47%
200,000	3	14.04%	26.50%
200,000	4	17.55%	44.05%
200,000	5	17.55%	61.60%
200,000	6	14.62%	76.22%
200,000	7	10.44%	86.66%
200,000	8	6.53%	93.19%
200,000	9	3.63%	96.82%
200,000	10 or more	3.18%	100.00%

**"Cycle" ≈ 40,000 hands.*

Tables showing jackpot and royal flush probabilities for varying numbers of cycles can also be constructed. Following is just such a table.

Jackpot or Royal Flush Probabilities by Cycles			
Number of Cycles	*Hits*	*Probability*	*Cumulative*
1	0	36.79%	36.79%
1	1	36.79%	73.58%
1	2	18.39%	91.97%
1	3	6.13%	98.10%
1	4	1.53%	99.63%
1	5	0.31%	99.94%
1	6 or more	0.06%	100.00%
3	0	4.98%	4.98%
3	1	14.94%	19.91%
3	2	22.40%	42.32%
3	3	22.40%	64.72%
3	4	16.80%	81.53%
3	5	10.08%	91.61%
3	6	5.04%	96.65%
3	7	2.16%	98.81%
3	8 or more	1.19%	100.00%
6	0	0.25%	0.25%
6	1	1.49%	1.74%
6	2	4.46%	6.20%
6	3	8.92%	15.12%
6	4	13.39%	28.51%
6	5	16.06%	44.57%
6	6	16.06%	60.63%
6	7	13.77%	74.40%
6	8	10.33%	84.72%
6	9	6.88%	91.61%
6	10	4.13%	95.74%
6	11	2.25%	97.99%
6	12 or more	2.01%	100.00%

If reasonable estimates are available for the average number of plays per hour for a given machine, calculating these probabilities over a given time is possible. Suppose, for example, a 729,000-cycle slot machine had not yielded a jackpot in three years. Constant play might produce about 400 decisions per hour, so assuming the machine was played about 50% of the time an average of 200 decisions per hour is reasonable. This activity level translates into a little over 5 million plays for a three-year period. The Poisson formula shows the probability of no jackpot over this many plays to be 0.00074, or about 1 in 1,350. If the machine was played constantly at 400 plays per hour for three years, there's only a 1 in 1.8 million chance of no jackpot during this time. At an average of 100 plays per hour it would be surprising but possible to see no jackpot in three years: the probability of which is 0.027 (about 1 in 37).

A CAVEAT

These discussions refer to the behavior of an individual machine. For a given device, the probability of an extreme outcome, such as no jackpot for a long period of time or several in a short period, is quite small. With many machines on the casino floor, however, there's a good chance that at least one of them will exhibit extreme behavior. To illustrate why this is so, assume there are 100 slot machines, each with a 729,000-play cycle. Over the course of 3 million plays, the probability of observing no jackpot on a given machine is 0.0163 (see the table on page 74) and so the probability of getting at least one jackpot on a given machine is 0.9837. For the collection of 100 machines, the probability they all produce at least one jackpot is $(0.9837)^{100} = 0.1933$ and so the probability at least one of these yields no jackpot over 3 million plays must be 0.8067. With a collection of 200 such machines, this probability rises to 96.3%, and with 300 devices, this probability is 99.3%. Even though there is a small probability that an individual device will exhibit a certain extreme behavior, when a large enough collection is considered, there's a good chance this behavior will appear in at least one of the collection.

> Grandpa: *I won 15 jackpots in a row; they flew me home first class just to get rid of me!*
>
> Drew: *I thought mom said you lost everything but your underwear and went home by bus?*
>
> Grandpa: *Oh, what does she know?*
>
> From the television show *Rugrats*

KENO

> ***House Advantage:***
> 27% (average)
> ***Typical Hold:***
> 27% (Nevada)

Keno generally is played by trying to guess some of twenty numbers[52] that will be randomly drawn from the field 1 through 80. Several types of keno tickets are available, depending on how many numbers the player marks, or selects, and the payoffs for how many of these selected numbers appear among the twenty drawn. The casino advantage on keno is large – 25% to 35% depending on the type of ticket and payout schedule. Nevada casinos retain a little over 27% of all money wagered at keno with winnings of about $90 million per year.[53]

Unlike many of the table games, heavy computer analysis is not required to analyze keno. Although a computer can ease the burden of the calculations, the relatively straightforward mathematics of combinatorial analysis – counting techniques of combinations and permutations covered in Chapter 1 – is all that is required to properly analyze keno games.

KENO ODDS

To see how keno odds are calculated, consider a simple 1-spot ticket in which the player only picks one number. At the MGM Grand in Las Vegas, the payout for a $2 1-spot ticket is $6. Since this is a relatively simple game, the use of combinations is not necessary, and the probability of winning is easily seen to be 20/80 or 1/4 (since the casino draws 20 numbers from a field of 80). Rather than factoring in the $2 cost for the ticket into the payoff (i.e., the net payoff if the player wins is $6−$2 = $4), a more common method is to calculate the expected return of this wager and then factor in the $2 cost at the end. For this wager, the expected return is:

$$ER = (+\$6)(1/4) = \$1.50.$$

This makes the expected value $1.50 − $2 = −$0.50, and so the player can expect to lose $0.50 of every $2 wagered, for a house advantage of 25%. Although unnecessary for this example, the probability of catching the one spot could be "computed" using the combinations formula:

> ***Expected Return* versus *Expected Value***
>
> For games like lotteries and keno where the wager is a "cost" taken from the bettor, it is more convenient to think in terms of *expected return,* rather than expected value, when analyzing the game. Whereas expected value represents the average net gain or loss, expected return is the average amount given back to the bettor after paying the cost. For example, if the expected return on a $1 lottery ticket is $0.60, the expected value is –$0.40.

52 In most cases, the casino will randomly draw twenty numbers, but keno variations range from four to twenty-two numbers drawn.

53 Nevada State Gaming Control Board.

$$\frac{\binom{20}{1} \times \binom{60}{0}}{\binom{80}{1}} = 1/4$$

A more complicated example is the MGM Grand's $2 6-spot keno ticket, where the combinations formula is virtually indispensable for calculating the house advantage. The payouts for this ticket are $3,000 if all six numbers are among the 20 drawn, $176 if the player catches 5 of his 6, $8 for catching 4 of 6, and $2 for catching 3 of the 6. To find the casino advantage, first compute the probabilities of catching 3, 4, 5, or 6 numbers. These are shown below:

$$P(catch\ 6) = \frac{\binom{20}{6} \times \binom{60}{0}}{\binom{80}{6}} = .000128985$$

$$P(catch\ 5) = \frac{\binom{20}{5} \times \binom{60}{1}}{\binom{80}{6}} = .003095639$$

$$P(catch\ 4) = \frac{\binom{20}{4} \times \binom{60}{2}}{\binom{80}{6}} = .028537918$$

$$P(catch\ 3) = \frac{\binom{20}{3} \times \binom{60}{3}}{\binom{80}{6}} = .12819548$$

The probabilities of catching 0, 1, and 2 numbers can be computed in a similar way. The expected return is found by multiplying each possible payout times its probability and summing. The following table summarizes the results.

MGM Grand $2 6-Spot Keno			
Catch	***Probability***	***Payout***	***Expected***
6	.000128985	3,000.00	.38695
5	.003095639	176.00	.54483
4	.028537918	8.00	.22830
3	.129819548	2.00	.25964
2	.308321425	0	0
1	.363494733	0	0
0	.166601753	0	0
		***ER* = 1.41973**	
		***HA* = 29.01%**	

With an expected return of $1.41973, the expected value for this $2 wager is –$0.58027 ($1.41973 minus $2), giving the casino a 29.01% (0.58027/2) advantage on this game.

The following table gives the summary for another example, the $7 Caesars Lucky Seven ticket found at Caesars Palace in Las Vegas.

$7 Caesars Lucky Seven Keno			
Catch	***Probability***	***Payout***	***Expected***
7	.000024403	77,777	1.89796
6	.000732077	1,777	1.30090
5	.008638505	77	0.66516
4	.052190967	7	0.36534
3	.174993241	0	0
2	.326654051	0	0
1	.315192505	0	0
0	.121574252	7	0.85102
		***ER* = 5.08038**	
		***HA* = 27.42%**	

Probability is like a wave. Because of the house advantage, over time the player dips lower and lower until he stops crossing the midpoint and ultimately loses all his money, unless he quits first.

Jess Marcum
Quoted in *Temples of Chance* (1992)

ROULETTE

House Advantage**:* 5.3% (double-zero) 2.7% (single-zero) ***Typical Hold: 24% (Nevada)

Roulette is a simple game to play: players merely try to guess which number will occur as the outcome of the spin of a ball around a numbered wheel. There are two types of wheels, the "double-zero," common in the United States, and the "single-zero," favored in Europe.

The double-zero wheel contains 38 pockets, numbered 1 through 36, 0 and 00. Of those numbered 1-36, 18 are red and 18 are black. The 0 and 00 are green. The single-zero wheel lacks the 00, and so has only 37 numbers. The game is played essentially the same regardless of which wheel is used, although the house advantage for the single-zero wheel is about half that of the double-zero game (2.7% vs. 5.3%). Except where otherwise specified, the discussion and analysis that follow are based on the double-zero game.

To play the game, players wager on a number, or combination of numbers. The following table shows the types of bets available, with their payoffs and true odds.

Roulette (Double-Zero Wheel)

Type of Bet	*True Odds*	*Payoff*
Straight-up (1 number)	37 to 1	35 to 1
Split (2 numbers)	36 to 2	17 to 1
Street (3 numbers)	35 to 3	11 to 1
Corner (4 numbers)	34 to 4	8 to 1
Five-Number Bet (5 numbers)	33 to 5	6 to 1
Double Street or Line (6 numbers)	32 to 6	5 to 1
Dozens/Columns (12 numbers)	26 to 12	2 to 1
Red/Black/Odd/Even/High/Low (18 numbers)	20 to 18	1 to 1

The dealer spins the wheel counterclockwise and releases a small ball clockwise on a track on the upper portion of the wheel. When the ball loses its momentum, it drops, bounces, and settles into one of the numbered pockets. Players win or lose depending on whether the winning number was among those they selected with their wager. As can be seen from the table above, the lower the probability of winning a particular wager, the higher the payoff. The casino makes money because the payoff odds are less than the true odds.

MATHEMATICS

The mathematics behind roulette is relatively straightforward, making it one of the easiest games in the casino to analyze. Assuming a balanced wheel, spins constitute independent trials. Every bet in the double-zero game will bring the house 5.26% in the long run, with the one exception of the five-number bet. This anomaly holds a 7.89% house advantage. Thus, the casino can expect to win about 5.3% of all money wagered in double-zero roulette.[54]

To see how the mathematics behind the game of roulette works, consider a \$1 straight-up bet on a single number. The probability of winning this wager is 1/38 (since there are 38 numbers on a double-zero wheel), and the probability of losing is 37/38. Notice that on the average this wager will win once and lose 37 times for every 38 spins (hence the 37 to 1 odds in the preceding table). To break even, the player would have to be paid \$37 for the one win to compensate for the 37 times he lost. That is, the true odds are 37:1. The payoff, however, is only 35:1, resulting in a loss to the player of \$2, or 5.26% of the \$38 wagered over the 38 spins. This 5.26% house advantage can be computed using the formula for expected value:[55]

$$EV = (+\$35)(1/38) + (-\$1)(37/38) = -\$0.0526.$$

This expected loss of \$0.0526 on a \$1 wager represents a house advantage of 5.26%. This, of course, is the same percentage edge that would be obtained regardless of the size of the wager. A \$5 single number bet, for example, has a player expectation of −\$0.263, or 5.26% of \$5.

Expectations and advantage for other types of roulette wagers can be computed in the same way. Except for the five-number bet, all will result in a 5.26% house edge. The calculations for a \$25 corner bet and a \$50 five-number bet are shown below.

\$25 Corner Bet

(Payoff = 8:1 = \$200, Probability = 4/38).

$$EV = (+\$200)(4/38) + (-\$25)(34/38) = -\$1.316.$$

$$\textit{House advantage} = \$1.316/\$25 = 0.0526, \text{ or } 5.26\%$$

\$50 Five-Number Bet

(Payoff = 6:1 = \$300, Probability = 5/38)

$$\textit{Expected Value} = (+\$300)(5/38) + (-\$50)(33/38) = -\$3.947.$$

$$\textit{House advantage} = \$3.947/\$50 = 0.0789, \text{ or } 7.89\%.$$

54 The overall house advantage in single-zero roulette is only 2.7%. The bets and payoffs are essentially the same as in the double-zero game (there is no five-number bet, and there may be a few bets offered that are not in the double-zero game), but there are only 37 numbers on the wheel. In addition, sometimes the "en prison" rule is offered in the single-zero game, lowering the house advantage on even-money bets (odd/even, black/red, high/low) to 1.35%. With this rule, rather than losing an even money wager when the 0 comes up, it remains for one more spin – in prison. If the desired even money event occurs on the next spin, the original wager is returned with no gain or loss; otherwise it loses. Occasionally a roughly analogous rule called "surrender" can be found in double-zero games. Under this rule, a player loses only half the wager for even money bets when 0 or 00 hits. Surrender lowers the house advantage for even money bets in double-zero roulette to 2.63%.

55 As noted in Chapter 1, the house advantage can also be computed directly from the true odds and payoffs: House Advantage = (37−35)/(37+1) = 2/38 = 0.0526, or 5.26%.

No pattern or combination of bets will alter the casino's 5.3% advantage.[56] As an illustration, suppose a player makes a combination bet totaling \$40 by putting \$20 on red, \$10 on column 2, and \$5 each on 0 and 00. The possible outcomes are:

- Lose \$40, with probability 10/38 (if a black number in columns 1 or 3 occurs – lose all bets),
- Lose \$10, with probability 8/38 (if a black number in column 2 occurs – win \$20 for column 2 bet, lose \$30 for the red, 0, and 00 bets),
- Lose \$0, with probability 14/38 (if a red number in columns 1 or 3 occurs – win \$20 for the red bet, lose \$20 for the 0, 00, and column 2 bets),
- Win \$30, with probability 4/38 (if a red number in column 2 occurs – win \$20 for the red bet, win \$20 for the column 2 bet, lose \$10 for the 0 and 00 bets),
- Win \$140, with probability 2/38 (if either 0 or 00 occurs – win \$175 for the winning single number bet, lose \$35 for the red, column 2 and losing single number bets).

The expected value is:

$$\begin{aligned} EV &= (-\$40)(10/38) + (-\$10)(8/38) + (\$0)(14/38) \\ &\qquad + (+\$30)(4/38) + (+\$140)(2/38) \\ &= (-\$10.526) + (-\$2.105) + (-\$0.000) \\ &\qquad + (+\$3.158) + (+\$7.368) \\ &= -\$2.105. \end{aligned}$$

Dividing by the total amount wagered, \$40, yields a 5.26% overall house advantage.

INDEPENDENT TRIALS AND A GAMBLER'S FALLACY

A common "gambler's fallacy" is to believe that black is more likely after a series of consecutive reds (black is "due") while another is that since red is "hot," red is more likely on the next spin. Both, of course, are wrong. The spins of a roulette wheel are random, independent events, and so the outcome of one spin does not affect the outcome of any other spin. If a long string of consecutive red outcomes has occurred, the next spin is no more or less likely to be red. The probability of red is 18/38 on each and every spin.

A flaw in the gambler's fallacy logic is mistaking the short run for the long run. While it is true that the probability of, say, ten consecutive reds is small, over the course of millions and millions of spins there will be many such sequences. If one were to examine all sequences of ten consecutive reds over the course of millions of spins, it would turn out that for about 47.37% (18/38) of these sequences, the next spin in the sequence (the eleventh) would be a black number, and for about 47.37% the next spin would be red (the remaining 5.26% would be green). Viewed in the

56 If the five-number bet is involved in a combination of bets, the overall house advantage for this combination will increase proportional to the fraction of the wager placed on the five-number bet.

context of the long run, a sequence of consecutive outcomes like this should not be surprising and does not change the probabilities of the various outcomes on subsequent spins.

As a by-product of the independence of outcomes in roulette, the multiplication rule for combining probabilities makes it relatively easy to calculate the probability of a series of consecutive outcomes. For example, the probability of getting two red numbers in two spins is (18/38) x (18/38) = .224. The probability of three consecutive reds is (18/38) x (18/38) x (18/38) = .106. The probability of ten consecutive reds is $(18/38)^{10}$ = .000568, or about 1 in 1,758. Similarly, the probability of getting the number 0 on three consecutive spins is $(1/38)^3$ = .0000182, or 1 in 54,872. The probability of getting three consecutive 8's is also .0000182. However, the probability of getting the same number (any number) on three consecutive spins is 38 x .0000182 = 000693, or 1 in 1,444.

Roulette Volatility

As noted in Chapter 1, the actual percentage won in a series of wagers will deviate from the expected win percentage due to statistical fluctuations. The degree of volatility depends on the number of wagers in the series and the type of wager.[57] In roulette, the greatest variation can be expected for the straight-up bet on a single number and the least variation for even-money wagers. This can be seen from the values of the per unit standard deviation for each wager. For the straight-up bet,

$$SD_{per\ unit} = \sqrt{(1/38)(-35-.0526)^2 + (37/38)(+1-.0526)^2} = 5.7626\,.$$

For an even-money wager,

$$SD_{per\ unit} = \sqrt{(18/38)(-1-.0526)^2 + (20/38)(+1-.0526)^2} = 0.9986\,.$$

Other roulette wagers have standard deviation values between these two extremes. All else equal, the larger the standard deviation, the greater the volatility and fluctuations that can be expected in a series of wagers.

> Percentages never lie. We built all these hotels on percentages. You can lose faith in everything, religion and God, women and love, good and evil, war and peace, you name it. But the percentage will always stand fast.
>
> Mario Puzo
> *Fools Die* (1978)

57 The wager amount does not affect the volatility associated with win percentage, although it is a factor when analyzing the volatility of the total win amount. See Chapter 1.

CRAPS

> ***House Advantage:***
> 1.41% (pass/come, no odds)
> 0.61% (pass/come, 2X odds)
> 1.52% – 16.67% (other bets)
> ***Typical Hold:***
> 14% (Nevada)

Craps is played by betting on the outcome of a roll of a pair of dice. There are a myriad of possible wagers that can be made, each with different odds, payoff and house advantage. To understand these wagers, familiarity with the probabilities associated with the various outcomes of tossing a pair of dice is helpful. The 36 possible outcomes and the probabilities of each possible total are given in the following table.

Outcomes When Rolling A Pair Of Dice	Total	Probability
(1,1)	2	1/36
(1,2) (2,1)	3	2/36
(1,3) (2,2) (3,1)	4	3/36
(1,4) (2,3) (3,2) (4,1)	5	4/36
(1,5) (2,4) (3,3) (4,2) (5,1)	6	5/36
(1,6) (2,5) (3,4) (4,3) (5,2) (6,1)	7	6/36
(2,6) (3,5) (4,4) (5,3) (6,2)	8	5/36
(3,6) (4,5) (5,4) (6,3)	9	4/36
(4,6) (5,5) (6,4)	10	3/36
(5,6) (6,5)	11	2/36
(6,6)	12	1/36

With more combinations (six) yielding a total of seven than any other, seven is the most likely total to be thrown, with probability 6/36, 1/6, or 16.67%. Working outwards from seven in either direction, the totals become progressively less likely until you get to two and twelve, the least likely outcomes, each with probability 1/36 or 2.78%. This effect is portrayed in the above table by the pyramid shaped list of outcomes.

THE MAIN BETS

The main bets in craps are the pass line, the come, the don't pass, the don't come and odds.

PASS LINE BET

When a game or new round of play begins, the roll is called the "come-out" roll. A bet on the pass line wins even money if the come-out roll is a 7 or 11, and loses if it is a 2, 3, or 12 (a roll of 2, 3, or 12 is called "craps"). If any other number is rolled, this number becomes the "point." Once a point is established, the pass wager is not resolved until the point is rolled again or a 7 is rolled. If the point is rolled a second time before a 7, the pass line wager wins even money. If a 7 occurs before the point is rolled a second time, the pass line wager loses.

The house advantage on the pass line bet is 1.414%. To see how this is calculated, note that there are two ways to win this bet – winning the come-out roll or making the point. Since there are six possible points (4, 5, 6, 8, 9, and 10), the number of ways to win the pass line bet can be further broken down into seven possibilities:

(1) win the come-out roll,
or (2) establish and make the point 4,
or (3) establish and make the point 5,
or (4) establish and make the point 6,
or (5) establish and make the point 8,
or (6) establish and make the point 9,
or (7) establish and make the point 10.

Similarly, the pass line bet could be lost in seven distinct ways. To calculate the expectation, the probabilities for each of these outcomes are needed.

Since there are eight ways to roll a 7 or 11 (winning numbers for the pass line bet), the probability of winning the come out roll is 8/36, or .222222. Since there are 4 ways to roll a 2, 3, or 12, the probability of losing the come-out roll is 4/36, or .111111. The probability of establishing and making a point depends on the particular point number. For example, the probability of establishing 4 as the point is 3/36 (three ways to roll a 4), and once established, the probability of making (passing) that point is 3/9 (there are six ways to roll a 7 and three ways to roll a 4). Thus the probability of establishing then making the point 4 is (3/36)(3/9) = 1/36. The probability of establishing then not making the point four is (3/36)(6/9) = 2/36. Similar reasoning can be used to find the probabilities associated with the points 5, 6, 8, 9, and 10. The following table summarizes the probabilities:

> Dice have their laws, which Courts of Justice cannot undo.
>
> St. Ambrose (*d.* 397)

Point Probabilities in Craps

Point	*Establish*	*Establish then Pass*	*Establish then don't pass*
4	3/36	3/36 x 3/9 ≈ .027778	3/36 x 6/9 ≈ .055556
5	4/36	4/36 x 4/10 ≈ .044444	4/36 x 6/10 ≈ .066667
6	5/36	5/36 x 5/11 ≈ .063131	5/36 x 6/11 ≈ .075758
8	5/36	5/36 x 5/11 ≈ .063131	5/36 x 6/11 ≈ .075758
9	4/36	4/36 x 4/10 ≈ .044444	4/36 x 6/10 ≈ .066667
10	3/36	3/36 x 3/9 ≈ .027778	3/36 x 6/9 ≈ .055556

Since winning the pass line bet pays even money, the expectation on a 1-unit pass line wager is:

$$EV = (+1)(.222222) + (+1)(.027778) + (+1)(.044444) + (+1)(.063131)$$

$$+ (+1)(.063131) + (+1)(.044444) + (+1)(.027778) + (-1)(.111111)$$
$$+ (-1)(.055556) + (-1)(.066667) + (-1)(.075758) + (-1)(.075758)$$
$$+ (-1)(.066667) + (-1)(.055556)$$

$$= (+1)(.492929) + (-1)(.507071) = -0.01414.$$

This represents a 1.414% house advantage on the pass line bet. The calculation above also shows the overall probability of winning the pass line bet is .492929 and the overall probability of losing the pass line bet is .507071.

After a point is established, a bet on the pass line cannot be removed or reduced, although it may be increased. This latter move is unfavorable to the player since the advantage on the pass line bet is on the come-out roll and the odds are against the pass line bet once a point is established. Also unfavorable, a player who misses the come-out roll may make a bet on the pass line after a point is established (called a *put bet*).

FREE ODDS

The odds bet is called free odds because … well, because it's free – the house advantage is zero. Odds can be taken after a point has been established on the pass line wager (odds can also be taken on come, don't pass, and don't come after a point is established – these bets are discussed on the next page). Like the pass line bet, if the point is rolled before a 7, the odds bet wins. If a 7 is rolled before the point, the odds bet loses. Unlike the pass line, which pays even money, a winning odds bet is paid according to the true odds. For example, suppose a $5 pass line bet is made and a 4 is thrown on the come-out roll. Odds may now be taken on the point 4. The amount of odds that can be taken varies depending on the casino, but for this example let's assume single odds are taken. This is an additional $5 bet that will win if a 4 is rolled before a 7 and lose if a 7 is rolled before a 4. Since there are six ways to roll a 7 and three ways to roll a 4, the true odds against winning this bet are 2 to 1. If the point passes (a 4 is rolled before a 7), the pass line will be paid $5 (even money) and the odds bet will be paid $10 (2 to 1). Because odds bets are paid according to the true odds, the expectation and house advantage are zero. For example, the expected value calculation for a $10 odds bet on the point 4 is:

$$EV(\$10) = (+\$20)(1/3) + (-\$10)(2/3) = \$0.00.$$

Because the house advantage is zero on the odds bet, taking odds when a point is established will lower the overall house advantage on the total wager, pass line and odds bet combined, from the 1.414% edge on the pass line alone. With single odds, the house advantage on the combined total wager is 0.848%, with double odds 0.606%, with triple odds 0.471%, with 5X odds 0.326%, and with 10X odds 0.184%. Many casinos are now allowing a 3/4/5X odds structure, where the player may take 3X odds on the 4 and 10, 4X odds on the 5 and 9, and 5X odds on the 6 and 8. This 3/4/5X odds structure leads to a 0.374% house advantage on the total wager (pass line or come plus odds).

The odds bet may be removed at any time. Come odds (see come bet below) are typically "off," or inactive, on a come-out roll, unless the bettor specifically asks that they be "on." Don't come odds are always on.

THE COME BET

The come bet gives the player the opportunity to make what is essentially a pass line bet after a point is established. A come bet turns the next roll of the dice into a come-out roll that wins on a 7 or 11, loses on a 2, 3, or 12, and otherwise establishes a point that will win if the point number is rolled again before a 7 and will lose if a 7 is rolled before the point. Since the come bet works like the pass line bet, the probabilities and resulting edge are the same as that for the pass line (1.414% with no odds, 0.848% with single odds, 0.606% with double odds, 0.471% with triple odds, 0.326% with 5X odds, 0.184% with 10X odds, and 0.374% for 3/4/5X odds).

THE DON'T PASS LINE BET

The don't pass wager works the opposite of the pass line bet, except that don't pass bettors are barred from winning on a come-out roll of 12. The don't pass line bet wins on a come-out roll of 2 or 3, loses on a 7 or 11, and ties on a 12. On point rolls, the don't pass wins if a 7 is rolled before the point and loses if the point is rolled before a 7. An analysis similar to the one for the pass line wager shows the house advantage on the don't pass bet to be 1.364%.[58]

Like the pass line bet, odds can be taken once a point is established. Odds on the don't pass line bet will win if a 7 rolls before the point and are paid at true odds – 1 to 2 for the 4 or 10, 2 to 3 for the 5 or 9, and 5 to 6 for the 6 or 8. Taking odds will reduce the house advantage on the combined wager, don't pass line plus odds, to 0.682% with single odds, 0.455% with double odds, 0.341% with triple odds, 0.227% with 5X odds, 0.124% with 10X odds, and 0.273% for 3/4/5X odds.[59]

Don't pass bets can be removed or reduced after a point is established.

THE DON'T COME BET

The don't come bet is essentially a don't pass bet made after a point has been established. It works just like the don't pass and has the same house advantage (1.364% with no odds, 0.682% with single odds, 0.455% with double odds, 0.341% with triple odds, 0.227% with 5X odds, 0.124% with 10X odds, and 0.273% with 3/4/5X odds).

OTHER CRAPS BETS

The pass line, don't pass line, come, and don't come wagers, along with the odds bet, represent the main wagers in craps. There are a plethora of other wagers that can

58 This is house advantage per roll. Since rolling 12 results in a push, the house advantage may also be expressed as 1.403% per resolved bet. There are other games, notably baccarat, in which the house advantage for certain wagers may be expressed either including or excluding ties.

59 These are house advantages per roll. The corresponding house advantages per resolved bet are 0.691%, 0.459%, 0.343%, 0.228%, 0.124%, and 0.274% for 1X, 2X, 3X, 5X, 10X, and 3/4/5X odds.

be made, all of which carry a higher house advantage, some significantly so. A brief summary of these wagers and their house advantages is given below.

PLACE AND DON'T PLACE BETS

A place bet on one of the point numbers, 4, 5, 6, 8, 9, or 10, is a bet that the number will roll before a 7. Payoffs are less than true odds, typically 9 to 5 for the numbers 4 and 10, 7 to 5 for the numbers 5 and 9, and 7 to 6 for the numbers 6 and 8. The house advantage is 6.67% for the 4 and 10, 4.00% for the 5 and 9, and 1.52% for the 6 and 8. A don't place bet is the opposite of a place bet – it's a bet that a 7 will roll before the number.[60] Like place bets, these wagers are paid off at less than true odds. The don't place on a 4 or 10 pays 5 to 11 (true odds are 1 to 2), a don't place on a 5 or 9 pays 5 to 8 (true odds 2 to 3), and a don't place on a 6 or 8 pays 4 to 5 (true odds 5 to 6). House advantages are 3.03% for the 4 and 10, 2.50% for the 5 and 9, and 1.82% for the 6 and 8. Expectation calculations and resulting advantages are shown below.

Place 4, Place 10

$$EV(\$5) = (+9)(3/9) + (-5)(6/9) = -0.3333,$$

$$HA = 0.3333/5 = 0.0667, \text{ or } 6.67\%.$$

Place 5, Place 9

$$EV(\$5) = (+7)(4/10) + (-5)(6/10) = -0.2000,$$

$$HA = 0.2000/5 = 0.0400, \text{ or } 4.00\%$$

Place 6, Place 8

$$EV(\$6) = (+7)(5/11) + (-6)(6/11) = -0.0909,$$

$$HA = 0.0909/6 = 0.0152, \text{ or } 1.52\%.$$

Don't Place 4, Don't Place 10

$$EV(\$11) = (+5)(6/9) + (-11)(3/9) = -0.3333,$$

$$HA = 0.3333/11 = 0.0303, \text{ or } 3.03\%.$$

Don't Place 5, Don't Place 9

$$EV(\$8) = (+5)(6/10) + (-8)(4/10) = -0.2000,$$

$$HA = 0.2000/8 = 0.0250, \text{ or } 2.50\%.$$

Don't Place 6, Don't Place 8

60 Don't place bets are not very common these days and not available in many casinos.

$$EV(\$5) = (+4)(6/11) + (-5)(5/11) = -0.0909,$$

$HA = 0.0909/5 = 0.0182$, or 1.82%.

BUY AND LAY BETS

Buy and lay bets are similar to the place and don't place bets except that the payoffs are at true odds and a 5% commission is charged on the amount of a buy bet and on the possible winning amount of a lay bet. All buy bets have a 4.76% house advantage. Lay bets carry house advantages of 2.44% for the 4 and 10, 3.23% for the 5 and 9, and 4.00% for the 6 and 8. Expectation and resulting house advantage calculations are shown below.

Buy 4, Buy 10

$$EV(\$21) = (+\$39)(3/9) + (-\$21)(6/9) = -\$1.00,$$

$$HA = 1.00/21 = 0.0476, \text{ or } 4.76\%.$$

Buy 5, Buy 9

$$EV(\$21) = (+\$29)(4/10) + (-\$21)(6/10) = -\$1.00,$$

$$HA = 1.00/21 = 0.0476, \text{ or } 4.76\%.$$

Buy 6, Buy 8

$$EV(\$21) = (+\$23)(5/11) + (-\$21)(6/11) = -\$1.00,$$

$$HA = 1.00/21 = 0.0476, \text{ or } 4.76\%.$$

Lay 4, Lay 10

$$EV(\$41) = (+\$19)(6/9) + (-\$41)(3/9) = -\$1.00,$$

$$HA = 1.00/41 = 0.0244, \text{ or } 2.44\%.$$

Lay 5, Lay 9

$$EV(\$31) = (+\$19)(6/10) + (-\$31)(4/10) = -\$1.00,$$

$$HA = 1.00/31 = 0.0323, \text{ or } 3.23\%.$$

Lay 6, Lay 8

$$EV(\$25) = (+\$19)(6/11) + (-\$25)(5/11) = -\$1.00,$$

$$HA = 1.00/25 = 0.0400, \text{ or } 4.00\%.$$

THE FIELD BET

The field bet is a bet that the next roll will be a 2, 3, 4, 9, 10, 11, or 12. The payoff is typically 2 to 1 for 2 and 12, and even money for 3, 4, 9, 10, and 11. Although this wager covers seven out of eleven numbers, the losing numbers (5, 6, 7, and 8) can be rolled more in more ways. The house advantage is 5.56%.

This can be seen with the usual expected value calculation for a $1 wager:

$$EV = (+\$2)(2/36) + (+\$1)(14/36) + (-\$1)(20/36) = -\$0.0556.$$

Some casinos offer a 3 to 1 payoff on either the 2 or 12 (but not both). A calculation similar to the one above will show that paying 3 to 1 for either the 2 or 12 lowers the house advantage to 2.78%. (Paying 3 to 1 on both the 2 and 12 would result in a house advantage of zero.)

BIG 6 AND BIG 8

The big 6 is a bet that the 6 will roll before a seven; the big 8 is that the 8 will roll before a 7. The payoff is even money and the house advantage is 9.09%, as can be seen from the following expectation calculation:

$$EV = (+1)(5/11) + (-1)(6/11) = -0.0909.$$

The big 6 and big 8, the place 6 and place 8, and the buy 6 and buy 8 are exactly the same wagers but with different payoffs, and hence different house advantages. The place 6 and place 8 pay 7 to 6 and have a house advantage of 1.52%; the buy 6 and buy 8 pay the true odds of 6 to 5 but charge a 5% commission, resulting in a house advantage of 4.76%; the big 6 and big 8 pay even money and have a house advantage of 9.09%.

HARDWAYS

There are four hardway wagers: hard 4, hard 6, hard 8, and hard 10. A hardway is a wager that the selected number will roll with doubles before the number is rolled any other way or a 7 is rolled. For example, a hard 6 wins if 3-3 comes up before an easy 6 (1-5, 5-1, 2-4, or 4-2) or a 7. Payoffs are 7 to 1 for hard 4 and hard 10, and 9 to 1 for hard 6 and hard 8. Expected value calculations are straightforward. For the hard 4 or hard 10, there are eight losing combinations (two easy ways to roll the selected number – 1-3, 3-1 or 4-6, 6-4 – and six ways to roll a 7) and only one winning combination (either 2-2 or 5-5).

Thus the expectation is:

$$EV = (+7)(1/9) + (-1)(8/9) = -0.1111.$$

For the hard 6 or hard 8, there are ten losing combinations (four easy ways to roll the selected number – 1-5, 5-1, 2-4, 4-2 or 2-6, 6-2, 3-5, 5-3 – and six ways to roll a 7) and only one winning combination (either 3-3 or 4-4). The expectation is:

$$EV = (+9)(1/11) + (-1)(10/11) = -0.0909.$$

House advantages are 11.11% for the hard 4 and hard 10, and 9.09% for the hard 6 and hard 8.

ANY CRAPS

Any craps is a one-roll bet that the next roll will be a 2, 3, or 12. If the next roll is not a 2, 3, or 12, the wager is lost. The payoff is 7 to 1[61] and the expected value is:

$$EV = (+7)(4/36) + (-1)(32/36) = -0.1111,$$

$$HA = 11.11\%.$$

ANY SEVEN

Another one-roll bet, any seven wins 4 to 1 if the next roll is a 7. The expected value is:

$$EV = (+4)(6/36) + (-1)(30/36) = -0.1667,$$

$$HA = 16.67\%.$$

CRAPS AND ELEVEN (C&E)

Craps and eleven (C&E) is a one-roll bet that the next roll will be a 2, 3, 11, or 12. Using payoffs of 3 to 1 on the 2, 3, and 12, and 7 to 1 on the 11, the expected value is:

$$EV = (+3)(4/36) + (+7)(2/36) + (-1)(30/36) = -0.1111,$$

$$HA = 11.11\%.$$

TWO OR TWELVE

These are two separate one-roll bets – a wager can be made that the next roll will be a two or a wager can be made that the next roll will be a twelve. The payoff is typically 30 to 1 for each.[62]

The expectation using this payoff is given by:

$$EV = (+30)(1/36) + (-1)(35/36) = -0.1389,$$

$$HA = 13.89\% \text{ (for either wager)}.$$

THREE OR ELEVEN

Like the wagers on two or twelve, these are two separate one-roll bets. Each typically pays 15 to 1.[63]

$$EV = (+15)(2/36) + (-1)(34/36) = -0.1111,$$

$$HA = 11.11\% \text{ (for either wager)}.$$

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61 Often the payoffs on the proposition bets in craps are posted in terms of "for" rather than "to." The payoff for the any craps wager, for example, is 8 for 1.

62 Some casinos only pay 29 to 1, yielding a house advantage 16.67%.

63 Some casinos only pay 14 to 1, yielding a house advantage 16.67%.

HORN BET

The horn is a one-roll bet that the next roll will be 2, 3, 11 or 12. It is really just four separate bets on these four numbers, so the bet is made in multiples of four. The horn returns the usual payoffs for the individual winning number: typically 30 to 1 for the 2 and 12, and 15 to 1 for the 3 and 11. With these payoffs and four units wagered on the horn, the bet will net 27 units for a hit on 2 or 12, and 12 units for a hit on 3 or 11. The expected value is:

$$EV = (+27)(2/36) + (+12)(4/36) + (-4)(30/36) = -0.50.$$

Since the bet was 4 units, the house advantage is 0.50/4 = 0.125, or 12.5%.[64]

HORN HIGH BET

The high horn is a horn bet made in multiples of five with the extra unit wagered on the "high" number designated by the bettor. For example, a $5 high twelve would put $1 each on the 2, 3, and 11, and $2 on the 12. With 30 to 1 payoffs for the 2 and 12, and 15 to 1 for the 3 and 11, the house advantage depends on which number is chosen to be "high." If 2 or 12 are designated high, the expected value is given by

$$EV = (+57)(1/36) + (+26)(1/36) + (+11)(4/36) + (-5)(30/36) = -0.6389.$$

If 3 or 11 are designated high, the expectation is:

$$EV = (+26)(2/36) + (+27)(2/36) + (+11)(2/36) + (-5)(30/36) = -0.6111.$$

These expectations yield house advantages of 12.78% (0.6389/5 = 0.1278) for the high two and high twelve, and 12.22% (0.6111/5 = 0.1222) for the high three and high eleven.[65]

HOP BET

The hop is a one-roll bet that the next roll will be a certain combination. There are two types, an "easy" or two-way hop, such as 6-1, and a "hard" or one-way hop, such as 2-2.[66] Typical payoffs are 15 to 1 for an easy hop and 30 to 1 for a hard hop. The expected value calculation for the easy hop is:

$$EV = (+15)(2/36) + (-1)(34/36) = -0.1111.$$

For the hard hop:

$$EV = (+30)(1/36) + (-1)(35/36) = -0.1389.$$

House advantages are 11.11% for the easy hop and 13.89% for the hard hop.[67]

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64 If payoffs are 29 to 1 on the 2 and 12, and 14 to 1 on the 3 and 11, the house advantage is 16.67%.

65 If payoffs are 29 to 1 on the 2 and 12, and 14 to 1 on the 3 and 11, the house advantages are 16.67% for all high horn bets.

66 A hard hop differs from a hardway bet in that the hard hop is a one-roll bet whereas the hardway is decided only if the roll is the designated total, easy or hard, or a seven.

67 Some casinos offer only 14 to 1 for an easy hop and 29 to 1 for a hard hop, raising the house advantages to 16.67% for both wagers.

SUMMARY OF HOUSE ADVANTAGES FOR CRAPS WAGERS	
WAGER	*HOUSE ADVANTAGE*
Pass Line/Come, no odds	1.41%
Pass Line/Come, 1X odds	0.85%
Pass Line/Come, 2X odds	0.61%
Pass Line/Come, 3X odds	0.47%
Pass Line/Come, 5X odds	0.33%
Pass Line/Come, 10X odds	0.18%
Pass Line/Come, 3/4/5X odds	0.37%
Don't pass/Don't Come, no odds (per roll)	1.36%
Don't pass/Don't Come, 1X odds (per roll)	0.68%
Don't pass/Don't Come, 2X odds (per roll)	0.45%
Don't pass/Don't Come, 3X odds	0.34%
Don't pass/Don't Come, 5X odds	0.23%
Don't pass/Don't Come, 10X odds	0.12%
Don't pass/Don't Come, 3/4/5X odds	0.27%
Place 4, Place 10	6.67%
Place 5, Place 9	4.00%
Place 6, Place 8	1.52%
Don't Place 4, Don't Place 10	3.03%
Don't Place 5, Don't Place 9	2.50%
Don't Place 6, Don't Place 8	1.82%
Buy (any)	4.76%
Lay 4, Lay 10	2.44%
Lay 5, Lay 9	3.23%
Lay 6, Lay 8	4.00%
Field Bet (2-1 on 2 and 12)	5.56%
Field Bet (3-1 on 2 or 12)	2.78%
Big 6, Big 8	9.09%
Hard 4, Hard 10	11.11%
Hard 6, Hard 8	9.09%
Any Craps	11.11%
Any Seven	16.67%
C&E	11.11%
Two, Twelve (30-1)	13.89%
Two, Twelve (29-1)	16.67%
Three, Eleven (15-1)	11.11%
Three, Eleven (14-1)	16.67%
Horn Bet (30-1 & 15-1)	12.50%
Horn Bet (29-1 & 14-1)	16.67%
Horn High 2, Horn High 12 (30-1 & 15-1)	12.78%
Horn High 3, Horn High 11 (30-1 & 15-1)	12.22%
Horn High (any, 29-1 & 14-1)	16.67%
Hop Bet – easy (15-1)	11.11%
Hop Bet – hard (30-1)	13.89%
Hop Bet – easy (14-1)	16.67%

Hop Bet – hard (29-1)	16.67%

BACCARAT

> ***House Advantage:***
> Player Bet: 1.24%
> (1.37% ignoring ties)
> Banker Bet: 1.06%
> (1.17% ignoring ties)
> Tie Bet: 14.36%
> ***Typical Hold:***
> 18% (Nevada)

Several things about James Bond movies give false impressions, including that baccarat requires some level of skill. Although the results of consecutive hands are mathematically dependent, it is not possible for the player to take advantage of this dependency and baccarat is for all practical purposes a game of pure chance. In addition, it is a relatively easy game to play since hitting and drawing rules are fixed and no decisions are required (or even possible) once play begins. Players need only decide whether to bet on the banker or the player (there is also a tie bet). The use of a pencil and pad to track hand outcome, which some players do, could be put to better use as a sketchpad.

Baccarat's overall house advantage is about 1.2% – one of the lowest in the casino.

RULES OF PLAY

Play proceeds as follows. After bets are made, two cards are dealt to each side (player and banker). All cards are community cards – there are no individual hands. Face cards and Tens are worth zero, Aces count as one, and all other cards are worth face value. The total for a hand is the sum of the values of the individual cards in the hand, modulo 10 (meaning that the first digit is dropped if the total is greater than nine).

> Monte Carlo has the right idea; fix a game where you are going to get people's money, but the people don't mind you getting it. A fellow can always get over losing money in a game of chance, but he seems so constituted that he can never get over money thrown away to a government in taxes.
>
> Will Rogers
> "A Visit to Monte Carlo"
> *A Will Rogers Treasury* (1982)

A total of 8 or 9 on the first two cards is called a natural. If either side has a natural, the hand is over and wagers are resolved. If neither side has a natural, a fixed set of rules (following) is applied to determine if either side takes a third card. Once the hands are complete, the side with the higher total wins. If the two sides have equal totals, player and banker bets are a push. The payoffs on winning player and banker bets are even money, but a 5% commission is charged to banker wins.[68] A winning tie bet pays 8 to 1.

Baccarat's hitting and standing rules are presented in the following table. To read the table, start with the player's first two cards total in the left column and proceed to the right, column by column to determine the drawing/standing rules.

68 The rules favor the banker and the casino's advantage on the banker bet would be negative without the 5% commission. With this commission, the house advantage on the banker bet is 1.06% (including ties).

Baccarat Drawing Rules*				
Player's Total (first two cards)	*Player's Rule*	*Banker's Total (first two cards)*	*Player's 3rd card*	*Banker's Rule*
0, 1, 2, 3, 4, 5	Draw	0, 1, 2	Anything	Draw
		3	1, 2, 3, 4, 5, 6, 7, 9, 0	Draw
			8	Stand
		4	2, 3, 4, 5, 6, 7	Draw
			1, 8, 9, 0	Stand
		5	4, 5, 6, 7	Draw
			1, 2, 3, 8, 9, 0	Stand
		6	6, 7	Draw
			1, 2, 3, 4, 5, 8, 9, 0	Stand
		7	Anything	Stand
6, 7	Stand	0, 1, 2, 3, 4, 5	--------	Draw
		6, 7, 8, 9	--------	Stand
**If either player or banker totals 8 or 9 on first two cards, no cards are drawn and hand is over.*				

As mentioned above, since the drawing rules are fixed, baccarat offers no opportunity for strategic decisions like blackjack does. Furthermore, although the game involves dependent trials (similar to blackjack), it has been shown that card-counting techniques are not effective in baccarat.[69]

ANALYSIS

Since baccarat is typically played with eight decks of cards, a proper mathematical analysis of baccarat can be performed by examining all possible combinations of player and banker hands for all possible six-card sequences in an eight-deck shoe. The analysis can be limited to six-card subsets since no hand can consist of more than six total cards for player and banker combined, and since hands comprised of less than six cards can be accounted for by summing the appropriate probabilities.

Given 416 cards in an eight-deck shoe, there are $\binom{416}{6}$, or almost seven trillion, six-card subsets, each of which must be examined in $\binom{6}{2}\times\binom{4}{2}\times 2 = 180$ ways. However, since suits and the distinction between Tens, Jacks, Queens, and Kings are irrelevant, only 5,005 distinguishable six-card subsets are possible, significantly easing the computational burden. A complete analysis, then, consists of (1) identifying the 5,005 distinguishable six-card subsets, (2) playing through the 180 possible hands for each of these subsets, (3) recording the player and banker scores and winning

69 See, for example, Peter Griffin, The Theory of Blackjack.

outcome (player, banker, or tie) for each hand, and (4) weighting each of the (5005 x 180) outcomes by the appropriate number of repetitions (corresponding to non-distinguishable six-card subsets).

The analysis described above shows that the player wins 44.625% of the hands, the banker wins 45.860% of the hands, and 9.516% end up tied. A complete breakdown by banker total is given in the following table.

Baccarat Outcomes by Banker Total

Banker Total	*Banker Win*	*Player Win*	*Tie*	*Total*
0	0.0000000	0.0829776	0.0057978	0.0887754
1	0.0048598	0.0603204	0.0041012	0.0692814
2	0.0089392	0.0561118	0.0040026	0.0690536
3	0.0145902	0.0537338	0.0044515	0.0727755
4	0.0326824	0.0534859	0.0072612	0.0934295
5	0.0433571	0.0494077	0.0079394	0.1007042
6	0.0538637	0.0479718	0.0192402	0.1210757
7	0.0768805	0.0312008	0.0203500	0.1284313
8	0.1060169	0.0110367	0.0109794	0.1280330
9	0.1174077	0.0000000	0.0110327	0.1284404
Total	0.4585974	0.4462466	0.0951560	1.0000000

The house advantage for the three types of bets can be computed as follows:

$EV_{Player} = (+1)(.4462466) + (-1)(.4585974) + (0)(.0951560) = -0.01235,$

$EV_{Banker} = (+0.95)(.4585974) + (-1)(.4462466) + (0)(.0951560) = -0.01058,$

$EV_{Tie} = (+8)(.0951560) + (-1)(.4585974) + (-1)(.4462466) = -0.14360.$

From these calculations, the casino advantages are about 1.24% on the player bet, 1.06% on the banker bet,[70] and 14.36% on the tie wager.

IGNORING TIES

As is the case with the don't pass and don't come wagers in craps, the player and banker advantages can be expressed either including or excluding ties. The 1.24% and 1.06% advantages reported above include ties. If ties are ignored, the player will win 49.3175% of the hands and the banker will win 50.6825%. The expected values per resolved bet for these two bets are:

$$EV_{Player} = (+1)(.493175) + (-1)(.506825) = -0.01364965,$$

and

70 Some casinos will charge only a 4% commission, lowering the edge on the banker bet to 0.60%. If a 3% commission is charged, the house advantage is 0.14%. A 2% commission would make the banker bet unfavorable to the house (the bettor would have the advantage).

$EV_{Banker} = (+0.95)(.506825) + (-1)(.493175) = -0.0116916.$

> At gambling, the deadly sin is to mistake bad play for bad luck.
>
> Ian Fleming
> *Casino Royale* (1953)

House advantages ignoring ties, then, are 1.36% and 1.17% for player and banker, respectively.[71] These advantages per resolved bet also could be obtained by dividing the house advantages per hand by the proportion of untied hands: $1.235\% / 0.9048 = 1.36\%$ and $1.058\% / 0.9048 = 1.17\%$.

Regardless of whether house advantage is expressed per hand or per resolved bet, expected casino earnings are the same. For example, if a player bets $25 a hand on the banker for 10,000 hands, the 1.06% house advantage per hand dealt results in a casino expected win of about $2,645 (.010579 x 10,000 hands dealt x $25 per hand). Applying the 1.17% house advantage per resolved bet to the untied hands yields – aside from rounding error – the same expected win (.011692 x 9,048 resolved bets x $25 per bet = $2,645). Obviously a similar equivalency exists for the player bet.

Volatility

Mathematical foundations of game volatility and analysis of fluctuations were discussed in Chapter 1, where basic examples using roulette and baccarat were presented. These issues are particularly vital for baccarat since it is a game favored by high rollers and represents one of the few situations where the casino can be put in a high-risk situation. This section elaborates further on fluctuations in casino win in baccarat. If needed, the interested reader may wish to review the volatility related sections of Chapter 1 before proceeding.

Standard Deviation

The following two tables show the calculations for the standard deviations associated with 1-unit wagers on banker and player.

Standard Deviation – $1 Banker Bet in Baccarat					
X	P(X)	X∗P(X)	X−EV	$(X-EV)^2$	$(X-EV)^2*P(X)$
+1	0.4462466	0.4462466	0.989421	0.978954	0.436855
−0.95	0.4585974	−0.4356675	−0.960579	0.922712	0.423153
0	0.0951560	0	−0.010579	0.000112	0.000011
	EV = +0.0105791			*VAR* = 0.860019	
				SD = 0.927372	

71 These advantages ignoring ties are based on the standard 5% commission on winning banker hands. If the commission is 4%, the banker advantage per resolved bet is 0.66%; with 3% commission it is 0.16%. A 2% commission gives the banker bet a negative house advantage (i.e., the bettor has the advantage).

Standard Deviation – \$1 Player Bet in Baccarat					
X	P(X)	X∗P(X)	X−EV	$(X-EV)^2$	$(X-EV)^2*P(X)$
+1	0.4585974	0.4585974	0.987649	0.975451	0.447339
−1	0.4462466	−0.4462466	−1.012351	1.024854	0.457338
0	0.0951560	0	−0.012351	0.000153	0.000015
EV = +0.0123508				*VAR* = 0.904691	
				SD = 0.951153	

Volatility is slightly greater for a bettor who makes all player bets (SD ≈ 0.95) compared to always betting the banker (SD ≈ 0.93).[72]

CONFIDENCE LIMITS FOR TOTAL HOUSE WIN

For the sake of simplicity, the analysis that follows assumes an overall game standard deviation of 0.94 per unit and a casino advantage of 1.15%, approximately the values for a large series of bets equally divided between player and banker.

Actual total casino win over the course of many hands will deviate from the expected win by an amount that depends on the per unit standard deviation, the amount wagered per hand, and the number of hands. Mathematically speaking, in repeated series of *n* equal wagers, the distribution of outcomes of total dollars won will be normally distributed (bell-shaped) around the expected win with standard deviation equal to $SD_{win} = unit\ wager \times \sqrt{n} \times SD_{per\ unit}$.[73] Confidence limits for the actual win can be obtained from the following:

$$Expected\ Win \pm (z \times SD_{win}) ,$$

where z is the appropriate standard normal value depending on the confidence level. For the usual 95% confidence, this is z = 1.96. Values of z corresponding to other confidence levels can found in a table of standard normal probabilities.[74]

In a series of 1,000 hands at \$500 per hand, for example, the expected casino win is \$5,750 (assuming a 1.15% house advantage), with an associated standard deviation of $0.94 \times \$500 \times \sqrt{1{,}000} \approx \$14{,}863$. The 95% confidence limits for the casino's win are given by \$5,750 ± 1.96(14,863), or −\$23,381 to +\$34,881.

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72 The standard deviations derived in the tables above apply to all hands, including ties. When computed on a per resolved bet basis, the standard deviations for player and banker wagers are 1.000 and 0.975, respectively.

73 Since the per unit standard deviation in baccarat is approximately equal to 1 (the average of banker and player standard deviations is about 0.94 if ties are included and about 0.99 if ties are ignored), the standard deviation for the number of units won is approximately equal to the square root of n.

74 See Table 1, Appendix.

Repeating this calculation for different numbers of hands leads to the following table:

95% Confidence Limits for Actual Win – $500 per hand					
	Number of Hands				
	1,000	*5,000*	*10,000*	*50,000*	*100,000*
EV_{win}	$5,750	$28,750	$57,500	$287,500	$575,000
SD_{win}	$14,863	$33,234	$47,000	$105,095	$148,627
95% Lower Limit	−$23,381	−$36,389	−$34,620	$81,513	$283,691
95% Upper Limit	$34,881	$93,889	$149,620	$493,487	$866,309

A chart displaying these confidence limits as a function of the number of hands played highlights how the volatility increases as the number of hands increases. Figure 5 shows the 95% confidence bands for the casino win for a series of $500 per hand wagers; Figure 6 shows these bands for $50,000 per hand bets.

FIGURE 5
BACCARAT - $500 per Hand
95% Confidence Bands for Total House Win

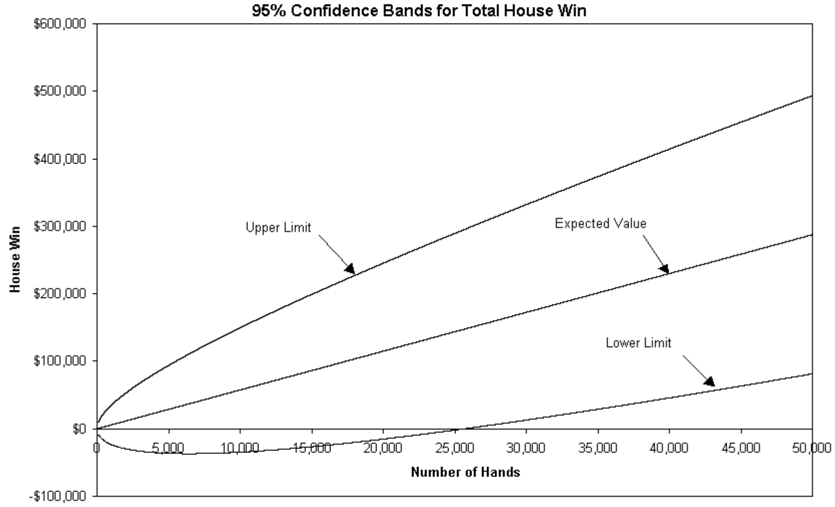
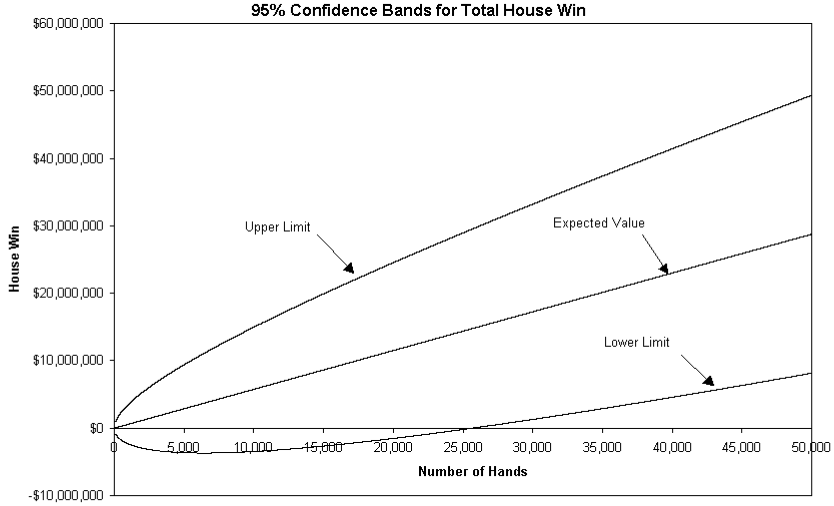
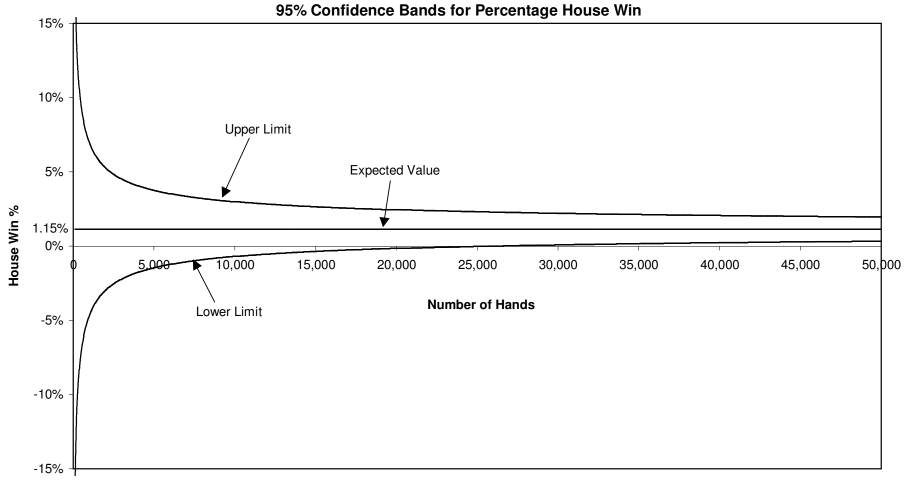


FIGURE 6
BACCARAT - $50,000 per Hand
95% Confidence Bands for Total House Win



As the number of hands played increases, the standard deviation associated with the expected win also increases (by a factor of $\sqrt{n}$), which results in a wider interval of possible win amounts in a 95% confidence band. This means that, in absolute dollar terms, the greater the number of hands played, the larger the potential fluctuations in casino win around the expected casino win. As discussed previously however, in percentage terms the fluctuations decrease as the number of hands increases. Specifically, the difference between the actual win percentage and theoretical win percentage will tend to get smaller as the number of hands increases. Figure 7 illustrates this effect.

FIGURE 7
BACCARAT - $500 per Hand
95% Confidence Bands for Percentage House Win



Note that in all of the above charts, the lower limit crosses the horizontal axis at about 25,000 hands. After this many hands, the house win is almost assuredly positive and there is very little risk that the casino will lose money. In a sense, the 25,000-hand mark serves as a lower limit "breakeven" value for a certain confidence level – there is only a 2½% chance that the casino will not at least break even after 25,000 hands played. By virtue of the law of large numbers, of course, there will always be such a breakeven value in a game with a negative player expectation. It is possible to determine this "breakeven" lower limit using the technique described in the following section, which explains how to calculate the number of hands for which the casino win percentage will be within any arbitrary amount of the theoretical win percentage with a certain specified probability.

DETERMINING NUMBER OF HANDS FOR DESIRED WIN PERCENTAGE ERROR

High rollers betting huge amounts of money per hand in baccarat represent a high-risk situation for casinos. A short-term lucky streak for the player could mean heavy losses for the casino. On the other hand, if the high roller plays long enough, the law of "averages" – or more appropriately stated, the law of large numbers – should minimize this risk. The following formula shows how to determine the number of hands necessary so that the probability is a certain specified level that the casino win percentage will be within a designated tolerable error of the theoretical win percentage.

NUMBER OF HANDS TO ACHIEVE SPECIFIED TOLERABLE ERROR*

The number of hands required so that the actual win percentage is within a designated tolerable error E of the theoretical win percentage with specified desired probability is:

$$n = \frac{z^2 (SD)^2}{E^2}$$

where z = the standard normal z value corresponding to the desired confidence level,

SD = wager standard deviation per unit,

and E = tolerable error for actual vs. theoretical win percent.

** assuming equal bets*

To illustrate the use of this formula, suppose the objective is to determine how many hands are necessary so that the probability is 95% that the casino win percentage will be within 0.15% of the theoretical win percent (i.e., between 1.00%

and 1.30%, assuming a 1.15% house advantage). Using a wager standard deviation of 0.94, the required number of hands is:

$$n = \frac{(1.96)^2 (0.94)^2}{(0.0015)^2} = 1{,}508{,}639 \, .$$

This means that the casino can be 95% confident that their actual win will be between 1.00% and 1.30% after about 1.5 million hands, (assuming a 1.15% theoretical win and equal wagers).[75] A similar calculation using z = 2.33 shows that about 2.1 million hands are required to be 98% confident the casino win percentage will be within 0.15% of theoretical win. The following table presents other values for number of required hands (assuming equal wagers) for selected confidence levels and desired tolerable error.

Hands Required to Achieve Desired Tolerable Error*		
Confidence Level	*Desired Tolerable Error*	*Number of Hands Required*
95%	.20%	848,609
95%	.15%	1,508,639
95%	.10%	3,394,438
95%	.08%	5,303,809
95%	.05%	13,577,751
95%	.04%	21,215,236
95%	.03%	37,715,975
95%	.02%	84,860,944
98%	.20%	1,199,244
98%	.15%	2,131,989
98%	.10%	4,796,976
98%	.08%	7,495,275
98%	.05%	19,187,904
98%	.04%	29,981,100
98%	.03%	53,299,734
98%	.02%	119,924,401
**Assumes baccarat wager standard deviation equal to 0.94 (including ties).*		

As noted above, this approach can be used to compute the number of hands for which the casino can be confident, with whatever specified probability, that the house win will not be negative, i.e., that the casino will not lose money. For example,

75 These calculations are in terms of house advantage including ties. If ties are excluded, volatility associated with the actual win percentage increases slightly (the wager standard deviation is about 0.99) and so for a given confidence level, the number of hands required so that the actual win percent is within the same tolerable error of the theoretical win percent – about 1.265% average ignoring ties – increases.

suppose the objective is to determine how many hands are needed so the probability is 97½% that the casino win will be positive. This required number of hands can be calculated with the same formula using a tolerable error of .0115 (assuming a house advantage equal to 1.15%) and z = 1.96 (since 1.96 corresponds to a "middle" area of 95%, the area to the right of the lower limit will be 97½%):

$$n = \frac{(1.96)^2 (0.94)^2}{(0.0115)^2} = 25{,}667 \text{ .}$$

This is the value referred to in the discussion on page 108 – about 25,000 hands. There is a 97½% probability that the casino win will be positive after 25,667 hands. Of course, another way to think about this is that there is only a 2½% chance the casino will have lost money (net) after 25,667 hands. The following table shows this "breakeven" number of hands for selected confidence levels.

Number of Hands to Achieve Specified Probability of Positive House Win*

Desired Probability	*Number of Hands Required*
80.0%	4,740
85.0%	7,180
90.0%	10,990
95.0%	18,080
97.5%	25,670
98.0%	28,190
99.0%	36,150
99.5%	44,340

**Assumes equal bets, house advantage of 1.15% and wager standard deviation of 0.94 (including ties). Values rounded.*

Final Notes

Previously discussed volatility issues are particularly important in baccarat because the bet sizes are frequently large, but the analyses presented here can also be applied to other games and/or wagers. In addition, although the methods above assume equal wagers, appropriate modifications can be made to accommodate variable betting.

Casino War

> ***House Advantage:***
> 2.88% (per hand)
> 2.68% (per unit wagered)
> 18.65% (bet on tie)

Casino War is a simple, fast paced game similar to the game many people play as kids. After the player places his bet, one card is dealt to the dealer and one to the player. If the player's card outranks the dealer's card, the player wins even money. If the dealer's card outranks the player's card, the player loses his money. If the two cards are equal in rank, the player has two options. He may surrender and forfeit half the original wager or he may "go to war." If he chooses to go to war, the player must put up an additional wager equal to the amount of the original bet. The dealer then burns three cards and deals a second card to himself and a second card to the player. If the dealer's second card is better than the player's second card, the player loses both the original and the second wager. If the player's second card is equal to or better than the dealer's second card, the player wins even money on the original bet only. The player may also make a side wager that the first two cards will be tied. If won, this tie bet pays 10 to 1.

Mathematics

The following table shows the outcomes for each possible dealer first card, based on six decks in play (typical for the casino version).

Casino War – Outcomes of First Two Cards			
Dealer Card	*# Ways to Win*	*# Ways to Lose*	*# Ways to Tie*
2	288	0	23
3	264	24	23
4	240	48	23
5	216	72	23
6	192	96	23
7	168	120	23
8	144	144	23
9	120	168	23
10	96	192	23
J	72	216	23
Q	48	240	23
K	24	264	23
A	0	288	23
	1872	1872	299

From the player's perspective, there is not much strategy involved in this game. The only decision to be made (aside from whether to play the game at all) is whether to go to war or surrender when there is a tie. Assuming the player wishes to maximize his expectation, the answer is simple – go to war. If the player always goes to war when there is a tie, the casino edge on the main bet is 2.88% per hand, or 2.68% per unit wagered since the average bet size under this strategy is 1.074 units. A player who always surrenders when there is a tie does worse – the casino edge with this strategy is 3.70%. As is the case with most tie bets, optimal strategy would dictate that players avoid the tie bet, which carries a whopping 18.65% house advantage. The relevant mathematics behind these numbers is discussed below.

THE MAIN BET

Using the figures from the table above, the probabilities of a player win, a player loss and a tie on the first two cards are computed as follows:

$$P(Win) = \sum_{x=2}^{Ace} P(Win\ given\ dealer's\ card\ is\ x) \cdot P(dealer's\ card\ is\ x)$$

$$= (288/311)(24/312) + (264/311)(24/312) + \ldots + (0/311)(24/312)$$

$$= 1872/4043 = 0.463023,$$

$$P(Lose) = \sum_{x=2}^{Ace} P(Lose\ given\ dealer's\ card\ is\ x) \cdot P(dealer's\ card\ is\ x)$$

$$= (0/311)(24/312) + (24/311)(24/312) + \ldots + (288/311)(24/312)$$

$$= 1872/4043 = 0.463023,$$

and

$$P(Tie) = (24/312)(23/311)(13)$$

$$= 299/4043 = 0.073955.$$

To find the expected value for the main wager under the assumption the player goes to war when there is a tie, it is necessary to find the probabilities of winning and losing a war. Careful thought (or brute force enumeration) would verify that these probabilities do not depend on the value of the first two tied cards. That is, the probability that the player will win (or lose) a war is the same regardless of whether the player's and dealer's tied cards are 2s, Jacks, or any other rank. To illustrate, consider the possible outcomes of a war when the player and dealer tie with 2s. These outcomes, broken down by the dealer's second card, are shown in the following table (assuming six decks).

Outcomes of War when Dealer and Player Tie with 2s					
Dealer's 2^{nd} Card	*# of this 2^{nd} Card*	*# of Winning Cards Left for Player*	*# of Losing Cards Left for Player*	*Total Winning Hands*	*Total Losing Hands*
2	22	309	0	6798	0
3	24	287	22	6888	528
4	24	263	46	6312	1104
5	24	239	70	5736	1680
6	24	215	94	5160	2256
7	24	191	118	4584	2832
8	24	167	142	4008	3408
9	24	143	166	3432	3984
10	24	119	190	2856	4560
J	24	95	214	2280	5136
Q	24	71	238	1704	5712
K	24	47	262	1128	6288
A	24	23	286	552	6864
	310			51,438	44,352

This table shows that when the dealer and player are both dealt first cards of 2 and the player goes to war, of the $310 \times 309 = 95{,}790$ possible second card combinations, 51,438 (or 0.536987) of these will be player winners and 44,352 (or 0.463013) will be player losers.

Since these figures hold regardless of the value of the first two tied cards, and since the first two tied cards are equally likely to be any of the 13 ranks, the fraction of all hands that will result in a tie and the player winning a war is $0.536987 \times 0.073955 = 0.039713$. Similarly, the fraction of all hands that result in a tie and the player losing a war is $0.463013 \times 0.073955 = 0.034242$.

The following table summarizes the outcomes and probabilities for the main bet.

Casino War – Main Bet			
Outcome	*Payoff*	*Probability*	*Expectation*
Win	+1	0.463023	0.463023
Lose	–1	0.463023	–0.463023
Tie and Win War	+1	0.039713	0.039713
Tie and Lose War	–2	0.034242	–0.068484
			E.V. = –0.028771

The expected value in the above table represents the player's expectation under the strategy of always going to war when there is a tie and is obtained using the usual calculation:

$$EV = (+1)(0.463023) + (-1)(0.463023) + (+1)(0.039713) + (-2)(0.034242) \approx -0.0288.$$

This is a 2.88% house advantage per hand. Since the probability of a tie on the initial two cards is 0.073955, the average bet size (assuming the player will always go to war) is 1.073955 units and the house advantage on per unit wagered basis is 2.8771%/1.073955 = 2.68%.

The house advantage against a player who always surrenders is 3.70% and is obtained from the following expected value calculation:

$$EV = (+1)(0.463023) + (-1)(0.463023) + (-0.5)(0.073955) \approx -0.0370.$$

In terms of the casino advantages for going to war versus surrendering when there is a tie, optimal strategy is to always go to war.

THE TIE BET

Using probabilities derived above, the expected value for the tie bet is given by:

$$EV = (+10)(0.073955) + (-1)(0.926045) \approx -0.1865.$$

Thus, the house advantage for the tie bet is 18.65%.

TECHNICAL NOTE ON THE PROBABILITY OF WINNING OR LOSING A WAR

One approach to calculating the probabilities of winning (0.039713) and losing (0.034242) a war is to condition on the value of the tied cards, then on the value of the dealer's second card (refer to the above outcomes table for war when the tied cards are 2s is laid out this way).

The formal probability analysis for this approach is enabled with the following conditional probability formulas:[76]

$$P(Win\ War) = \sum_{x=2}^{Ace}\sum_{y=2}^{Ace} P(Win \mid T_x \ and \ D_y) \cdot P(D_y \mid T_x) \cdot P(T_x)$$

and

$$P(Lose\ War) = \sum_{x=2}^{Ace}\sum_{y=2}^{Ace} P(Lose \mid T_x \ and \ D_y) \cdot P(D_y \mid T_x) \cdot P(T_x)\ ,$$

where T_x represents the event that the initial tied cards are x's and D_y is the event that the dealer's second card is a y.

In the above formulas, the inner summations on y yield the probabilities of winning and losing the war given the value of the tied cards:

$$\sum_{y=2}^{Ace} P(Win\ War \mid T_x \ and \ D_y) \cdot P(D_y \mid T_x) = P(Win\ War \mid T_x) = 0.536987\ ,$$

and

76 Recall P(A|B) is the probability of A given B.

$$\sum_{y=2}^{Ace} P(Lose\ War \mid T_x\ and\ D_y) \cdot P(D_y \mid T_x) = P(Lose\ War \mid T_x) = 0.463013.$$

With this in mind, the probability of winning a war becomes:

$$P(Win\ War) = \sum_{x=2}^{Ace} P(Win\ War \mid T_x) \cdot P(T_x)$$

$$= \sum_{x=2}^{Ace} (0.536987) \times [(24/312)(23/311)]$$

$$= 13 \times (0.536987) \times (0.005689) = 0.039713,$$

and

$$P(Lose\ War) = \sum_{x=2}^{Ace} P(Lose\ War \mid T_x) \cdot P(T_x)$$

$$= \sum_{x=2}^{Ace} (0.463013) \times [(24/312)(23/311)]$$

$$= 13 \times (0.463013) \times (0.005689) = 0.034242.$$

Cards are war, in disguise of a sport.

Charles Lamb
Essays of Elia (1832)

SIC BO

House Advantage: 2.78% (small/low, big/high)   7.87% - 18.98% (others)

Sic Bo (also known as Chinese Chuck-a-luck) is a game played with three dice. Players can place a variety of wagers on the combinations that might occur. The dealer spins a cage containing the dice, and wagers are paid according to the payoffs given in the following table, which summarizes the various wagers available and the casino advantages for each.

SIC BO – Wagers and House Advantages		
Wager	***Payoff***	***H.A.***
SINGLE-DIE WAGER		
One-of-a-kind	1 to 1 on one die	7.87%
	2 to 1 on two dice	
	3 to 1 on three dice	
TWO-DICE WAGERS		
Two of a kind	10 to 1	18.52%
Two-dice combination	5 to 1	16.67%
THREE-DICE WAGERS		
Three of a kind	180 to 1	16.20%
Any three of a kind	30 to 1	13.89%
Total = 4 or 17	60 to 1	15.28%
Total = 5 or 16	30 to 1	13.89%
Total = 6 or 15	17 to 1	16.67%
Total = 7 or 14	12 to 1	9.72%
Total = 8 or 13	8 to 1	12.50%
Total = 9 or 12	6 to 1	18.98%
Total = 10 or 11	6 to 1	12.50%
Small/Low (triples lose)	1 to 1	2.78%
Big/High (triples lose)	1 to 1	2.78%

MATHEMATICS

Casino advantages in the table above are derived using straightforward probability calculations. As an example, consider the one-of-a-kind wager. This bet is made on a single number and wins if the chosen number appears on one or more of the dice. Payoffs are 3 to 1 if the number appears on all three dice, 2 to 1 if it appears on two of the three dice, and 1 to 1 if it appears on only one of the dice. Suppose a wager is made on the number 4. The relevant probabilities are:

P(4 appears on all three dice) = (1/6)(1/6)(1/6) = 1/216,

P(4 appears on two dice) = (1/6)(1/6)(5/6)x3 = 15/216,

and P(4 appears on one die) = (1/6)(5/6)(5/6)x3 = 75/216.

Using these probabilities and payoffs, then, the usual expected value calculation for this wager is:

$$EV = (+3)(1/216) + (+2)(15/216) + (+1)(75/216) + (-1)(125/216) = -0.0787.$$

This calculation applies regardless of the chosen single number, and so the casino advantage for the one of a kind wager is 7.87%.

The three-of-a-kind wager has the largest payoff, 180 to 1, but with true odds of 215 to 1, this bet has a 16.20% casino advantage:

$$EV = (+180)(1/216) + (-1)(215/216) = -35/216 = -0.1620.$$

Bets with the largest casino advantage are wagers on a total of 9 or 12, each with house advantage of 18.98%:

$$EV = (+6)(25/216) + (-1)(191/216) = -41/216 = -0.1898.$$

Bets with the lowest house advantage are the small (or low), which covers totals of 4 through 10, and the big (or high), which covers totals of 11 through 17, except that these bets lose if triples are rolled. The casino advantage on either of these wagers is 2.78%. To see this, first note that there are 107 ways of rolling a total of 4 through 10 (or 11 through 17). Now subtract the two triples in this set – triple 2 and triple 3 (or triple 4 and triple 5) – to obtain 105 ways to win this wager. The expected value for one of these wagers is then:

$$EV = (+1)(105/216) + (-1)(111/216) = -6/216 = -0.0278.$$

House advantages for other wagers in Sic Bo can be calculated in a similar way.

THE BIG SIX WHEEL (WHEEL OF FORTUNE)

House Advantage:
19.8% (average)
Typical Hold:
44% (Nevada)

The big six wheel, or wheel of fortune, is a carnival type game where players wager on the outcome of the spin of a large wheel by placing bets on one of the symbols on the wheel. The typical wheel has 54 positions featuring different denominations of currency and two special symbols, which are often casino logos or jokers (or one of each). The following table shows the number of positions, payoffs, odds, and house advantage for such a wheel.

BIG SIX WHEEL				
Symbol	*# of Positions*	*Payoff*	*True Odds*	*H.A.*
Casino Logo	1	40 to 1	53 to 1	24.07%
Joker	1	40 to 1	53 to 1	24.07%
$20 bill	2	20 to 1	26 to 1	22.22%
$10 bill	4	10 to 1	25 to 2	18.52%
$5 bill	7	5 to 1	47 to 7	22.22%
$2 bill	15	2 to 1	13 to 5	16.67%
$1 bill	24	1 to 1	5 to 4	11.11%
TOTAL	54			19.84%

A bet on one of the special symbols (casino logo or joker) does not win if the other one comes up.

The mathematical analysis for a big six wheel is straightforward. To illustrate the house advantage calculation, consider a $1 bet on one of the special symbols:

$$EV = (+40)(1/54) + (-1)(53/54) = (-13/54) = -0.2407.$$

In the long run a player can expect to lose about $0.2407 of every $1 wagered on a special symbol, which equates to a house advantage of 24.1%. The advantages for other wagers are calculated analogously.

Assuming an equal amount of money is wagered on each of the seven symbols, the overall house advantage is 19.84%. If the amount wagered on each symbol is proportional to the number of positions the symbol occupies on the wheel, the overall casino advantage is 15.53%.

POOLED GAMES

Pooled games of pure chance are rare in the casino environment, with notable exceptions for bingo, and progressive jackpots on table games and slot machines. Outside the casino, they are the basis of both lotteries and the numbers game, and pari-mutuel wagering on horse and dog racing and jai alai. In both these examples, players pay a sum for the opportunity to win a portion of the pooled proceeds from all players. In pure pooled games with no minimum payouts, the operator is guaranteed a return. For example, a lottery operator may take 40% of all ticket sales and return 60% to be distributed among winning ticket holders.

This is different from most forms of keno. For example, if only one person buys a keno ticket for a dollar and that person correctly picks all 9 numbers on his ticket, then he is entitled to a payout of about $50,000. The casino that banks this game must pay the player this amount. In a pure pooled lottery, if only one player buys a ticket for one dollar and that ticket wins, he would be entitled only to 60% of the pool, or 60 cents from his one-dollar bet.

In practice, however, lotteries and other pooled games are not pure pooled, or pari-mutuel games because they have smaller house banked bets or provide a guaranteed minimum payout.

PROGRESSIVE SLOTS

Casinos can increase the theoretical paybacks on slots by either increasing the number of winning combinations or increasing the amounts of paybacks on winning combinations. A casino can do the former on many slot machines by modifying the computer chip that operates the game so as to produce more or higher paying winning combinations. A casino can do the latter by increasing the paybacks on combinations. For example, the payout on lining up four bars may be increased from 100 coins to 200 coins. In either case, over time, the casino decreases the theoretical win, pays out more in winnings and retains less revenue as a percentage of handle.

Slot machines with progressive meters do not have a set theoretical win in the same sense as other slot machines. On non-progressive slot machines, the theoretical payout to players is the same on every play because the opportunity to win, the winning combinations, and the amounts of the payouts on winning combinations remain constant. On progressive machines, the progressive meter increases with each pull of the handle. For example, on a dollar progressive machine, a nickel of every dollar played may be added to the progressive jackpot. Therefore, on any pull of the handle, the theoretical pay back will increase as the progressive meter increases.

If no player wins the progressive jackpot, at some point, the increasing size of the jackpot creates a theoretical payback that will result in a statistical advantage to the players. From this point to the time a player wins the jackpot, the total amount of the paybacks to the players including the progressive jackpot should exceed the total amount of money played by the players. This is shown in the following chart.

Payout Distributions for a Progressive Slot Machine on Three Different Dates Assuming No Player Wins The Progressive							
On 1st Day			On 30th day			On 60th day	
Payout	*Hits*		*Payout*	*Hits*		*Payout*	*Hits*
0	566,966		0	566,966		0	566,966
2	296,827		2	296,827		2	296,827
5	20,624		5	20,624		5	20,624
10	50		10	50		10	50
25	50		25	50		25	50
40	50		40	50		40	50
50	40		50	40		50	40
100	30		100	30		100	30
150	25		150	25		150	25
180	20		180	20		180	20
200	15		200	15		200	15
300	12		300	12		300	12
400	10		400	10		400	10
500	8		500	8		500	8
1,000	5		1,000	5		1,000	5
1,200	3		1,200	3		1,200	3
Progressive 100,000	1		Progressive 150,000	1		Progressive 200,000	1
Payback %	94.5%		Payback %	100.2%		Payback %	105.8%

Slot Teams

This slight statistical advantage is what attracts the slot teams. They find progressive jackpots that have arisen to a level that favors the players. Only certain progressive slot machines will meet the slot team's criteria. First, the number of slot machines that are linked to the progressive jackpot must be manageable. The team needs to monopolize all the slot machines to avoid the risk that a non-member will win the jackpot. Second, the statistical frequency of hitting the jackpot must be consistent with the slot team's bankroll. No slot team has an unlimited bankroll. Based on probabilities, the team needs to have enough cash on hand to play all the machines on the carousel until one hits the progressive jackpot.[77] For example, the previous chart shows a progressive slot machine that has a cycle of 884,736 (and a probability of a little better than one in one million of hitting the progressive jackpot). If the

77 The Poisson probability distribution can be used to determine the probability of hitting the jackpot over the course of a specified number of plays. See the section on Slot Machines earlier in this chapter.

probability of hitting the progressive jackpot is one in ten million, the slot team would need an astronomical bankroll to have reasonable assurances that it would hit the jackpot before running out of money. Therefore, slot teams avoid the "mega" jackpot progressive that have very infrequent hits or winners. To minimize their potential exposure, they concentrate on progressive carousels that feature lower jackpots with a higher frequency of payouts.

Slot teams must know when progressive jackpots are at a level that the theoretical payback favors the player. The team leader determines this using spotters who visit various casinos to monitor the progressive jackpots. When a jackpot hits a predetermined amount, the organizer calls the team members to descend on the carousel.

Then they move in and take over every machine in the carousel that is linked to the progressive meter. Most teams are very aggressive. If a non-team member is playing a machine, the team is responsible for moving that person off the machine. Team members can accomplish this by being rude to the player, situating a team member with poor hygiene next to the non-member, or other method. Once they monopolize the carousel, they continue to play until a team member wins the progressive jackpot (or the team runs out of money.)

Usually one person organizes and finances the team. This person will recruit others to play the machines for an hourly or set fee. The organizer that provides all the money to play the machines takes all the risk and receives all the reward. The team will play until it wins the jackpot. They then leave, only to return when other people feed the progressive jackpot to acceptable levels.

A slot team only benefits the team's organizers. Casinos do not like slot teams because they win large jackpots that would otherwise go to their regular players. No favorable consequences result from a slot team winning the jackpot. They do not go around telling their friends that they won a large jackpot at a particular casino. Regular players and tourists do not benefit from slot teams. They fuel the progressive jackpots, but are left with no chance to win the big jackpots because they are intimidated away from the machines by the slot teams.

An easy solution would be for the casinos to ban slot teams, just as they can ban card counters. This is not possible in some states like Nevada where regulations imply the amount of the progressive jackpot are public funds that are being held in trust until a member of the public wins the jackpot.[78] While a court would not likely find this regulation to create a right to play slot machines in a casino, many Nevada casinos do not want to incur the costs of a lawsuit to defend their rights to exclude slot teams. So they go to plans B and C.

Plan B is a simple strategy. Here the casino can post signs that it retains the discretion to limit players to playing one machine. Thus, when a slot team tries to monopolize a carousel, they will need more team members and incur greater expense in doing so.

Plan C is to reduce the increment with which each play of the slot machine will increase the progressive jackpot. If, for example, five cents of every dollar normally goes to fund the progressive, this can be reduced to one cent. This substantially

78 A questionable proposition based on either law or mathematics.

increases the risk to the slot team organizer because he will not receive as much of his money back in the form of the jackpot. Therefore, the jackpot has to be much higher before he will cannibalize the carousel. To implement plan C, however, the casino cannot misrepresent what portion of each pull goes toward the progressive jackpot.

BINGO

Bingo is derived from the European game of lotto. At first, it was primarily a carnival game, but later was used by movie theaters during intermissions between double features.

Bingo usually is played with printed cards that are arranged in a five-square by five-square arrangement. The first column is the “B” column, the second is the “I” column, the third is the “N” column, the fourth is the “G” column and the fifth is the “O” column. The “B” column on the card contains five randomly chosen numbers typically from 1-15.[79] The “I” column on the card contains five numbers from 16-30. The “N” column on the card contains five numbers from 31-45. The “G” column on the card contains five numbers from 46-60. The “O” column on the card contains five numbers from 61-75. The center square usually is marked as a “free square.”

The objective of bingo is to achieve a particular pattern on your card before any of the other players. The most common game is to award a cash prize to the first player to obtain a straight line, either vertical, horizontal or diagonal. Other games include a cover-all, where the objective is to be the first person to cover all of the numbers on the card. Many other variations of bingo exist including "Letter X", "Six Pack", "Coverall" and "Indian Style Papoose".

A bingo game starts when the first numbered object or ball is selected at random and called. The ball or object draw can occur in many different ways. The first keno games were played by drawing tiles, numbered 1-75. More commonly, the operator draws keno balls by hand or using a machine that blows the balls into a tube. Random number generators have replaced the ball draw in many places. Like the columns, numbers 1-15 are called using the “B” for reference. Thus, instead of calling number “6,” the operator will call “B6.” This is the same for the numbers in the other columns. In the traditional bingo game, play continues by the operator drawing one ball after another until one player completes a "bingo" by covering five numbers in a vertical, horizontal or diagonal row on one of their cards. That player would then win the “pot,” i.e. all the money bet on the game less the operator’s commission.

> I shall never believe that God plays dice with the world.
>
> Albert Einstein
>
> God not only plays dice. He also sometimes throws the dice where they cannot be seen.
>
> Stephen Hawking (*Nature*, Vol. 257)

79 Some bingo games have used more than 75 numbers, but these are rare.

GAMES WITH A SKILL COMPONENT

5

Casino games in which the player can have an affect on the outcome using strategy decisions are treated in this chapter. Of these games with a skill component, the most popular by far is blackjack. It is the only house banked game that can be "beaten" without cheating by the player, although to do so requires counting cards. Other banked games with a skill component covered in this chapter include four poker type derivatives (Let It Ride, Caribbean Stud, Pai Gow Poker, and Three Card Poker) and Red Dog.

Casino cardroom poker, where players are competing against one another and the house takes a percentage (the "rake") of each pot, is a non-banked game in which skill plays a major role.

BANKED GAMES

Conceptually, a casino risks its money against the player's money in a house-banked game. But, two things fundamentally work against the player in house-banked games with a skill component – the game's built-in house advantage and the player's inability to always play the game with optimum skill. Except for card-counters, the first guarantees that, over time, the casino will end up a winner. The latter allows the casino to obtain results in excess of their house advantage.

BLACKJACK

House Advantage:
0.5% (basic strategy, 6 decks)
2.0% (average player)
Typical Hold:
13% (Nevada)

Blackjack, or "21," is the most popular and largest revenue generating casino table game. In Nevada in 2000, blackjack accounted for casino winnings of approximately $1.2 billion – almost one-half of all table games and second only to slot machines in overall casino revenue. There are seven times as many blackjack tables (about 3,700 in Nevada) as any other single table game and twice as many as all other table games combined. Blackjack's popularity stems from its simple play, that the player to some extent can guide his own destiny, the (correct) perception that the game can be beaten,[80] and the camaraderie among players in the friendly competition against the casino.

RULES OF PLAY

The objective of blackjack is to beat the dealer by having a total higher than the dealer's without exceeding 21. The object is not, as some people think, to get as close to 21 as possible.

CARD VALUES

Cards 2 through 10 are worth their face value, face cards (Jacks, Queens, Kings) are worth 10, and Aces can be counted as either 1 or 11. Card suits are meaningless. In the discussion that follows, the term "Ten" will be used to refer to any card – face card or 10 – that is a ten-valued card in blackjack. The value of a hand is the total of the values of the individual cards making up the hand. The name of the game is derived from the best possible hand, an Ace-Ten combination on the first two cards. This combination, called a *blackjack* or *natural*,[81] totals 21 and is unbeatable. A dealer blackjack beats all player hands, even those with three or more cards totaling 21, except a player who also received a natural on the first two cards.

HARD, SOFT, PAT AND STIFF HANDS

A *hard* hand is one that either does not contain an Ace or if it does, the Ace counts as 1. A *soft* hand is one that contains an Ace counted as 11. For example, A-6 is a soft 17, A-2-3 is a soft 16, A-6-8 is a hard 15, and 9-7 is a hard 16. When a hand value exceeds 21, it is *busted*. Hands totaling 17 through 21 are called *pat* hands – generally players will stand pat with these totals. Hands with hard totals of 12 through 16 are known as *stiff* hands – these hands can be busted by drawing an additional card.

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80 Ever since Dr. Edward Thorp's Beat the Dealer was published in 1962, the world has known that correct strategy can make blackjack the lowest house advantage (zero, with favorable rules) game in the casino, and could even be beaten with card-counting.

81 They must be the first two cards. A two-card Ace-Ten combination occurring after a split is not a blackjack, merely a total of 21.

PROCEDURE

To begin the game, each player makes a bet, then the dealer deals each player a card, starting with the player on the dealer's left, and one card to himself. This is repeated so that each player and the dealer have two cards. One of the dealer's cards is face up and the other is face down. The former is known as the *upcard* and the latter as the *hole card*.

If the dealer's upcard is an Ace, he may offer the players *insurance*, an optional side bet on whether the dealer has a natural. Players may wager up to one-half their original wager. If the dealer has a natural, insurance bets are paid 2 to 1 and then the original hands are settled. If the dealer does not have a natural, insurance bets are lost and play on the main hands proceeds as usual.

> **Caveat**
>
> Probabilities and house advantage in blackjack vary with the number of decks in play and the rules in effect. Unless stated otherwise, all ensuing discussions assume a single deck game with **old Las Vegas Strip rules**: dealer must stand on soft 17, the player may double down on any first two cards only (no doubling after splitting), non-Ace pairs may be split up to four times, split Aces receive only one card, and no surrender.

If the dealer's upcard is a Ten, he may check the hole card to see if he has a natural before proceeding. If the dealer has a natural, all players' hands lose, except if a player also has a natural, in which case it's a push (tie) and that player neither wins nor loses. If the dealer does not have a natural, play then proceeds as usual.

If the dealer does not have a natural, then beginning with the player to the dealer's left, each player completes his hand by exercising one of the following options: stand, hit, double down, split, or (if offered) surrender.

A player may *stand* if satisfied with his original two-card total, or may choose to *hit* his hand by taking an additional card to try to improve his total. The player can take as many cards as he likes until his total exceeds 21 (called *busting* or *breaking*) or is satisfied with the total and stands. If the player busts, he automatically loses his wager. The *double down* option allows the player to double (or less) the original wager in favorable situations. When this option is exercised, the player receives one, and only one, additional card. The hand is settled in the usual way with payouts or losses based on the new total wager. Most casinos allow the player to double down on any first two cards, although this can vary. Restrictions on doubling down and their effect on the house advantage are discussed later in this section. If the player's first two cards are equal in value, he may elect to *split* the pair and play two independent hands each starting with one of the original two cards. Each hand is played as usual, except that split Aces receive only one card each (and if this card is a Ten, the total is 21 but it is not a natural). Rules on re-splitting and doubling down after a split vary.

Surrender is an option offered by some casinos allows the player, after seeing his first two cards and before taking any other action, to forfeit half the original wager rather than playing out the hand. When offered, surrender is almost always late surrender, which means it is only an option if the dealer does not have a natural.

Once all players have finished their hands, the dealer plays his own hand. The dealer has no options in playing his hand and must adhere to a fixed set of rules. These rules usually require the dealer to hit all totals of 16 or less and to stand on totals of 17 or more. In some casinos, notably those in downtown Las Vegas and northern Nevada, the dealer must hit a soft 17.[82]

If the dealer busts, all players remaining in the game (who did not bust or surrender) win and are paid even money regardless of their total, except that naturals are paid 3 to 2.[83] If the dealer does not bust, the dealer's hand is compared with each of the remaining player's hands to resolve their bets. If the player's total exceeds the dealer's total, the player wins – again the payoff is even money for all hands except naturals, which are paid 3 to 2. If the dealer's total is higher, the player's wager is lost. If the totals are equal, the hand is a push (tie). Once all bets are resolved, new wagers are made and the process begins again.

Dealer Probabilities

Since the dealer's rules are fixed, it is possible to compute the probabilities of the dealer making various final totals. The two tables that follow show the percentage of time the dealer will make each total and the probabilities of making these final totals given different dealer upcards. The notation "BJ" refers to a natural and "21" refers to a hand totaling 21 with three or more cards.

Dealer Final-Hand Probabilities*

17	18	19	20	21	BJ	Bust
14.58%	13.81%	13.48%	17.58%	7.36%	4.83%	28.36%

**Figures for single-deck games.*

Dealer Final Totals By Upcard*

	Dealer's Final Total						
Dealer's Upcard	*17*	*18*	*19*	*20*	*21*	*BJ*	*Bust*
2	.139	.132	.132	.124	.121	0	.353
3	.130	.131	.124	.123	.116	0	.376
4	.131	.114	.121	.116	.115	0	.403
5	.120	.124	.117	.105	.106	0	.429
6	.167	.107	.107	.101	.098	0	.421
7	.372	.139	.077	.079	.073	0	.260
8	.131	.363	.129	.068	.070	0	.239
9	.122	.104	.357	.122	.061	0	.233
Ten	.114	.113	.115	.329	.037	.078	.214
Ace	.126	.131	.130	.132	.052	.314	.117

**Figures for single-deck games.*

A look at these tables will help in understanding the proper strategy to play the game. The dealer is most likely to bust with upcards of 2-6, particularly 4-6.

82 The dealer hitting soft 17 increases the house advantage by about 0.2%.

83 In some casinos, player naturals are paid immediately if the dealer does not also have a natural.

Generally, players seeking to maximize expected gain will tend to be more aggressive with double downs and pair splits, and will tend to stand with stiff totals, hoping the dealer will bust, against these dealer upcards. Against upcards of 7-A, the player should hit stiff totals to try to improve his hand since with these upcards the dealer has a good chance of making a pat hand. The set of exact strategy rules that maximizes the player's expectation (without counting cards) is called "basic strategy."

THE BASICS OF THE HOUSE ADVANTAGE

Why is it the house has the advantage in blackjack? Simply put, it is because the player must act first. Because of this, the player will lose anytime there is a "double bust" in which both player and dealer totals exceed twenty-one. To get a rough idea of how much this is worth, consider a player who plays his hand exactly like the dealer must play. This player and the dealer will each bust about 28% of the time and so there will be a double bust about 8% of the time (.28 x .28 ≈ .08). All else being equal, this gives the casino roughly an 8% advantage. All else, however, is not equal. The 3 to 2 payoff for a player natural is worth about 2.3%. Thus, a player who mimics the dealer is giving the casino about a 5.7% edge.[84] The player can further reduce the house edge by taking advantage of the strategy options not available to the dealer. The approximate potential benefits of each of these strategy options are:[85]

- Proper hitting and standing, worth ≈ 3.5%.
- Doubling down when advantageous, worth ≈ 1.7%.
- Splitting when advantageous, worth ≈ 0.5%.

Additionally, surrender, if allowed, is worth a fraction of a percent (late surrender is worth ≈ 0.06%, early surrender is worth ≈ 0.62% but is rarely offered) and insurance can be a positive expectation move for a player who is tracking cards. Proper use of these options can, for most games offered, whittle the casino advantage down to anywhere between 0.0% (single deck, Las Vegas Strip rules) and 1.0% (six decks, dealer hits soft 17, double down on ten and eleven only), depending on the number of decks in play and the particular rules in use. A casino advantage of 0.5% is a reasonable representative figure that can be applied to expert basic strategy for most games. For the average player, the casino advantage is about 2%.

BASIC STRATEGY

Basic strategy is the player's strategy that maximizes his expectation, or average gain, playing one hand against a freshly shuffled pack, without keeping track of the cards. It is a complete set of decision rules that tells the player whether to hit, stand, double down, split, or surrender (if offered) depending only on the player's present cards and the dealer's upcard. It does not depend on other players' cards or previously played cards. Basic strategy was developed by analyzing all possible combinations of decisions and outcomes for a given set of player's cards and dealer's upcard. The decision resulting in the largest expected win (or smallest average loss) is the basic

84 Peter Griffin reports 5.5% in The Theory of Blackjack (6th ed., 1999)

85 Aggregate estimates based on various sources.

strategy decision. In the long run, a player who follows basic strategy perfectly will maximize his average gain. Basic strategy charts[86] for single and multiple deck games under typical rules (surrender in these charts refers to late surrender) are given below.

Effect of Multiple Decks

Other things being equal, a blackjack game with multiple decks will result in a larger house advantage than a single deck game. The reasons for this are:

- ♣ Naturals are less prevalent in multiple deck games (dropping from 0.0483, or once every 20.7 hands, for a single deck, to 0.0475, or once every 21.1 hands for six decks), reducing the player advantage on 3 to 2 payoffs for naturals.[87]
- ♣ Doubling down is less effective.
- ♣ Judicious standing with stiff totals is less successful.

These outcomes result because the impact of removing individual cards is less for a multiple deck than for a single deck. To see this, suppose the player doubles down with a (6,4) against the dealer's upcard of 5. The player is hoping for a Ten or Ace. With the three cards already in play, the probability of drawing a Ten or Ace in a single deck game is 20/49 = .408, since there are still 16 Tens and 4 Aces in the remaining 49 cards. For an eight-deck game, however, this same probability is only 160/413 = 0.387. In terms of the chances of receiving the desired Ten or Ace hit card on the double down, the effect of removing the three cards is more dramatic for a single deck than for the eight decks. For similar reasons, the player is less successful when standing with stiff totals. In this situation the player is hoping the dealer will bust, but with eight decks, there are more small cards available (percentage-wise) for the dealer to avoid busting than with a single deck.

Spanish 21

A relatively new game on casino floors is a variation on blackjack called Spanish 21. This game is played exactly like regular blackjack except that the ten-spot cards (i.e., the cards in rank between the nines and the Jacks, but not the face cards) have been removed. A variety of rule changes are added to compensate players for the removal of the ten-spot cards. These changes include one where a player 21 always beats a dealer 21, liberal doubling rules, a double-down rescue rule that allows a player to keep the doubled portion of a bet and forfeit the original wager, and bonuses for certain specific hands. Basic strategy for Spanish 21 is not the same as that for regular blackjack. The house edge using correct strategy is about 0.8%, but the game tends to earn much more than that for the casino since many players are not familiar with the correct strategy or use regular blackjack basic strategy.

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86 Basic strategy charts differ slightly depending on assumed rules and level of detail. For example, a composition dependent strategy – one that takes into account the specific denominations of cards in the player's hand – would indicate to hit (6,2) vs. dealer's 5 or 6, but to double down on (5,3) vs. dealer's 5 or 6 in the single-deck game. Performance differences associated with these modifications are very slight.

87 Additionally, since the effect of removing the player's blackjack cards is greater in a single deck, the chances the dealer also has a natural when the player does is slightly greater in multiple deck games.

BLACKJACK BASIC STRATEGY – SINGLE DECK										
Player's Hand	*DEALER'S UPCARD*									
	2	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*Ten*	*A*
A/A	SP	SP	SP	SP	SP	SP	SP	SP	SP	SP
10/10	–	–	–	–	–	–	–	–	–	–
9/9	SP	SP	SP	SP	SP	–	SP	SP	–	–
8/8	SP	SP	SP	SP	SP	SP	SP	SP	SP	SP
7/7	SP	SP	SP	SP	SP	SP	SP/H	H	–	H
6/6	SP	SP	SP	SP	SP	SP/H	H	H	H	H
5/5	D	D	D	D	D	D	D	D	H	H
4/4	H	H	SP/H	SP/D	SP/D	H	H	H	H	H
3/3	SP/H	SP/H	SP	SP	SP	SP	H	H	H	H
2/2	SP/H	SP	SP	SP	SP	SP	H	H	H	H
Soft 21	–	–	–	–	–	–	–	–	–	–
Soft 20	–	–	–	–	–	–	–	–	–	–
Soft 19	–	–	–	–	D/S	–	–	–	–	–
Soft 18	–	D/S	D/S	D/S	D/S	–	–	H	H	–
Soft 17	D	D	D	D	D	H	H	H	H	H
Soft 16	H	H	D	D	D	H	H	H	H	H
Soft 15	H	H	D	D	D	H	H	H	H	H
Soft 14	H	H	D	D	D	H	H	H	H	H
Soft 13	H	H	D	D	D	H	H	H	H	H
Hard 21	–	–	–	–	–	–	–	–	–	–
Hard 20	–	–	–	–	–	–	–	–	–	–
Hard 19	–	–	–	–	–	–	–	–	–	–
Hard 18	–	–	–	–	–	–	–	–	–	–
Hard 17	–	–	–	–	–	–	–	–	–	–
Hard 16	–	–	–	–	–	H	H	H	SR/H	SR/H
Hard 15	–	–	–	–	–	H	H	H	SR/H	H
Hard 14	–	–	–	–	–	H	H	H	H	H
Hard 13	–	–	–	–	–	H	H	H	H	H
Hard 12	H	H	–	–	–	H	H	H	H	H
11	D	D	D	D	D	D	D	D	D	D
10	D	D	D	D	D	D	D	D	H	H
9	D	D	D	D	D	H	H	H	H	H
8 or less	H	H	H	H	H	H	H	H	H	H

– = Stand; H = Hit.
SR/H = Surrender if allowed, otherwise hit; SR/S = Surrender if allowed, otherwise stand.
D = Double down if allowed, otherwise hit; D/S = Double down if allowed, otherwise stand.
SP = Split; SP/H = Split if double down after split allowed, otherwise hit; SP/D = Split if double down after split allowed, otherwise double down.

Blackjack Basic Strategy – Multiple Decks										
Player's Hand	*Dealer's Upcard*									
	2	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*Ten*	*A*
A/A	SP	SP	SP	SP	SP	SP	SP	SP	SP	SP
10/10	–	–	–	–	–	–	–	–	–	–
9/9	SP	SP	SP	SP	SP	–	SP	SP	–	–
8/8	SP	SP	SP	SP	SP	SP	SP	SP	SP	SP
7/7	SP	SP	SP	SP	SP	SP	H	H	H	H
6/6	SP/H	SP	SP	SP	SP	H	H	H	H	H
5/5	D	D	D	D	D	D	D	D	H	H
4/4	H	H	H	SP/H	SP/H	H	H	H	H	H
3/3	SP/H	SP/H	SP	SP	SP	SP	H	H	H	H
2/2	SP/H	SP/H	SP	SP	SP	SP	H	H	H	H
Soft 21	–	–	–	–	–	–	–	–	–	–
Soft 20	–	–	–	–	–	–	–	–	–	–
Soft 19	–	–	–	–	–	–	–	–	–	–
Soft 18	–	D/S	D/S	D/S	D/S	–	–	H	H	H
Soft 17	H	D	D	D	D	H	H	H	H	H
Soft 16	H	H	D	D	D	H	H	H	H	H
Soft 15	H	H	D	D	D	H	H	H	H	H
Soft 14	H	H	H	D	D	H	H	H	H	H
Soft 13	H	H	H	D	D	H	H	H	H	H
Hard 21	–	–	–	–	–	–	–	–	–	–
Hard 20	–	–	–	–	–	–	–	–	–	–
Hard 19	–	–	–	–	–	–	–	–	–	–
Hard 18	–	–	–	–	–	–	–	–	–	–
Hard 17	–	–	–	–	–	–	–	–	–	–
Hard 16	–	–	–	–	–	H	H	SR/H	SR/H	SR/H
Hard 15	–	–	–	–	–	H	H	H	SR/H	H
Hard 14	–	–	–	–	–	H	H	H	H	H
Hard 13	–	–	–	–	–	H	H	H	H	H
Hard 12	H	H	–	–	–	H	H	H	H	H
11	D	D	D	D	D	D	D	D	D	H
10	D	D	D	D	D	D	D	D	H	H
9	H	D	D	D	D	H	H	H	H	H
8 or less	H	H	H	H	H	H	H	H	H	H

– = Stand; H = Hit.
SR/H = Surrender if allowed, otherwise hit.
D = Double down if allowed, otherwise hit; D/S = Double down if allowed, otherwise stand.
SP = Split; SP/H = Split if double down after split allowed, otherwise hit.

Effect of Rule Variations

The previous section explained why the house advantage increases with more decks. This section addresses the effect of different rule variations. In a single deck game with old Las Vegas Strip rules (dealer stands on soft 17, double down on any first two cards but not after splits, split non-Ace pairs up to four times, split Aces receive only one card, no surrender), basic strategy results in essentially a zero house advantage. The following table shows the changes in the house advantage resulting from common rule variations.

*Effect Of Common Rule Variations**	
HOUSE FAVORABLE	
Two decks	+0.32%
Four decks	+0.48%
Six decks	+0.54%
Eight decks	+0.57%
Dealer hits soft 17	+0.20%
No soft doubling	+0.13%
Double down only on 10 or 11	+0.26%
No doubling on 9	+0.13%
No doubling on 10	+0.52%
Double down only on 11	+0.78%
No re-splitting of pairs (non-Aces)	+0.03%
Dealer wins ties	+9.00%
Blackjacks pay 6:5	+1.39%
Blackjacks pay even money	+2.32%
Player loses splits and double downs to dealer natural	+0.11%
PLAYER FAVORABLE	
Double down on any number of cards	–0.24%
Double down after splitting pairs	–0.14%
Late surrender	–0.06%
Early surrender	–0.62%
Re-split Aces	–0.06%
Draw to split Aces	–0.14%
Blackjack pays 2-1	–2.32%
Six card winner	–0.15%
Player's 21 pushes dealer's 10-up blackjack	–0.16%
**Percentages will vary slightly depending on the number of decks in play.*	

6-5 for Naturals

A common variation found in single-deck blackjack games today is paying only 6-to-5 for naturals, a variation that increases the house advantage by about 1.39%.

For each rule variation in effect in a game, add the corresponding percentage from the table to the 0.00% benchmark value to arrive at the approximate overall house advantage for the game. For example, a 6-deck game where dealer stands on soft 17 and double down is allowed on 10 and 11 only, the house advantage is 0.54% + 0.26% = 0.80%. This same game with dealer hitting soft 17 would have a 1.00% house advantage.

Effect of Individual Cards

The table below shows the relative importance to the player of each individual card in the deck. The first part gives the change in basic strategy expectation if a single card of a certain denomination is removed from the deck. The second part gives the player's expectation when the dealer's upcard is a certain denomination. The third part shows the player's expectation as a function of his own first card.[88]

Relative Importance Of Individual Cards									
Effects of Removal (Change in Player's Expectation)									
2	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*T*	*A*
.38%	.44%	.55%	.69%	.46%	.28%	.00%	–.18%	–.51%	–.61%
Player's Expectation when Dealer Exposes									
2	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*T*	*A*
10%	14%	18%	24%	24%	14%	5%	–4%	–17%	–36%
Player's Expectation when his First Card is									
2	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*T*	*A*
–12%	–14%	–16%	–19%	–18%	–17%	–9%	0%	13%	52%

Besides helping to better understand basic strategy, the relative importance of different cards is important for card-counting (see below) and certain cheating techniques.[89]

Insurance

The insurance bet in blackjack is offered when the dealer's upcard is an Ace. This side bet is made for up to one-half the value of the original wager and pays 2 to 1 if the dealer has a natural. This wager amounts to a bet on whether the dealer's hole card is a Ten (value) and is completely independent of the original wager. The casino

88 Peter Griffin, The Theory of Blackjack (6th ed., 1999).

89 See, for example, Bill Zender, How to Detect Casino Cheating at Blackjack (1999).

advantage on the insurance bet depends on the number of decks in play. The reasoning is illustrated for the case of a single-deck game.

If the dealer shows an Ace and the player is holding two 10s, then there are 14 Tens and 35 non-Tens among the 49 remaining unseen cards. In this case the probability the dealer has a natural is 14/49, the probability the dealer does not have a natural is 35/49, and the expected value of the insurance wager is:

$$EV = (+2)(14/49) + (-1)(35/49) = -0.14286.$$

Similarly, the expectations for the cases when the player is holding a Ten and non-Ten, and two non-Tens are:

$$EV = (+2)(15/49) + (-1)(34/49) = -0.08163,$$

and $$EV = (+2)(16/49) + (-1)(33/49) = -0.02041.$$

Thus, the house advantage on the insurance wager is 14.286% when the player is holding two Tens, 8.163% when the player is holding a Ten and a non-Ten, and 2.041% when the player is holding two non-Tens. To find the overall house advantage, these three values must be weighted by the probabilities the player will be in the three situations. These probabilities are (for a single deck game):

Player holds two Tens: $(16/51)\times(15/50) = 0.094118$.

Player holds a Ten and a non-Ten: $(16/51)\times(35/50)\times 2 = 0.439216$.

Player holds two non-Tens: $(35/51)\times(34/50) = 0.466667$.

The overall house advantage for the insurance wager, then, is:

$$HA = .094118(14.286\%) + .439216(8.163\%) + .4666667(2.041\%) = 5.88\%.$$

Similar reasoning can be applied to multiple deck games. The following table summarizes the results for 1, 2, 4, 6, and 8 decks.

HOUSE ADVANTAGE ON INSURANCE BET				
# of Decks	*Player Holds Two Tens*	*Player Holds a Ten and a Non-Ten*	*Player Holds Two Non-Tens*	*Overall H.A.*
1	14.29%	8.16%	2.04%	5.88%
2	10.89%	7.92%	4.95%	6.80%
4	9.27%	7.80%	6.34%	7.25%
6	8.74%	7.77%	6.80%	7.40%
8	8.47%	7.75%	7.02%	7.47%

“EVEN MONEY”

Insuring a natural (for the full half-unit insurance wager) is sometimes referred to as taking “even money.” The reasoning for this is as follows. If the dealer does not have a natural, the player loses the half-unit insurance bet but wins 1½ to 1 on the original wager for a net profit of 1 unit. If the dealer does have the natural, the player wins 1 unit (2 to 1 on the half-unit wagered) for the insurance wager and pushes on the original bet, for a 1-unit profit. Thus, insuring a natural results in the player being paid even money regardless of whether or not the dealer also has a natural.

Taking even money may result in a 100% return to the player, but insurance is still a negative expectation play even when holding a natural. The figures above show the disadvantage on the insurance bet (alone) to be about 8% in this situation.

When the original bet and the insurance bet are considered together, the overall expected value of not taking insurance when holding a natural is:

$$EV = (0)(15/49) + (+1.5)(34/49) = +1.0408.$$

That is, not taking insurance will return 104% to the player in the long run compared to 100% by taking even money.

These computations on insurance bets assume the player has no knowledge of cards other than his own first two cards and the dealer's upcard. For card counters, however, the insurance bet could be favorable since they'll know when there are more Tens than usual remaining to be played.[90]

Volatility

The approximate probabilities of the various player outcomes from a single hand of blackjack are given in the following table:[91]

Player Outcome	*Approximate Probability*
±2	.10
+1.5	.05
±1	.75
0	.10

The average squared outcome yields the variance (since the expected value is effectively zero) for a single hand of blackjack:

$$\text{Variance} \approx (\pm 2)^2(.10) + (1.5)^2(.05) + (\pm 1)^2(.75) + (0)^2(.10) \approx 1.26.$$

The standard deviation is the square root of 1.26, or about 1.1 (i.e., 1.1 units or 110% of the original wager). Using this per unit standard deviation, confidence limits can be constructed for the actual win in a large series of independent blackjack wagers. The standard deviation of the win percentage for n units wagered is:

$$SD_{win\%} = \frac{SD_{per\ unit}}{\sqrt{n}} = \frac{1.1}{\sqrt{n}}.$$

90 If more than one-third of the unseen cards are Tens (in value), insurance is a positive expectation bet.

91 Peter Griffin, The Theory of Blackjack (6th ed., 1999).

For 1,600 hands, for example, the standard deviation is 0.0275, or 2.75%. This is equal to 44 units (2.75% of 1,600). Assume now a player betting $10 per hand and a 1.0% house advantage. Then the expected casino win is 16 units (1% of 1,600) or $160, with an associated standard deviation of $440. Adding and subtracting one standard deviation from the expected win, about 68% of the time the house win from this player (after 1,600 hands) should be between −$280 and +$600. Adding and subtracting two standard deviations from the expected win indicates that about 95% of time the casino win will be between −$720 and $1,040.

This type of analysis was used to construct the following table showing approximate 95% confidence limits for casino units won for various numbers of wagers assuming expert basic strategy against a rather typical 0.5% house edge game.

Approximate 95% Confidence Limits for Units Won by House
(Basic strategy player at 0.5% house advantage)

	1,000 Wagers	*10,000 Wagers*	*100,000 Wagers*	*1,000,000 Wagers*	*10,000,000 Wagers*
Single Wager EV	.005	.005	.005	.005	.005
Single Wager SD	1.1	1.1	1.1	1.1	1.1
Units EV	5	50	500	5,000	50,000
Units SD	34.8	110.0	347.9	1,100.0	3,478.5
Lower Limit (95%)	−65	−170	−196	+2,800	+43,043
Upper Limit (95%)	+75	+270	+1,196	+7,200	+56,957

The limits in the preceding table give a rough guide as to the variation that can be expected. For example, the casino may lose as much as $17,000 or win as much as $27,000 for a $100 per hand basic strategy bettor over the course of 10,000 hands. A loss or win outside this range may be due to something other than normal statistical fluctuations. Similarly, over the course of 1,000 hands, 95% of the time a $100 per hand player will find his winnings between −$7,500 (casino win) and $6,500. For a given number of hands played, win values occurring between the lower and upper confidence limits are to be expected. Win amounts outside these 95% confidence should alert the casino to the possibility that something may be amiss.[92] An unusually large win by a player over the course of this many hands, for example, may be due to cheating or card-counting.[93]

92 The level at which an alert is triggered can be adjusted by using a confidence level other than 95% to derive the confidence limits.

93 These confidence limits apply to one series of n hands. When viewed in terms of many players each betting n hands, chances are fairly high at least one will win or lose outside the range of the confidence limits. Thus, an unusually large player win (or loss) will occasionally occur just due to normal statistical fluctuations. See related discussion on the likelihood of an extreme outcome with many slot machines on page 76.

The Unskilled Player

Odds in blackjack vary significantly depending on the skill level of the player, which can range from a slight advantage for a skilled card counter (see below) to a substantial disadvantage for a new player with no knowledge of basic strategy. Most casinos should expect that over time actual win percentage will exceed the theoretical house advantage because many, if not most, blackjack players will not always play with the highest player expectation (i.e., according to basic strategy ignoring card-counting). How much this benefits the casino will depend on many factors, such as whether the casino's clientele are predominately tourists versus locals, the maturity of the market, and the size of the table limits. This is because local blackjack players tend to have a better grasp on average of basic strategy than tourists. The same generalities can be said of players in mature markets as opposed to new markets, and those who play at higher limit games than lower limit games.

Card-Counting

Because successive hands in blackjack are dependent trials, cards that have been played on previous hands affect the probabilities of what will happen on later hands. This is not true for games such as roulette and craps, where the spins of the wheel and the throws of the dice are independent trials. A series of nine consecutive red numbers in roulette does not make red any more or less likely on the tenth spin. Similarly, the probability of rolling a seven is the same, 1/6, regardless of what was thrown on the previous roll (or series of rolls) of the dice. For cards being dealt from the same deck or shoe as in blackjack, however, the statistical environment is quite different. Once a card is removed, the composition of the remaining deck changes. If four Aces have been dealt (without replacement) from a single deck, the probability that the next card will be an Ace is zero. A knowledgeable player can beat blackjack using a memory-based card-counting system because the probabilities of different outcomes change depending on what cards have already been played.[94]

The General Idea

The following table, displayed previously and reproduced here for convenience, shows the change in player's expectation from basic strategy expectation by removing one card of each value from a single deck.

Effects of Removal of Individual Cards
(Change in player's expectation)

2	3	4	5	6	7	8	9	T	A
.38%	.44%	.55%	.69%	.46%	.28%	.00%	−.18%	−.51%	−.61%

94 Although baccarat is also a game of dependent trials, it has been shown that card-counting techniques are not effective in the game of baccarat. See, for example, Peter Griffin, The Theory of Blackjack.

Changes in player expectation in the above table reveal that, generally, removing low cards from the deck will increase the player's expectation while removing high cards will reduce the player's expectation. Thus, when a preponderance of low cards has been played and the remaining deck is rich in high cards, subsequent hands will tend to favor the player. Conversely, if a preponderance of high cards has been played, subsequent hands will tend to favor the dealer. These observations form the basis for card-counting strategies. A player keeping track of the cards that have been played can recognize when the remaining deck is rich in high cards, and can then take advantage of the increase in player expectation at these times by increasing the bet size and/or altering playing decisions. When the remaining deck contains proportionally more small cards, the counter should lower the bet. With proper use of such a strategy, it is possible to play blackjack with a positive overall expectation (i.e., the casino advantage will be negative).

Why does a deck rich in high cards favor the player? The reasons are:

- Naturals are more prevalent.
- The dealer is more likely to bust when hitting stiffs.
- Player double downs are more effective.
- Player splits are more effective.
- Insurance can be effective.

To elaborate on these reasons, although both dealer and player will get more naturals when the deck is rich in Tens and Aces, the player is paid a 3 to 2 premium (the dealer is not). The dealer must hit stiff totals (hard 12-16), and is more likely to bust if there is a preponderance of Tens remaining, whereas the player can choose not to hit these stiff hands. Player double downs and splits are more effective since the player is often looking for a high hit card in these situations (and also because in many of these situations the dealer's upcard is weak, leading to more dealer busts). More Tens than usual also means that insurance can be a positive expectation wager for the player. Standard expected value calculations show that insurance is advantageous if more than one-third of the remaining cards are Tens.

CARD-COUNTING SYSTEMS

Many card-counting systems have been devised since Dr. Edward Thorp first published *Beat the Dealer* in 1962 explaining how blackjack could be beaten using a memory-based system. All systems are designed to keep track of high and low cards to identify favorable and unfavorable situations. The player then adjusts bet size and strategy decisions accordingly. The systems differ in the point values assigned to particular cards, whether a side count is kept for certain cards (Aces, for example), adjustments for number of decks, and associated strategy.

One of the simplest yet effective systems is the Hi-Lo (or Plus/Minus) count, first proposed by Harvey Dubner in 1963 and later refined by Julian Braun, who used computers to determine the correct playing and betting strategies for the system.[95] In the Hi-Lo count, small cards (2, 3, 4, 5, and 6) are assigned a value of +1, Tens and Aces are assigned a value of −1, and 7's, 8's, and 9's are assigned 0. As cards are

95 This Hi-Lo count system appeared in the second edition of Edward Thorp's Beat the Dealer in 1966.

revealed through play, the player assigns the appropriate value to each and keeps a running total, called the running count, of the values of the exposed cards. The running count is divided by the number of decks remaining in the shoe to get the true count. The true count is used to make decisions about when to deviate from basic strategy, when to raise and lower the bet, and when to take insurance. When the count is high, the remaining deck is rich in high cards – favorable to the player – and the bet is raised. When the count is low, there is a preponderance of small cards left and the bet is lowered. Note that a card-counter does not need to memorize each specific exposed card – tracking the sum of the values of these cards is all that is necessary. An easy way to accomplish this is to pair high (−1 value) and low (+1 value) cards, which cancel to zero, and then total the excess cards.

The Hi-Lo count is a level-1, balanced, single-parameter system. The level of a system refers to the highest absolute point value assigned to any single card. The system is balanced because the sum of the card values for the entire deck is equal to zero. A multi-parameter system is one in which an additional side count is required, such as for Aces. Since no such separate side count is required in the Hi-Lo count, it is a single-parameter system. Despite its simplicity, the Hi-Lo is an effective system advocated by numerous experts. Stanford Wong espouses the Hi-Lo system in his classic book, *Professional Blackjack*, and Bill Zender recommends the Hi-Lo for multiple deck games in his excellent book, *Card-counting for the Casino Executive*. For single-deck games, Zender suggests the Hi-Opt I, another level-1, balanced system that differs from the Hi-Lo in assigning a point value of zero to Aces and 2s.[96]

ADVANTAGES GAINED BY CARD-COUNTING

The edge gained by a card-counter depends on several factors, including the number of decks in play, the penetration (how far into the pack cards are dealt before reshuffling), the rules of the game, the bet spread and the system used. A skilled counter will typically have a 0.5% to 1.5% advantage over the house.[97] More decks and shallow penetration (not dealing very far into the pack before reshuffling) tend to decrease the counter's advantage. Any advantage created through card-counting, of course, is negated when the deck is shuffled. Possible casino countermeasures against card-counting include preferential or at-will shuffling.[98]

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96 Hi-Opt I can be attributed to Charles Einstein, who proposed a count system in 1968 that was later refined by Julian Braun, an anonymous Mr. G., and Lance Humble and Carl Cooper. A detailed treatment of the Hi-Opt I can be found in Humble and Cooper's The World's Greatest Blackjack Book, (1980).

97 Martin Millman, in "A Statistical Analysis of Casino Blackjack" (American Mathematical Monthly, 1983), estimated that with four decks the counter could gain 1.35% and with six decks 0.91%.

98 See Bill Zender's Card-counting for the Casino Executive for further details on casino countermeasures.

CARIBBEAN STUD POKER

House Advantage:
5.22% (per hand)
2.56% (per unit wagered)
Typical Hold:
27% (Nevada)

Caribbean Stud poker is played with a single standard deck of 52 cards and uses the usual five-card hand rankings in poker. Players make an ante bet before the cards are dealt, and may also make an optional $1 progressive jackpot side bet. Each player is then dealt five cards face down. The dealer is also dealt five cards, one of which is face up. Players evaluate their own cards and then either fold and forfeit the ante bet, or call the dealer by placing an additional call bet equal to exactly twice the ante. The dealer then turns over his other four cards to make a five-card poker hand. If the dealer's hand is an Ace-King or better, the dealer is said to qualify. If the dealer does not qualify, any player who made the call bet is paid even money on only their ante. If the dealer qualifies, the value of his hand is compared to that of each player who made the call bet. If the dealer wins, the player loses both the call bet and the ante bet. If the player wins, he is paid even money on the ante bet and the call bet is paid according to the following table:

Caribbean Stud Poker Payoffs

Hand	*Call Bet Pays*	*Jackpot (typical)*
Royal Flush	100 to 1	100%
Straight Flush	50 to 1	10%
Four of a Kind	20 to 1	$100
Full House	7 to 1	$75
Flush	5 to 1	$50
Straight	4 to 1	0
Three of a Kind	3 to 1	0
Two Pair	2 to 1	0
One Pair or less	1 to 1	0

Jackpot payoffs, which may vary from the amounts listed above, are given to players who made the $1 progressive jackpot side bet regardless of the dealer’s hand.

> The gambling known as business looks with austere disfavor upon the business known as gambling.
>
> *The Devil’s Dictionary*, 1906
> Ambrose Bierce

Mathematics

The following table shows the theoretical distribution of five-card poker hands.[99]

Distribution Of Five-Card Poker Hands			
Hand	*Frequency*	*Probability*	*Cum. Prob.*
Royal Flush	4	0.00000154	0.00000154
Straight Flush	36	0.00001385	0.00001539
4 of a Kind	624	0.00024010	0.00025549
Full House	3744	0.00144058	0.00169606
Flush	5,108	0.00196540	0.00366146
Straight	10,200	0.00392465	0.00758611
3 of a Kind	54,912	0.02112845	0.02871456
Two Pair	123,552	0.04753902	0.07625358
One Pair	1,098,240	0.42256903	0.49882261
Ace-King	167,280	0.06436421	0.56318681
Nothing	1,135,260	0.43681319	1.00000000
TOTAL	2,598,960	1.00000000	

The figures in the above table represent the usual frequencies and probabilities applicable to any five-card hand randomly selected from a single deck of 52 cards. Less than 8% of hands are better than one pair, and a royal flush will occur only once every 650,000 hands. In Caribbean Stud, the dealer will qualify 56.3% of the time and the jackpot side bet will pay out about once every 270 hands.

Caribbean Stud optimal strategy requires determining which hands should be folded and which should not. To shed light on this issue, consider two cases – a player holding nothing (less than Ace-King), and a player holding the best Ace-King hand.

Holding Less Than Ace-King (The Cost of "Bluffing")

Since the dealer will qualify 56.3% of the time, a player who makes the call bet, or "bluffs," when holding less than Ace-King will win one unit about 43.7% of the time and lose three units about 56.3% of the time (the exact values depend on which five cards the player is holding). The expected value of a bluff, then, is roughly:

$$EV = 0.437(+1) + 0.563(-3) = -1.25.$$

Compared to a one-unit loss for just folding, bluffing costs about 25% more in the long run. A player holding less than Ace-King should simply fold.

Holding the Best Ace-King Hand

Now consider what happens if a player calls with the best Ace-King hand, A-K-Q-J-9. Here the player wins one unit if the dealer does not qualify, wins three units if

99 See the section on cardroom poker later in this chapter for more details on poker mathematics.

the dealer has an Ace-King hand (ignoring a tie), and loses three units if the dealer has a pair or better. The expected value of calling with this strongest of Ace-King hands is about:

$$EV = 0.437(+1) + 0.064(+3) + 0.499(-3) = -0.868.$$

This average loss of 0.87 units is about 13% better than the one-unit loss that results from folding. Thus, proper strategy with the best Ace-King hand is to call. Obviously this means the player should call if holding better than Ace-King.

THE BASIC STRATEGY AND HOUSE ADVANTAGE

The preceding discussion points to an optimal strategy that says to fold when holding less than an Ace-King hand and call when holding better than Ace-King. This accounts for 93.6% of all hands. The appropriate decision for the remaining 6.4% of the hands that are Ace-King depends on the other three player cards and the dealer's exposed card. Details of the complete optimal strategy are rather complex, but can be found in Griffin and Gwynn,[100] or Ko.[101] Use of perfect strategy would result in a casino advantage of 5.22% (in terms of the ante bet).

SIMPLIFIED STRATEGY

Since the full optimal strategy is unwieldy, several authors have suggested relatively simple nearly optimal versions. Griffin and Gwynn's simplified strategy is to make the call bet with a hand of A-K-J-8-3 or better. This approach, which ignores the dealer's upcard, results in a house advantage of about 5.3%.

Vancura[102] proposes a set of five rules to achieve a house advantage virtually equal to the theoretical minimum of 5.22%:

- ♥ Always fold if holding nothing.
- ♥ Always call if holding a pair or better.
- ♥ If holding A-K, call if one of the other three cards matches the dealer's upcard.
- ♥ If holding A-K-Q or A-K-J, call when the dealer's upcard is any of the five cards you're holding.
- ♥ If holding A-K-Q-x-y (x greater than y), call when the dealer's upcard is less than x.

AVERAGE BET SIZE

The advantages previously quoted are in terms of the ante bet. Thus, a player using optimal strategy can expect to lose 5.22% of the ante bet in the long run. Since the player will add to this ante bet when making the call bet, however, the average amount of money wagered is greater than the ante amount. For example, a player betting $10 on the ante will add another $20 when making the call bet, putting a total of $30 at risk. Players using optimal strategy will call about 52% of the time, making the average amount wagered 2.04 units. Relative to the average wager, then, the house advantage is 5.22% divided by 2.04, or about 2.56%.

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100 Peter Griffin and John M. Gwynn, Jr., "An Analysis of Caribbean Stud Poker", in Finding the Edge: Mathematical Analysis of Casino Games (2000).

101 Stanley Ko, Mastering the Game of Caribbean Stud Poker (1997).

102 Olaf Vancura, Smart Casino Gambling (1996).

ADVANTAGE PER HAND VS. ADVANTAGE PER UNIT WAGERED

When discussing the house advantage for Caribbean Stud, is it 5.22% (per hand) or 2.56% (per unit wagered)? In terms of expected player loss (or expected casino win), the two methods of expressing the house advantage are equivalent. Using a $10 ante bettor as an example, applying the 2.56% edge to the average bet ($20.40), the casino can expect to win 52.2 cents per hand (.0256 x $20.40 = $0.522). This equates to 5.22% relative to the ante bet (.0522 x $10 = $0.522). Thus, both house advantages are correct in terms of what each is expressing.

Many, however, consider the 5.22% per hand house advantage to be the more appropriate and useful figure. The main reason for this is that it is the more realistic measure of how much the casino will earn, and what the player can expect to lose, as the result of a single hand. Additionally, it can be misleading to quote the advantage per unit wagered without also reporting the average bet size, and using this figure makes it easier to rate players for purposes of comps.

THE PROGRESSIVE JACKPOT BET

The advantages quoted above do not include the jackpot side bet. A glance at the distribution of five-card poker hands in the above table shows the overall probability of winning something on the $1 progressive jackpot bet is 0.00366 – about 0.37% or once in every 270 hands. The house advantage on this side bet depends on the payout schedule, which varies from casino to casino, and the level of the progressive jackpot. For a $100,000 jackpot with the typical payout schedule shown in the table on page 143, the player's expected return is $0.523 for a house advantage of 47.7%. A $200,00 jackpot raises the player's expected return to $0.815 (18.5% house advantage), and at $250,000 the expected return is $0.961 (3.9% edge). The breakeven jackpot amount, where the expected return on the jackpot side bet is equal to the $1 wagered and the house advantage equals zero, is about $263,000 (for this payout schedule).

From the player's perspective, since the ante bet will cost about 5.22% of each $1 ante wager, for a given jackpot amount and payout schedule, the overall house advantage for the combined ante plus jackpot side bet is greater than that for just the jackpot bet. Because of this, the jackpot amount needed to make the combined ante plus jackpot side bet a breakeven proposition must be greater than the breakeven amount for the jackpot bet alone. For the payout schedule considered earlier, and a $5 ante wager, a jackpot of about $352,000 is needed to make the overall expectation, ante plus jackpot bet combined, equal to zero. A jackpot greater than this amount results in a positive expectation for the player.

Note also that the amount of the ante bet plays a role in the combined expectation. The larger the ante bet, the greater the breakeven jackpot amount. A $10 ante bettor, for example, would need about a $442,000 jackpot to have an overall breakeven proposition (with the payout schedule above).

LET IT RIDE

House Advantage: 3.51% (per hand) 2.87% (per unit wagered) ***Typical Hold:*** 22% (Nevada)

Let It Ride is another poker type game played with a single standard deck of 52 cards. Players place three bets of equal size before the cards are dealt. An optional bonus side bet of $1 may also be made. Each player is then dealt three cards and the dealer two cards, all face down. After examining the three cards, each player may take back one of the three bets, or "let-it-ride" and keep all three bets active. The dealer then turns over one of his two cards, which acts as the fourth card for each player. The players must again make the decision to either take back one of their bets (of the two or three that remain depending on their first decision) or let all remaining bets ride. The dealer then exposes the second community card and remaining wagers are resolved and paid according to:

Hand	***Payoff***
Royal Flush	1000 to 1
Straight Flush	200 to 1
Four of a Kind	50 to 1
Full House	11 to 1
Flush	8 to 1
Straight	5 to 1
Three of a Kind	3 to 1
Two Pair	2 to 1
Pair of Tens or better	1 to 1

Players are not playing against the dealer or other players. The player is simply trying (hoping) to get a good poker hand using his three cards and the dealer's two community cards. A five-card hand with a pair of 10s or better wins at least even money on the remaining wagers. Although the player always starts with three units bet, he always has control of two of the three bets.

MATHEMATICS

Computer analysis of all possible hand combinations shows that optimal play results in a 3.51% house advantage, relative to the one unit base bet. For a player wagering a $5 base bet (starting with three $5 bets), the casino can expect to earn about 17½ cents per hand (.035 x $5 = $0.175).

Depending on how much the player decides to let ride, however, the final amount wagered can be one, two, or three units. Thus, as happens in Caribbean Stud poker, the average bet is greater than the base bet. The frequency of occurrence of the final wager amount depends on how the hand is played, but proper strategy results in a one-unit final wager 84.57% of the time, two units 8.50% of the time, and three units 6.94% of the time.[103] Based on these figures, the average bet size is 1.224 units.

Let It Ride Bet Frequencies	
Bet Size	*Frequency*
1 unit	0.8457
2 units	0.0850
3 units	0.0694
Average Bet Size	1.224 units

Expressed in terms of this average bet, the casino advantage is 2.87% (0.0351/1.224 = 0.0287).

The two methods for expressing the house advantage – 3.51% per hand and 2.87% per unit wagered – are equivalent. To see that this is so, note that multiplying the latter by the average bet size yields the former: $0.0287 \times 1.224 = 0.0351$. Thus, for example, regardless of which way the house advantage is expressed, the expected casino win for a $5 base bet optimal strategy player is $0.176 per hand:

$$0.0351 \times \$5 \textit{ base bet} = 0.0287 \times \$6.12 \textit{ average bet} .$$

BASIC STRATEGY

A basic strategy with 3.51% per hand house advantage is given below:[104]

- ♣ Three-Card Strategy: Let the first bet ride if holding
- ♣ A winning hand
- ♣ A straight flush with lowest card better than a 2
- ♣ A suited, one-gap straight with at least one high card (10 or better)
- ♣ A suited two-gap straight with at least two high cards (10 or better)

Four-Card Strategy:[105] Let the second bet ride if holding

- ♦ A winning hand
- ♦ A flush
- ♦ An outside straight with at least one high card (10 or better)

LET IT RIDE BONUS

The Let It Ride bonus bet is a $1 side bet similar to the jackpot bet in Caribbean Stud. The following table shows some common payoff schedules and the house advantage associated with each.

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103 Kilby and Fox, Casino Operations Management.

104 Stanley Ko, Mastering the Game of Let It Ride.

105 Several hands with expected value equal to zero could be included in the four-card second bet strategy without changing the overall house advantage. Examples are an outside straight with no high cards and a one-gap straight with four high cards. Not including such hands will have the effect of reducing the fluctuation in the player's bankroll.

Let It Ride Bonus Bet					
	Payoff 1	*Payoff 2*	*Payoff 3*	*Payoff 4*	*Payoff 5*
Royal Flush	20,000	20,000	25,000	20,000	20,000
Straight Flush	2,000	1,000	2,500	2,000	1,000
4 of a Kind	100	100	400	400	400
Full House	75	75	200	200	200
Flush	50	50	50	50	50
Straight	25	25	25	25	25
3 of a Kind	9	4	5	5	5
Two Pair	6	3	0	0	0
High Pair	0	1	0	0	0
House advantage	13.77%	23.73%	24.07%	25.53%	26.92%

A casino is a mathematics palace set up to separate players from their money. Every bet made in a casino has been calibrated within a fraction of its life to maximize profit while still giving the players the illusion that they have a chance.

Nicholas Pileggi
Casino (1995)

PAI GOW POKER

House Advantage:
2.84% (average, non-banker)
Typical Hold:
21% (Nevada)

Pai Gow poker is played with 53 cards – a standard single deck plus one Joker. The Joker can be used to fill a straight or flush, otherwise it is an Ace. After bets are placed, each player and the dealer are dealt seven cards, used to make a five-card hand and a two-card hand. The five-card hand must be of equal or higher rank than the two-card hand. Hand rankings are as follows:

Pai Gow Poker Hand Rankings
Five Aces
Royal Flush
Straight Flush (A-2-3-4-5 is highest)
Four of a kind
Full House
Flush
Straight (A-2-3-4-5 is second highest)
Three of a kind
Two pair
One pair
High card

Note that the highest-ranking hand is five Aces, not the royal flush, and the second highest straight is (in most casinos) A-2-3-4-5.

One person is designated the banker. In the casino version of the game, the dealer usually acts as banker although a player may be the banker.[106] A player who acts as banker must be able to cover all bets at the table if all other players win.

Once all hands are set, the player's two hands are compared to the banker's two hands. If the player's five-card hand outranks the banker's five-card hand and the player's two-card hand outranks the banker's two-card hand, the player wins. If both of the banker's hands outrank both of the player's hands, the player loses. If one of the player's hands outranks the corresponding banker's hand and one of the banker's hands outranks the corresponding player's hand, it is a push. The payoff on winning hands is even money, but the casino takes a 5% commission on the winning bets.[107]

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106 How frequently a player can bank varies from casino to casino.

107 The casino still collects the commission when a player acts as banker.

Expected Casino Win

Little has been written on the mathematics and strategy of Pai Gow poker. The discussion here is based on Stanford Wong's *Optimal Strategy for Pai Gow Poker*.

The house advantage against the player in Pai Gow Poker is due to two factors: commission and (for the non-banking player) copies.

♦ Effect of Commission: 1.57%.

Since commission is not collected on losing hands or pushes, the cost to the average player of the casino's 5% commission on winning hands is about 1.57%.

♦ Effect of Copies: 1.27%.

Copies – where the banker and player have the same two-card hand or the same five-card hand – go to the banker. Copies occur about 2.54% of the time, giving the banker a natural advantage of about 1.27% over the average player.

The combined effects of the commission and the banker's natural advantage give the casino a 2.84% advantage over the average non-banking player. This edge can be reduced by about 0.3%, to 2.54%, if the player is skilled in setting hands.[108]

Player Acting as Banker

A player who acts as banker reclaims the 1.27% banker's natural advantage from copies, but the casino still collects commission on winning hands. Under certain conditions, the banker's natural advantage can offset the effective cost of commission and a skillful player can even have a 0.2% to 0.4% edge when banking. Wong suggests this can occur when a player banks against multiple opposing players whose total bets are six to fourteen times the player-banker's bet when not banking (which should be minimum). Typical casino rules allow a player to bank no more than one out of two hands, so this edge would only apply every other hand at most.

Expected house win is about 2.84% of monies wagered when the casino is acting as banker. When a player acts as banker, the expected casino win will be slightly less because the casino loses the natural advantage due to copies. The banker pays commission only on net win, which depends on the number of other players and their betting levels.

The effective cost of commission to a player-banker tends to be less when there are more opposing players. For hands in which a player acts as banker at a table with no other players, the effective cost of commission is 1.63%.[109] Coupled with the 1.27% banker natural edge, the net result is a 0.36% casino edge on these hands. A skilled player who banks every other hand and bets equal amounts as player and banker (typical casino rules) would lose 2.54% on one hand and 0.36% the next, for an average casino win of 1.45%. If there are other non-banking players at the table, the expected casino win will be higher. Thus, depending on the number of players at the table, their skill level, and how often the players bank, the casino can expect to

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108 Optimal hand-setting strategy is not presented here, but can be found in Stanford Wong, Optimal Strategy for Pai Gow Poker (1992).

109 Wong reports that out of $1000 wagered, players win about $314, lose $326, and push $360. Thus a player banking at a table with no other players will pay on average .05($326) = $16.30 in commission to the casino.

win between 1.45% and 2.84% of the money wagered in Pai Gow poker. When there are multiple players, even if a player is banking, this figure is closer to 2.84%.[110]

110 Additionally, when there are many players, the risk to the casino is less when a player acts as banker, even though the expected casino win is the same as when the casino banks. See Wong, Optimal Strategy for Pai Gow Poker, (1992) pp. 22-25.

THREE CARD POKER

House Advantage:
Ante/Play: 3.37% (per hand)
2.01% (per unit wagered)
Pair Plus: 2.32%
Typical Hold:
18% (Mississippi)

Three Card poker is a relatively new poker-based game that has become popular due to its simplicity, fast action and reasonably player friendly (compared with other non-traditional games) house advantage. The game is played with a single deck of 52 cards. Since, as the name implies, each player's hand is made up of only three cards, hand rankings are slightly different from the usual five-card poker rankings. Some types of hands that occur in five-card poker are not possible in three-card poker (four-of-a-kind, full-house, two pair), and the relative rankings of three-of-a-kinds, straights and flushes are reversed: a three-of-a-kind occurs less often and so is higher ranking than a straight, and a straight occurs less often and is better than a flush.

RULES OF PLAY

Each player and the dealer receive three cards. Players may wager on either one or both of two independent propositions: ante/play and pair plus.

ANTE/PLAY WAGER

If betting on the ante/play, the player places the ante wager and then decides to play or fold after examining his three cards. If the player folds, the ante wager is forfeited. If the player wishes to play, he places a second bet equal in size to the ante. The dealer then checks his three-card hand to see if it "qualifies" with a Queen or better. If the dealer's hand does not qualify, the player wins even money for the ante wager and the play wager is returned. If the dealer does qualify, the player's hand is weighed against the dealer's hand. If the player's hand outranks the dealer's, the player wins even money on both the ante and play wagers. If the dealer's hand outranks the player's, the player loses both the ante and play wagers. If the player and dealer hands are equal in rank, it is a push.

ANTE BONUS

Regardless of the outcome of the ante/play wager, and regardless of whether the dealer qualifies, if the player has a straight or better, a bonus is paid on the ante wager (but not the play wager). This bonus pays 1 to 1 for a straight, 4 to 1 for three of a kind, and 5 to 1 for a straight flush. It is possible that the player loses the ante and play wagers but wins the bonus. This would happen, for example, if the player has three of a kind and the dealer has a straight flush.

PAIR PLUS WAGER

The pair plus wager is a bet on the value of the player's cards and is independent of the dealer's hand. If the pair plus wager is made and the player has a pair or better, he will win according to: 1 to 1 for one pair, 4 to 1 for a flush, 6 to 1 for a straight, 30 to 1 for three of a kind, and 40 to 1 for a straight flush.

Mathematics

The hand rankings and their frequency of occurrence for three-card poker are given in the following table.

Distribution Of Three-Card Poker Hands			
Hand	*Frequency*	*Probability*	*Cum. Prob.*
Straight Flush	48	0.002172	0.002172
Three of a Kind	52	0.002353	0.004525
Straight	720	0.032579	0.037104
Flush	1,096	0.049593	0.086697
One Pair	3,744	0.169412	0.256109
Queen-high or better	9,720	0.439819	0.695928
All Others	6,720	0.304072	1.000000
Total	22,100	1.000000	

As mentioned previously, these hand rankings are slightly different from the usual five-card poker rankings.

House Advantage – Pair Plus Wager

The expected value of the pair plus wager can be found using the probability distribution above:

$$EV = (+40)(0.002172) + (+30)(0.002353) + (+6)(0.032579) + (+4)(0.049593) + (+1)(0.169412) + (-1)(0.743891) = -0.02317.$$

Thus, the house advantage for the pair plus wager, for which there is no strategy, and which will win 25.61% of the time (see the above table), is 2.32%.

Strategy and House advantage – Ante/Play Wager

Since the player will lose one unit by folding, the approach is, in principle, straightforward. To determine the optimal strategy for the ante/play wager, identify the weakest three-card hand for which the expectation of making with the play wager exceeds −1, and make the play wager for this hand or better.

Ignoring the ante bonus (a straight or better hand should obviously not be folded), the expected value of making the play wager is given by:

$$EV = (+2)\times P(W_q) + (+1)\times P(W_{nq}) + (-2)\times P(L),$$

where $P(W_q)$ is the probability of beating the dealer when the dealer qualifies, $P(W_{nq})$ is the probability of beating the dealer when the dealer does not qualify, and $P(L)$ = the probability of losing to the dealer when the dealer qualifies. Computing these probabilities, starting with the lowest three-card hand and working up, reveals

Q-6-4 as the weakest hand for which the above expected value exceeds −1. For Q-6-3 the expectation is −1.0031 and for Q-6-4 it is −0.9939.

Thus, the optimal strategy is simple: play when holding a Q-6-4 or better. The house advantage using this strategy is 3.37% per hand (i.e., relative to the ante bet).[111]

AVERAGE BET SIZE

Since the average bet size on the ante/play wager is greater than the ante bet, the house advantage can also be expressed relative to the number of units wagered. Using optimal strategy the player will "play" 67.42% of the time, so the average bet size on the ante/play wager is 1.6742 times the ante. This means the house advantage is 2.01% per unit wagered (3.37% divided by 1.6742).

COMPARING ANTE/PLAY AND PAIR PLUS WAGERS

Note that when comparing the pair plus wager to the ante/play wager, the expected casino win is greater for the ante/play wager (for bets of equal size). In terms of expected casino win, the relevant comparison is 2.32% for the pair plus versus 3.37% (not 2.01%) for the ante/play.

COST OF "BLUFFING"

The previous table shows that the dealer will qualify 69.59% of the time. Thus, a player who "bluffs" by making the Play wager when holding a poor hand (less than Queen-high) will lose two units about 69.59% of the time and win one unit about 30.41% of the time. The expected value of bluffing, then, is:

$$EV(Bluff) = 0.3041(+1) + 0.6959(-2) = -1.0877.$$

Compared to the 1-unit lost by folding, bluffing will cost an extra 8.77% of the ante in the long run.

> Those who defy the laws of probability the gods of chance destroy.
>
> David Spanier
> *Inside the Gambler's Mind* (1994)

111 Many of the figures cited here are adapted from Mastering the Game of Three Card Poker (1999) by Stanley Ko, which contains further details on the analysis of Three Card Poker.

RED DOG

> ***House Advantage:***
> 2.80% (per hand)
> 2.37% (per unit wagered)

Red Dog is a simple card game whose object is to guess if the value of a third card will fall between those of two previously selected cards. Sometimes referred to as Acey-Deucey, or In-Between, this game is no longer found in many American casinos, but is popular in Canada and on the Internet.

RULES OF PLAY

Red Dog is usually played with six decks. Suits are irrelevant and cards are ranked in the usual way – deuces are lowest and Aces are highest.

After players place their bets, the dealer draws two cards face up. There are three possibilities:

- ♥ If the two cards are consecutive in rank, the hand is a push.
- ♥ If the two cards are equal in rank, a third card is drawn. If this third card is equal in rank to the first two, the players win 11 to 1. If the third card does not match the rank of the first two, the hand is a push.
- ♥ If the two cards are not consecutive or equal, the dealer announces the spread between the two cards and the players have the option to double their bet. After players have either doubled or declined to double, the dealer draws a third card. If the third card falls between the first two, the players win. Payoffs depend on the spread between the first two cards. 5 to 1 for a spread of one, 4 to 1 for a spread of two, 2 to 1 for a spread of three, and 1 to 1 for spreads of four through eleven. If the third card is equal to either of the first two cards or falls outside the range between the first two cards, players lose all money wagered.

MATHEMATICS

Mathematical analysis of Red Dog is relatively straightforward and can be accomplished using basic probability calculations. The analysis involves first determining the spreads for which the player should double his bet, and then computing the associated house advantage.

OPTIMAL STRATEGY

To identify the spreads for which the player should double his bet, first compute the expectation associated with each spread.

When the spread is one, the probability of winning is (assuming six decks) 24/310 = 0.07742 and so the expectation is:

$$EV = (+5)(0.07742) + (-1)(0.92258) = -0.53548.$$

When the spread is two, the probability of winning is 48/310 = 0.15484 and the expectation is:

$$EV = (+4)(0.15484) + (-1)(0.84516) = -0.22581.$$

Similar calculations yield the expectations associated with each of the remaining spreads, which are summarized in the following table.

Red Dog – Spread Expectations (Six Decks)			
Spread	*Payoff*	*P(Win)*	*Expectation*
1	5	0.07742	–0.53548
2	4	0.15484	–0.22581
3	2	0.23226	–0.30323
4	1	0.30968	–0.38065
5	1	0.38710	–0.22581
6	1	0.46452	–0.07097
7	1	0.54194	0.08387
8	1	0.61935	0.23871
9	1	0.69677	0.39355
10	1	0.77419	0.54839
11	1	0.85161	0.70323

Since the expectation is positive only for spreads of seven or more, optimal strategy is to double the bet only for spreads of seven or more.

HOUSE ADVANTAGE

To determine the overall house advantage, it is first necessary to compute how often each of the various spreads will occur.

The probability the initial two cards are consecutive (i.e., the spread is zero) is equal to

$$(24/312)\times(24/311)\times 2\times 12 = 0.14247\,.$$

The probability the spread is one is equal to

$$(24/312)\times(24/311)\times 2\times 11 = 0.13060\,.$$

This reasoning continues through the remaining possible spread values to the probability that the spread is equal to eleven, which is given by:

$$(24/312)\times(24/311)\times 2\times 1 = 0.01187\,.$$

Finally, the probability the first two cards are equal in rank is:

$$(24/312)\times(23/311)\times 13 = 0.07395\,.$$

These probabilities are given in the following table, which summarizes the mathematical details associated with Red Dog optimal strategy play.

Red Dog – Casino Advantage (Six Decks)						
Spread	*P(Spread)*	*Payoff*	*Bet*	*P(Win)*	*EV*	*Avg. Bet Size*
0	0.14247	0	1	0.00000	0.00000	0.14247
1	0.13060	5	1	0.07742	–0.06993	0.13060
2	0.11872	4	1	0.15484	–0.02681	0.11872
3	0.10685	2	1	0.23226	–0.03240	0.10685
4	0.09498	1	1	0.30968	–0.03615	0.09498
5	0.08311	1	1	0.38710	–0.01877	0.08311
6	0.07123	1	1	0.46452	–0.00506	0.07123
7	0.05936	1	2	0.54194	0.00996	0.11872
8	0.04749	1	2	0.61935	0.02267	0.09498
9	0.03562	1	2	0.69677	0.02803	0.07123
10	0.02374	1	2	0.77419	0.02604	0.04749
11	0.01187	1	2	0.85161	0.01670	0.02374
Equal	0.07395	11	1	0.07097	0.05773	0.07396
	1.00000				–0.02798	1.17808

As can be seen from this table, assuming six decks, the house advantage under optimal strategy is 2.80% (per hand).

Similar analyses can be performed for other numbers of decks. The casino advantage is larger with fewer decks (3.16% for 1 deck, 3.08% for 2 decks, 2.88% for 4 decks) and slightly smaller for 8 decks (2.75%).

Average Bet Size

As is the case with some other games (e.g., Caribbean Stud Poker, Three Card Poker, Let It Ride, Casino War), the average amount wagered in Red Dog is larger than the original bet. The above table shows that using optimal strategy, the player will wager an average of 1.18 units. Relative to this average bet size, the casino advantage for six decks is 2.37% per unit wagered (2.80%/1.18). The analogous per-unit advantages for one, two, four and eight decks are 2.67%, 2.61%, 2.45% and 2.34%, respectively.

> "Well, I'm in a gambling mood Sammy. I'll take a glass of whatever comes out of that tap."
>
> "Looks like beer, Norm."
>
> "Call me Mister Lucky."
>
> Norm, from *Cheers*

POOLED GAMES

POKER

> ***House Rake:***
> 5% – 10%
> (Casino does not play)

Unlike other casino games, cardroom poker does not pit the player against the casino. Instead, players compete against each other[112] and money won or lost merely is transferred from one player to another. The casino provides a dealer, who does not play, and makes its money by (a) taking a percentage of each pot, (b) charging an hourly fee, or (c) collecting a flat amount for every hand. The first of these is most common, and the percentage extracted is called the rake. A rake of 5% to 10% is typical.[113]

BRIEF HISTORY OF POKER

Poker is a widely played card game, considered by some experts to be the most popular card game among Americans,[114] America's national game,[115] and the most popular international card game in history.[116] The original inspiration for the game may well be an ancient Persian pursuit called *Âs-Nâs*, played with 20 cards during its popularity in 18th century Paris. A descendant of Âs-Nâs, known as *poque* in French, came to the United States with the French colonists of New Orleans (hence the word "poker"). An English version of the game is Brag and another French offshoot is named Ambigu. It is generally agreed that the full 52-card deck version of poker was born in the early part of the 19th century in the gambling saloons of New Orleans, quickly spread north on the Mississippi riverboats, then west to the gold fields and on to the rest of the world. Today more than 70 million Americans play poker and hundreds of millions more play the game worldwide.[117]

> **Poker is not a game of cards. It's a game of people.**
>
> Doyle Brunson
> World Poker Champion

Poker is really a generic name for a collection of literally hundreds of games, varying widely with respect to specific rules of play and betting structure.[118] Poker can be played almost anywhere and in many different forms. A variety of poker games can be found in casino cardrooms, but the two most popular by far are Texas Hold'em and

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112 Because the participants play against each other, card room poker is an exception to the Nevada regulation that prohibits persons managing a gaming establishment – including any owner, director, officer, or key employee – from gambling at their own establishment. NGC Reg. 5.013.

113 In Nevada, the "rake-off" may not exceed 10% of the total amount wagered in a hand. NGC Reg. 23.050.

114 Richard A. Epstein, The Theory of Gambling and Statistical Logic, 201 (1995).

115 A. Alvarez, Poker: Bets, Bluffs, and Bad Beats, 22 (2001).

116 John Scarne, Scarne's Guide to Modern Poker, 13 (1980).

117 See, e.g., Greg Dinkin and Jeffrey Gitomer, The Poker MBA, xi (2002); Alvarez, Poker: Bets, Bluffs, and Bad Beats, 23.

118 The variations include high games (the highest hand wins), low games (lowest hand wins), high-low split, fixed limit, spread limit, pot limit, no limit, stud, draw, games in which the hands are closed, and those in which some of the players' cards are exposed. Jokers, wild cards and special rules can be used to produce even more variations. See, e.g., David Sklansky, The Theory of Poker, 2-4 (1999).

Seven-Card Stud. Bigger rooms may offer a few other variations such as Lowball and Omaha.

THE SKILL FACTOR

The object of poker is not necessarily to have the best hand, but to win the most money. Having the best hand usually helps, but does not guarantee victory because of the importance of skill and psychology. Indeed, it is fair to say that skill is more of a factor in poker than in any other casino game. For skilled players, poker can be a positive expectation game and it attracts many professional and semi-professional players. There are not many people who earn a living or consistently make money gambling, but the majority of those who do are poker players.

> One of the finest illustrations of the laws of chance is furnished by the game of poker. It is not a game of pure chance, like dice and roulette, but one involving a large element of skill or judgment.
>
> Horace Levinson
> *Chance, Luck, and Statistics* (1963)

With most casino games – roulette, craps, baccarat and keno are examples – it is not mathematically possible to win in the long run. Blackjack and video poker offer the opportunity for a slight player edge (one to two percent) and a few players can make money at these games with perfect play (and counting cards in the case of blackjack). The going is tough, however, and the earnings are small. Skill does play a part in these games, but luck is a large factor. In cardroom poker, skill is a much greater factor and the better players will win out over the weaker players in the long run.

BETTING STRUCTURES

Casino poker games are played table stakes, which means a player may bet only with the chips (or money) he has on the table during a hand. If a player runs out of chips when calling or betting, he cannot add more until the hand is over, and must go all-in to stay in the hand. When a player goes all-in, all subsequent wagers by other players go into a separate side pot in which the all-in player has no interest – he may win the main pot, to which he contributed, but may not win the side pot even if his hand is the best.

The limits, or absence of limits, on how much a player may bet and raise will dramatically affect the game dynamics, including players' decisions and strategies, and the relative balance of luck versus skill in the game. A variety of betting structures may be found:

FIXED LIMIT.

In a fixed-limit game, no bet or raise may exceed a specified amount. This amount usually varies with the betting round, with later rounds allowing higher bets and raises than early rounds. In a $5–$10 fixed-limit game, for example, players may bet or raise exactly $5 in early rounds and exactly $10 in later rounds

SPREAD LIMIT.

Spread limit games are similar to fixed-limit, but allow any bet between the two amounts at any time. Thus in a \$10–\$20 spread-limit game, bettors may make wagers of any amount between \$10 and \$20 at any time, with the provision that raises must be at least equal to the preceding bet.

POT LIMIT.

In pot-limit games, bets or raises are limited only by the amount of money in the pot at the time the wager is made.

NO LIMIT.

In no-limit games, a player may bet or raise any amount he has in front of him (table stakes limit betting in a hand to the chips and money on the table).

Pot limit and no limit formats are generally used only for more serious games (no-limit is used in the World Series of Poker, the premier high-stakes tournament). In most limit games, a bet and either three or four raises per betting round are permitted.

CARDROOM GAME STRUCTURE MIX

With a variety of betting structures that could be offered in the various cardroom poker games, what combination might provide the best opportunity for a successful operation?[119] While both skill and luck play a role in poker games, some games and betting structures emphasize the skill component more than others. No-limit, pot-limit, and spread-limit games with a large spread compared to the bet significantly reduce the luck factor and highlight the skill factor. Weaker players in these types of games will usually lose all or most of their money to the more experienced and skilled players. To be successful, casino cardrooms should endeavor to offer games that provide a good balance of luck and skill. These types of games will allow poor players to win often enough that they will continue to return, but also permit strong players to do well in the long run. This can best be achieved in limit games with two fixed tiers of betting such as \$10–\$20 hold’em or \$15–\$30 stud. In addition, spreading a limit that is too close in size to another limit (for example, spreading a \$15–\$30 hold’em game when you are also spreading \$10–\$20 and \$20–\$40 hold’em) may have the effect of reducing the total number of games the cardroom will spread in the long run.[120]

POSTING OF RULES

Casino cardrooms must typically post rules regarding fees and betting limits. Nevada regulations, for example, require that the casino post the rules of each card game in a place legible from each card table, and that the rules include (a) the maximum rake-off percentage or other fee charged, (b) the maximum number of

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119 This and other cardroom operational issues are discussed in “Cardroom Theory – A Two-Way Street,” by Donna Harris and Mason Malmuth in Finding the Edge: Mathematical Analysis of Casino Games, (2000).

120 Id.

raises allowed, (c) the monetary limit of each raise, (d) the amount of the ante, and (e) other necessary rules.[121]

Mathematics of Poker

Although cardroom poker differs from most casino games in that the players are not playing against the house – the "house edge" comes from a rake or fee in the form of a percentage of each pot or hourly charge for the casino providing the table, dealers, cards, etc. – a look at the mathematics relating to poker is appropriate here for purposes of further explaining how probabilities and odds are computed and used in gambling games. We first show the probabilities for the five-card hand rankings and then provide a brief discussion of odds in the most popular form of poker today, Texas Hold 'Em.

Poker Hand Rankings

Poker hand rankings are determined by their likelihood of occurrence when five cards are dealt at random from a shuffled deck of 52 cards. The highest-ranking hand is the least likely; the second highest-ranking hand is the second least likely, and so on. The following table summarizes the hand rankings and their probabilities of occurring in a five-card hand dealt from a deck of 52 cards.

Poker Hand Rankings (5-card)

	Hand	*Probability*	*Approximately*
1.	Royal flush	0.000002	1 in 649,740
2.	Straight flush*	0.000014	1 in 72,193
3.	Four of a kind	0.000240	1 in 4,165
4.	Full house	0.001441	1 in 694
5.	Flush	0.001965	1 in 509
6.	Straight	0.003925	1 in 255
7.	Three of a kind	0.021129	1 in 47
8.	Two pairs	0.047539	1 in 21
9.	One pair	0.422569	1 in 2.4
10.	High card	0.501177	1 in 2

** Excluding royal flushes*

121 NGC Reg. 23.080.

COMPUTING PROBABILITIES OF POKER HANDS

The probabilities of various poker hands (given in the table above) are determined using the usual probability rules and methods of counting, including combinations and permutations. The calculations for the probabilities of the different types of five-card poker hands are outlined below.[122]

First note since there are $\binom{52}{5}$ = 2,598,960 possible five-card hands that can be dealt from a single deck of 52 cards, the probability associated with a particular type of poker hand can be computed by counting the possible number of such hands and dividing by 2,598,960. The calculations of these probabilities are shown below.

1. *Royal Flush*

$$\frac{4}{2{,}598{,}960} = 0.00000154 \ (\approx 1 \text{ in } 649{,}740)$$

2. *Straight Flush* (excluding royal flushes)[123]

$$\frac{36}{2{,}598{,}960} = 0.0000139 \ (\approx 1 \text{ in } 72{,}193)$$

3. *Four of a Kind*

$$\frac{13 \times \binom{48}{1}}{2{,}598{,}960} = \frac{624}{2{,}598{,}960} = 0.000240 \ (\approx 1 \text{ in } 4{,}165)$$

4. *Full House*

$$\frac{13 \cdot \binom{4}{3} \times 12 \cdot \binom{4}{2}}{2{,}598{,}960} = \frac{3{,}744}{2{,}598{,}960} = 0.001441 \ (\approx 1 \text{ in } 694)$$

5. *Flush*

$$\frac{4 \times \left[\binom{13}{5} - 10\right]}{2{,}598{,}960} = \frac{5{,}108}{2{,}598{,}960} = 0.001965 \ (\approx 1 \text{ in } 509)$$

122 The interested reader may wish to refer back to Chapter 1 for a review of combinations, permutations, and probability rules.

123 Note that if the royals are included, there are 40 straight flushes, so the probability of any straight flush is 40/2,598,960 = .0000154, or 1 in 64,974.

6. *Straight*

$$\frac{10\times\left(\frac{20\cdot 16\cdot 12\cdot 8\cdot 4}{5!}\right)-40}{2{,}598{,}960} = \frac{10{,}200}{2{,}598{,}960} = 0.003925 \quad (\approx 1 \text{ in } 255)$$

7. *Three of a Kind*

$$\frac{13\times\left[\binom{4}{3}\cdot\frac{48\cdot 44}{2!}\right]}{2{,}598{,}960} = \frac{54{,}912}{2{,}598{,}960} = 0.021128 \quad (\approx 1 \text{ in } 47)$$

8. *Two Pair*

$$\frac{13\cdot 12\ \times\left\{\left[\binom{4}{2}\cdot\binom{4}{2}\div 2!\right]\times 44\right\}}{2{,}598{,}960} = \frac{123{,}552}{2{,}598{,}960} = 0.047539 \quad (\approx 1 \text{ in } 21)$$

9. *One Pair*

$$\frac{13\times\binom{4}{2}\times\frac{48\cdot 44\cdot 40}{3!}}{2{,}598{,}960} = \frac{1{,}098{,}240}{2{,}598{,}960} = 0.422569 \quad (\approx 1 \text{ in } 2.4)$$

10. *Worse Than One Pair*

By subtraction, there are 1,302,540 possible hands with worse than one pair. (2,598,960 – (40+624+3744+5108+10200+54912+123552+1098240) = 1,302,540), so the probability of being dealt worse than one pair is:

$$\frac{1{,}302{,}540}{2{,}598{,}960} = 0.501177 \quad (\approx 1 \text{ in } 2)$$

EXAMPLE OF OTHER POKER PROBABILITIES AND ODDS – TEXAS HOLD 'EM

> Poker is a game of many skills: you need card sense, psychological insight, a good memory, controlled aggression, enough mathematical know-how to work out the odds as each hand develops, and what poker players call a leather ass – i.e., patience.
>
> A. Alvarez
> "No Limit" (1994)

To further illustrate how probability is used in poker, consider the most popular form of poker today, Texas Hold 'Em (usually referred to as just *Hold 'Em*). In Hold 'Em, each player is dealt two cards face down, and then a total of five community cards are dealt face up in the center of the table. Each player uses the five community cards in combination with his two hole cards to form the best five-card hand. After the first two cards are dealt to each player there is a round of betting.[124] After the first round of betting, the first three community cards (the *flop*) are exposed, followed by a second round of betting. After the second betting round, the fourth community card (the *turn*) is exposed, followed by another round of betting, and then the fifth and final community card (the *river*) is exposed, followed by a final round of betting. Each betting round begins with the first active player to the left of the dealer (or in a game dealt by a house dealer, the first active player to the left of the *button* used to indicate dealer position).[125]

Suppose a player has four cards to flush after the flop. Consider the following three questions:

(a) What is the probability he will make the flush on the turn?

(b) If he doesn't make the flush with the turn card, what is the probability he will make it on the river?

(c) What is the probability he will make the flush on either the turn or river?

Note that these are three different questions (with, of course, three different answers).

To be more specific, suppose the player holds two spades and two of the three flop cards are spades. Since the player has seen five cards – his two hole cards and the three flop cards – there are 47 remaining unseen cards, of which nine are spades (i.e., there are nine *outs*, or cards that will complete the flush). Thus the probability he will make his flush on the turn card is 9/47 = .191, for odds against of 38 to 9, or about 4.2 to 1, answering question (a). To answer (b), note that if he doesn't make the flush on the turn, there are still 9 spades left in the 46 remaining cards, so the probability he makes it on the river is 9/46 = .196, for odds against of 37 to 9, or 4.1 to 1. To answer (c), first compute the probability he doesn't make the flush on the turn or river, then

124 In the typical Hold 'Em game, two blind bets are posted before the cards are dealt – a small blind by the player to the dealer's immediate left and a large blind by the next player to the left of the small blind. A blind is a forced bet made before the player sees his cards used to start the pot and stimulate action. The small blind is usually equal to one-half the amount of the big blind. Since the deal rotates around the table (even in a casino where the dealer is not a player, a button used to signify the nominal dealer rotates after each hand), all players participate equally in the posting of any forced blind bets. We will refrain from discussing further details regarding betting amounts and structure as it is not necessary for this example.

125 Because the first two players to the left of the dealer (or button) have already acted by putting in blind bets, the player one to the left of the big blind is the first with any choices (to call, raise, or fold in the first round of betting) on the pre-flop betting round.

subtract from one: $1-\left(\frac{38}{47}\times\frac{37}{46}\right)=1-0.6503=0.3497$. The probability of making the flush on either the turn or river is almost 35%, for odds against of 1.86 to 1.

This last calculation illustrates how some probabilities can be easier to determine by first computing the probability of the opposite (complement), then subtracting the result from one.

Note that the probability that the flush is made with one card to come depends on whether we look at making the flush on the turn card or the river card (having not made the flush on the turn). In the example above, the former probability is .191; the latter is .196.[126] The following tables show probabilities and odds for making hands with a given number of outs (cards that will make the desired hand), with the "one card to come" probabilities and odds computed assuming that one card is the final river card.

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126 This explains apparent discrepancies that might be found in comparing poker books. Some list the probability (or odds) with one card to come assuming 47 cards left (making the hand on the turn), others with 46 cards left (making the hand on the river, given it was not made on the turn).

Hold 'Em Odds – Expressed as Odds Against		Odds Against	
Number of Outs	*Drawing to (Example)*	*2 Cards to Come*	*1 Card to Come**
21		.43 to 1	1.19 to 1
20		.48 to 1	1.30 to 1
19		.54 to 1	1.42 to 1
18		.60 to 1	1.56 to 1
17		.67 to 1	1.71 to 1
16		.75 to 1	1.88 to 1
15	Open straight flush draw (Straight, flush, straight flush)	.85 to 1	2.07 to 1
14		.95 to 1	2.29 to 1
13		1.08 to 1	2.54 to 1
12		1.22 to 1	2.83 to 1
11		1.40 to 1	3.18 to 1
10		1.60 to 1	3.60 to 1
9	Four flush (Flush)	1.86 to 1	4.11 to 1
8	Open straight draw (Straight)	2.18 to 1	4.75 to 1
7		2.59 to 1	5.57 to 1
6		3.14 to 1	6.67 to 1
5		3.91 to 1	8.20 to 1
4	Gutshot straight (Straight)	5.07 to 1	10.50 to 1
3		7.01 to 1	14.33 to 1
2	Pocket pair (3 of a kind)	10.88 to 1	22.00 to 1
1	3 of a kind (4 of a kind)	22.50 to 1	45.00 to 1

** River (fifth community card) to come.*

A man who can play delightfully on his guitar and keep a knife in his boot would make an excellent poker player.

W.J. Florence
Handbook on Poker (1891)

Hold 'Em Odds – Expressed as Probabilities		Probability	
Number of Outs	*Drawing to (Example)*	*2 Cards to Come*	*1 Card to Come**
21		69.9%	45.7%
20		67.5%	43.5%
19		65.0%	41.3%
18		62.4%	39.1%
17		59.8%	37.0%
16		57.0%	34.8%
15	Open straight flush draw (Straight, flush, straight flush)	54.1%	32.6%
14		51.2%	30.4%
13		48.1%	28.3%
12		45.0%	26.1%
11		41.7%	23.9%
10		38.4%	21.7%
9	Four flush (Flush)	35.0%	19.6%
8	Open straight draw (Straight)	31.5%	17.4%
7		27.8%	15.2%
6		24.1%	13.0%
5		20.4%	10.9%
4	Gutshot straight (Straight)	16.5%	8.7%
3		12.5%	6.5%
2	Pocket pair (3 of a kind)	8.4%	4.3%
1	3 of a kind (4 of a kind)	4.3%	2.2%

** River (fifth community card) to come.*

SPORTS BETTING

Betting on sports is a popular form of entertainment for many Americans. Office pools and friendly wagers among acquaintances on the outcome of organized sporting contests are commonplace. Every day the latest "lines" can be found in newspapers and other media outlets across the country. Radio talk shows hype the games; hosts and callers argue the wisdom of the point spreads for the upcoming weekend games. Hundreds of offshore online gambling sites accept wagers on sporting events. There is little doubt sports betting is the largest form of gambling in the United States, but most of this wagering is not, strictly speaking, legal.

At present, Nevada is the only state in which a legal bet can be placed in a *sports book* – an entity that takes wagers from both sides to make a profit – and online wagering is a legal gray area at best.[127] In round numbers, Nevada sports books accept $2 billion in wagers each year, resulting in $120 million in revenue for the book.[128] As far as other types of sports betting, officials estimate between $100 billion and $400 billion is wagered on sports in the United States each year.[129] The focus of this section is legal sports betting and how it works.

FACTS ABOUT SPORTS BETTING

There are many myths about sports wagering in Nevada – the only state where it is legal, regulated, policed, and taxed. The following are current facts, according to the *American Gaming Association*,[130] about sports wagering:

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127 At present there is disagreement between the U.S. Department of Justice and the U.S. courts regarding the legal status of Internet gambling in the United States. At issue are two laws: the U.S. Wire Act of 1961, which prohibited gambling over the "wires," and the Professional and Amateur Sports Protection Act of 1992 (PASPA), which banned sports wagering in all states except those with pre-existing operations (Nevada, Oregon and Delaware). The Justice Department in both the Clinton and Bush administrations has expressed the view that the U.S. Wire Act of 1961 applies to all forms of Internet gambling, and therefore it is illegal under existing law. The U.S. Court of Appeals for the Fifth Circuit, however, interpreted the Wire Act differently. In Thompson v. MasterCard International et al., the appeals court in 2002 affirmed a lower court ruling that under federal statutes sports betting conducted over the Internet is illegal, but casino games are legal. Separately, the Word Trade Organization (WTO) issued an interim ruling in 2004 that found the aggressive efforts of the U.S. government to curb Internet gambling in violation of WTO commercial services accords. The ruling, which gives the WTO the authority to impose trade sanctions against the United States, stems from a complaint filed in 2003 by the Caribbean island nation of Antigua and Barbuda. American Gaming Association (http://www.americangaming.org).

128 Nevada State Gaming Control Board Revenue Reports.

129 See, e.g., Kaitlin Gurney, "Sports Wagering to Get New N.J. Push," Philadelphia Inquirer, April 12, 2004, page B01.

130 The American Gaming Association (AGA) opened its office in Washington, D.C., in June 1995 with the fundamental goal of creating a better understanding of the gaming entertainment industry by bringing facts about the industry to the general public, elected officials, other decision makers and the media through education and advocacy. The AGA represents the commercial casino entertainment industry by addressing federal legislative and regulatory issues affecting its members and their employees and customers, such as federal taxation, regulatory issues, and travel and tourism matters. In addition, the AGA has an aggressive public education program designed to bring the industry's message to target audiences both in the nation's capital and across the country. The AGA provides leadership in addressing newly emerging national issues and in developing industry wide programs on critical issues such as disordered and underage gambling. The association also serves as the industry's first national information

- Individuals must be physically present and over 21 years of age to wager in Nevada. Calling from out-of-state is prohibited.
- Nevada's legal sports wagering represents less than 1 percent of all sports betting nationwide. In 2000, $2.3 billion was legally wagered in Nevada's sports books; the National Gambling Impact Study Commission (NGISC) estimated that illegal wagers are as much as $380 billion annually.
- Gross revenue for Nevada's sports books was $123.8 million in 2000; 95 percent of all bets placed in 2000 were returned to patrons in winnings.
- About one-third of all sports bets in Nevada are placed on college sporting events.
- Wagering on high school sports in Nevada never been legal.
- A new Nevada Gaming Commission regulation bans any future Olympic bets. Prior to the rule's implementation, Olympic betting had been limited to games involving the U.S. men's basketball "Dream Teams," for which professional athletes played in the 1990s.
- Point-shaving scandals have plagued college basketball since the 1940s and 1950s, well before Nevada's modern sports books existed.
- Nevada's sports books and regulators assisted in catching those responsible for point-shaving incidents in the 1990s. The Arizona State head basketball coach was notified during a game's halftime of a possible fix because Nevada's sports books had alerted the PAC Ten conference and FBI of irregularities in betting.
- Nevada's sports books are not the only source of point spreads appearing in newspapers. Danny Sheridan, a top sports analyst producing the "line" for *USA Today*, is based in Alabama.
- Legal sports wagering helps bring more than 30 million visitors to Nevada each year and provides employment for thousands of people. The Las Vegas Convention and Visitors Authority estimated that the 2000 Super Bowl weekend generated $79.2 million in non-gaming economic impact.
- The "grandfather clause" in current law was expressly included in the Professional and Amateur Sports Protection Act of 1992 (PASPA) out of respect for all states that had authorized sports wagering prior to the enactment of the federal ban. According to the Senate Judiciary Committee report on the bill that became PASPA, Sen. Rpt. 102-248, "(The committee) has no wish to apply this new prohibition retroactively."

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clearinghouse, providing the media, elected officials, other decision makers and the public with timely, accurate gaming industry data. Information taken from the AGA website (http://www.americangaming.org), August 24, 2004.

- ♣ The NCAA never recommended a ban on wagering to the NGISC. The NCAA did recommend nearly all of the provisions included in alternative legislation supported by the gaming industry.
- ♣ The NGISC Final Report cited a study by the University of Michigan showing widespread illegal gambling by student-athletes "despite NCAA regulations prohibiting such activities."

How Sports Books Make Money

In sports wagering, the bettor is not playing against the casino as in games like blackjack, roulette, or craps. Sporting contests are not governed by the laws of probability theory and so the house has no built-in mathematical edge on sports bets the way it does for other casino games.[131] Instead, a sports book makes money by acting as a middle-man or broker in essence for bettors on opposite sides of a proposition and then charging a commission for this service. Simply put, a sports book offers two equally attractive betting options in an attempt to ensure that equal amounts of money will be bet on both sides of the wager.[132] The commission charged on every bet made is known as the *vigorish* or *vig* or *juice*. If the total amount wagered on each side (e.g., on each team) is the same, the book is assured a profit from the commission but if more money is wagered on the winning side, the sports book may incur a loss in paying off the winning wagers.[133] To minimize the risk of incurring such a loss, the book use a betting *line*, or method of handicapping to make the *underdog* in the sporting contest as attractive to bet as the *favorite*. There are two basic types of betting lines, the *point spread* and the *money line*.

Point-Spread Betting

In point-spread betting, a certain number of points – called the *point spread*, the *line*, the *number*, the *price*, or simply the *spread* – is given to the underdog for purposes of deciding the bet. A wager on the favorite wins only if the favorite wins the game by more than the point spread (i.e., they *cover the spread*). A bet on the underdog wins if either the underdog wins the game outright or if the favorite wins by less than the spread. If the favorite wins by exactly the spread amount, the game is a tie for betting purposes and wagers that were made at that spread amount are returned. The favorite is said to be *laying* points (e.g., "I'll take Denver and lay the 3½ points."); the underdog is *getting* points (e.g., "He's betting Oakland and *taking* the 3½ points."). When making a bet with a point spread, the bettor pays 11 to win 10 – this is how the sports book makes its money.

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131 Cardroom poker is another exception in which the casino has no built-in mathematical edge. Like sports bettors, poker players in a casino poker room are not playing against the house; the casino makes money by taking a percentage of the pots, called the rake, or charging a (typically hourly) time fee.

132 Strictly speaking, for the standard point-spread type wager the sports book aims for equal amounts bet on each side, but for a money-line wager the casino desires a certain ratio of money bet on each side.

133 See previous note.

The following examples for professional football (National Football League) and college basketball show how point spreads are typically listed in a newspaper.

NFL		
Favorite	*Line*	*Underdog*
DENVER	3½	Oakland
Dallas	2½	PHILADELPHIA
MIAMI	6	New York Jets

COLLEGE BASKETBALL		
Favorite	*Line*	*Underdog*
KENTUCKY	5	Massachusetts
SYRACUSE	11	Villanova
Georgetown	4½	VIRGINIA

In lines like those in the above tables, home teams are usually indicated by all capital letters. For the first of the NFL football lines, Oakland is a 3½-point underdog playing at Denver, and if you were to bet on Denver, they would have to win by more than 3½ points to cover the spread in order for you to win. In the other NFL games, Philadelphia is getting 2½ points at home against Dallas, and Miami is laying 6 points at home against New York. For the basketball examples, Kentucky is laying five points at home against Massachusetts, Villanova is getting 11 points at Syracuse, and Virginia is getting 4½ points at home against Georgetown.

MATHEMATICS OF POINT-SPREAD BETTING

When making a point-spread bet, the bettor must pay $11 to win $10 (or some multiple of these amounts).[134] If you win the wager, you receive $21, the $11 you wagered plus the $10 you win. If you lose, your $11 disappears. Assuming a 50% chance of picking the correct team with a point-spread bet, the expected value and house edge can be computed in the usual way. For example, the expected value of a $110 bet assuming a 50% chance of success is:

$$EV = (+100)(.5) + (-\$110)(.5) = -\$5.00.$$

This –$5.00 expectation on a $110 wager equates to a house advantage of 0.04545, or about 4.55%.

One way to think about the house edge on a point-spread bet is to consider what would happen if you were to make a wager of equal amounts, say $110, on each of the two teams. Your total wager amount would be $220 and assuming no tie, you would win one bet and lose one bet, receiving $210 back for the winning wager ($110 from

134 Sports books typically do not require bets to be made in multiples of $11 (or $5.50), but doing so makes payoffs easier and experienced players do so. If the bet amount is not a multiple of $11, the payoff is typically rounded down. For example, the payoff on a winning $50 bet will be $45 instead of the 10:11 payoff of $45.45.

the wager plus $100 you won) for a net loss of $10. That is, you are guaranteed to lose $10, or 4.55%, on the $220 put at risk.

It should not be difficult to see why the sports book endeavors to induce equal amount of money to be bet on both sides of a point-spread wager. Suppose for example, the Denver Broncos are a 3½-point favorite over the Oakland Raiders and suppose the betting were even on the two teams, say $110,000 on the Broncos and $110,000 on the Raiders. If the Broncos win by more than 3½ points, then the sports book pays out $100,000 to the Broncos bettors (and returns their $110,000 in wagers), leaving the book with a $10,000 profit. If the Raiders win or lose by less than 3½ points, then the Raiders bettors are paid $100,000 (and returned their $110,000 in wagers), again leaving the casino with a $10,000 profit. It may be helpful to think about the winning bettors being paid from the wagers (all but $10,000 in this example) of the losing bettors. In either case, if equal amounts are wagered on both teams, the casino retains 4.5% ($10,000/$220,000) of the money wagered. If, on the other hand, more money is wagered on one team than the other, the casino may take a loss. Suppose for example, $220,000 is bet on the Raiders and only $11,000 on the Broncos. If the Raiders win the game or lose by less than 3½ points, the casino will not have enough (losing) Broncos money to off-set the $200,000 in winnings it needs to pay the Raiders bettors.

The Magic Number is 52.4%...
To Be a Winner at Point-Spread Betting

In order to overcome the house edge in point-spread betting, the player would need to win 11 times for every 10 losses (assuming equal size bets), or 11 out of every 21 bets, or 52.4% of the time, to off-set the 10:11 payoff: 10(–$11) + 11(+$10) = 0.

When more money is being wagered on one team, the sports book may change the line in order to encourage bettors to wager on the other team and balance its book. This is why you will sometimes see the line move up or down from its original spread amount up until game time. If, for example, the line has the Raiders as a 3½-point underdog to the Broncos and a much larger amount of money has been bet on the Raiders than the Broncos, the casino may lower the line – perhaps so the Broncos are favored by only 2½ or 1½ points, or less if necessary – to stimulate betting on the Broncos.

If the line changes, it has no effect on bets that have already been placed. That is, once a player makes a bet at a certain point spread, the bet is "locked into" that spread even if the line subsequently changes. If you place a bet on the Broncos when the line is Broncos –3½ (Broncos favored by 3½), and the line moves to Broncos –2½, your bet is still tied to the 3½-point spread; if the Broncos win by 3 points, you will lose your bet (but anyone who made a bet at the new price of 2½ points would win).

Money-Line Betting

Betting the money-line allows the player to wager on the actual outcome of the game – the team that wins outright – without factoring in a point spread. Money-line betting is commonly used in baseball, hockey, and boxing, but is also offered in football and basketball (in addition to the point-spread betting option).

Here are some examples of how a money line might be listed.

BASEBALL		
Favorite	*Line*	*Underdog*
Yankees	–150/+130	RED SOX
DODGERS	–140/+120	Giants
Phillies	–130/+110	ROCKIES

BOXING		
Favorite	*Line*	*Underdog*
Lewis	–400/+300	Tyson

The money line is composed of two odds, one for the favorite and one for the underdog. In the money line examples in the table above, the numbers with the minus sign are the odds for the favorite and the numbers with the plus sign are the odds for the underdog.[135] The negative value for the favorite tells you how much money you have to wager to win $100; the plus value for the underdog tells you how much you will win if you wager $100. In the baseball example, to bet the Yankees you must wager $150 to win $100; to bet the Red Sox you put up $100 to win $130. Of course the bets need not be exactly theses amounts; you could, for example, bet $75 to win $50 on the Yankee or $50 to win $65 on the Red Sox. If a decimal point is inserted two places from the right in the money-line odds values in the table above (as is sometimes done), they may be interpreted as the amount that must be wagered to win $1 for the favorite and the amount that will be won if $1 is wagered on the underdog. Using this method for the boxing example, to bet Lewis you must wager $4.00 to win $1.00 (laying 4-to-1) and to bet Tyson you wager $1.00 to win $3.00 (getting 3-to-1).

MATHEMATICS OF MONEY-LINE BETTING

To see in general how the casino makes its revenue on the money-line bets, consider the (usual) case when the favorite is listed with a positive odds value and the underdog with a negative value.[136] For example, suppose you make a bet at $-130/+110$ on each team in the Phillies/Rockies game. You bet $130 on the Phillies (to get $100 if they win) as the favorite and $100 on the Rockies (to win $110) as the underdog, giving the casino a total of $230. If the Phillies win, you will receive $230, the $130 you originally wagered plus $100 for winning, and the house will realize zero profit. If the Rockies win, the casino returns to you $210, the $100 originally wagered plus $110 as winnings, leaving $20 for the casino sports book as profit. As in

135 This is usually the case with money line odds (that the odds value for the underdog is a positive value), but it is possible that both numbers in a money line are negative; this occurs when the teams are closely matched (or, more to the point, when the betting public perceives that the contest is closely matched). For example, a money line might be –105/–105, meaning to bet either team you must wager $105 to win $100. Note that the usual pay $11 to win $10 structure in point spread betting is equivalent to a –110/–110 money line.

136 Id.

this example, in general the casino makes its profit on money-line bets when the underdog wins. The house advantage on one particular money line bet depends on the probability of winning the bet (for example, if you bet on the Phillies, the probability the Phillies win the game), while the overall advantage the casino enjoys on money lines for a contest depends on both the probability of each team winning and the amount of money bet on each.

CONVERTING WIN PROBABILITIES TO MONEY LINES

To understand more precisely how the house advantage works with a money line, consider a game between the Denver Broncos and Oakland Raiders, and assume there is a 60% chance the Broncos will win. That is, if this game were to be played under identical conditions a large number of times, Denver would win 60% of the time. A fair money line would then be $-150/+150$ (with Denver, of course, the favorite). It is fair because at this line, both the Denver bettor and the Oakland bettor would have expected winnings equal to zero:

Denver Bettor: EV = .60(+\$100) + .40(–\$150) = 0
Oakland Bettor: EV = .40(+\$150) + .60(–\$100) = 0

In general, the fair money line can be computed using the ratio of the win probabilities for the two teams:

Fair Money Line

$$\textit{Fair Money Line} = \pm\frac{p}{1-p}\times 100,$$

where p is the probability of winning for the favorite. That is, the fair money line values for the favorite and underdog, respectively, are:

$$FML_{fav} = -\frac{p}{1-p}\times 100 \qquad \text{and} \qquad FML_{dog} = +\frac{p}{1-p}\times 100.$$

Using the above formula with p = .60 yields the $-150/+150$ fair money line for the Denver/Oakland example above.

When the teams are considered to be exactly evenly matched, the fair money line is $-100/+100$. The farther the win probability for the favorite is from .50, the greater will be the values in the fair money line. If, for example, if one team is only slightly favored and, say, p = .52, the fair money line would be (approximately) $-108/+108$; if the favorite is strongly favored to win and, say, p = .80, the fair money line would be $-400/+400$.

EXPECTED VALUE AND HOUSE EDGE ON MONEY LINE BETS[137]

The casino does not use the fair money line, of course, since by definition this is the line for which the expected value of the wager is equal to zero. To create a house edge, the fair money line must be adjusted. One way to do this is to subtract a small amount from each of the odds.[138] For example, subtracting 10 points from each of the two values in the original Denver/Oakland example (where p = .60 and the fair money line is $-150/+150$) results in the line $-160/+140$. Compared to the fair money line, with this *20-cent* line a Denver bettor must wager a little more ($160 versus $150) to win $100 and the Oakland bettor will win a little less ($140 versus $150) for a $100 bet.

Using the 60/40 ratio of win probabilities for Denver and Oakland, this 20-cent line results in the following expected values for the two types of bettors:

Denver Bettor: EV = .60(+$100) + .40(–$160) = –$4.00
Oakland Bettor: EV = .40(+$140) + .60(–$100) = –$4.00

These expected values[139] convert to the following house advantages:

House Edge Favorite (Denver): $4/$160 = .025, or 2.50%
House Edge Underdog (Oakland): $4/$100 = .040, or 4.00%

Although the house will keep on average $4 of each type of bet (assuming the .60 probability of winning for the favorite), it is $4 of $160 wagered on the favorite compared to $4 of $100 wagered on the underdog. From the bettor's perspective, then, the house edge with this type of money line is less for a bet on the favorite due to the larger amount of money put at risk.

If the ratio of money bet on the favorite to money bet on the underdog is consistent with the money-line odds (for the –160/+140 line, this would mean $160

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137 This section focuses on the situation where a money line is derived by subtracting a small amount from each part of the fair money line, and where the resulting line is one in which the underdog value is positive. This approach covers a great many cases and serves to illustrate the basic mathematics involved.

138 When the contest is closely matched, this approach of subtracting the same value from each side of the fair money line may lead to a "line" in which the odds value for the underdog is less than 100. In this situation, the casino sports book uses a different method in which the money line values for both sides are negative. For example, when p=.50, rather than subtracting 10 points from both sides of the fair money line of –100/+100 and offering a –110/+90 line, the casino may offer a –105/–105 line.

139 It is important to realize that these calculations assume the probability of winning for the favorite is the presumed .60 we started with. If this probability is different from .60, then the wager expectations would change. If you felt the line was "off" because the favorite had a smaller (or greater) probability of winning, then a bet on the underdog (favorite) may even result in a positive expectation. Suppose, for example, the line is –160/+140 with Denver as the favorite, but you felt that Denver only had a .55 probability of winning. Then the expected values of the two bets would be:

Denver Bettor: EV = .55(+$100) + .45(–$160) = –$17.00

Oakland Bettor: EV = .45(+$140) + .55(–$100) = +$8.00

Thus if you are correct that Denver's probability of winning is only .55, a bet on Oakland at the –160/+140 money line would carry an 8% advantage for the player (while a bet on Denver in this situation would carry a 10.6% player disadvantage). Similarly, if the probability of the favorite winning is really greater than the presumed .60, then a bet on the favorite could be a positive expectation wager.

bet on the favorite for every \$100 bet on the underdog),[140] then the overall advantage for the casino sports book is 3.08%. This can be obtained either by taking a weighted average of the two house advantages, with weights equal to the wager amounts, or by noting that the casino will keep on average \$8 for every \$260 worth of bets divided between the two teams in the \$160/\$100 ratio.

Overall House Edge on –160/+140 Money Line*

$$\frac{\$160(.025)+\$100(.040)}{\$160+\$100}=\frac{\$8}{\$260}=0.0308\text{, or } 3.08\%$$

*Assuming a 160/100 favorite-to-underdog ratio of money bet and win probability for favorite equal to .60.

Performing the same analysis but subtracting 5 points from each of the fair money-line values would result in a *10-cent*, or *dime* line of –155/+145, with associated house advantages of 1.29% for the favorite, 2.00% for the underdog, and 1.57% overall assuming the 160/100 favorite-to-underdog ratio of money bet.

Generally the process described above can be used for any probability of winning p and any adjustments to the fair money line, with the resulting house advantages computed in a similar way. Note, however, that when the contest is closely matched (i.e., when p is close to .50), the actual money line offered by the casino will be one in which both values are negative and appropriate modifications to the process described above would be needed. For example, for two teams who are considered exactly evenly matched, the probability either team wins is .50 and the fair money line is $-100/+100$. In this case the casino may offer a line such as $-105/-105$ (a 10-cent line), or $-110/-110$ (a 20-cent line). Similarly, if p = .5122, so one team is slightly favored, then the line might be a 10-cent $-100/-110$ or 20-cent $-105/-115$ line.

Assuming a c-cent line is created by subtracting $c/2$ from each of the two values in the fair money line, where p represents the probability of the favorite winning, then the expected loss for a bet on the favorite is equal to the expected loss for a bet on the underdog (as in the Denver-Oakland example), and this expected loss is given by:

$$EV = -\tfrac{1}{2}(1-p)c.$$

In the Denver-Oakland example above, p = .60, c = 20, the 20-cent line is $-160/+140$, and EV = –4.00. For this same example a 10-cent line would be $-155/+145$, with an expected loss for either favorite or underdog equal to $-\tfrac{1}{2}(1-.60)(10) = -2.00$.

General formulas for the house advantages of each wager and, assuming the appropriate ratio of money wagered on each side,[141] the overall house advantage under this approach are given by:

140 On money-line wagers, the casino desires the appropriate ratio of money bet on each side. For a –160/+140 line, for example, this ratio would be \$160 to be bet on the favorite for every \$100 bet on the underdog.

141 Id.

$$HA_{fav} = \frac{c(1-p)^2}{c + p(200-c)}$$

$$HA_{dog} = \frac{c(1-p)}{200}$$

$$Overall\ HA = \frac{2c(1-p)^2}{200 + c(1-p)}$$

Analogous formulas can be developed in terms of the money line values. Using *d* and *f* to represent the money-line odds for the underdog and favorite, respectively, and assuming *d* is positive and *f* negative,[142] these formulas are as follows:

House Advantages for Favorite and Underdog

$$HA_{fav} = \frac{100(d+f)}{f(d-f+200)}$$

$$HA_{dog} = \frac{f+d}{f-d-200}$$

$$Overall\ HA = \frac{200(f+d)}{(f-100)(d-f+200)}$$

where HA_{fav} and HA_{dog} are the house advantages, and $f < 0$ and $d > 0$ are the money-line odds, for the favorite and underdog, respectively.

Note too that it is possible to derive the presumed probability of winning for the favorite given a certain money line. Again, assuming $f > 0$ and $d < 0$ are the money lines for the favorite and underdog, respectively, and the line is derived by subtracting the same amount from both sides of the fair line, the presumed probability of the favorite winning can be computed as follows:

$$p = \frac{d-f}{d-f+200}$$

For example, if the money line is –140/+130, then $f = -140$, $d = +130$, and $p = (130+140)/(130+140+200)$ = 0.574. To verify the value of *p* for the Denver/Oakland example from above, note that $f = -160$, $d = +140$, and so $p = (140+160)/(140+160+200)$ = 0.600.

142 Analogous formulas can be derived for the (less common) case when d is negative.

The tables below show money lines and associated house advantages for a variety of scenarios.

House Edge - 10-cent Money Line Bets					
f	*d*	*p*	**HAfav**	**HAdog**	**Overall HA**
-105	-105	0.5000	2.381%	2.381%	2.381%
-110	+100	0.5122	2.217%	2.439%	2.323%
-115	+105	0.5238	2.070%	2.381%	2.215%
-120	+110	0.5349	1.938%	2.326%	2.114%
-125	+115	0.5455	1.818%	2.273%	2.020%
-130	+120	0.5556	1.709%	2.222%	1.932%
-135	+125	0.5652	1.610%	2.174%	1.850%
-140	+130	0.5745	1.520%	2.128%	1.773%
-145	+135	0.5833	1.437%	2.083%	1.701%
-150	+140	0.5918	1.361%	2.041%	1.633%
-155	+145	0.6000	1.290%	2.000%	1.569%
-160	+150	0.6078	1.225%	1.961%	1.508%
-165	+155	0.6154	1.166%	1.923%	1.451%
-170	+160	0.6226	1.110%	1.887%	1.398%
-175	+165	0.6296	1.058%	1.852%	1.347%
-180	+170	0.6364	1.010%	1.818%	1.299%
-185	+175	0.6429	0.965%	1.786%	1.253%
-190	+180	0.6491	0.923%	1.754%	1.210%
-195	+185	0.6552	0.884%	1.724%	1.169%
-200	+190	0.6610	0.847%	1.695%	1.130%

House Edge - 15-cent Money Line Bets					
f	*d*	*p*	**HAfav**	**HAdog**	**Overall HA**
-105	-110	0.5062	3.367%	3.586%	3.479%
-115	+100	0.5181	3.143%	3.614%	3.362%
-120	+105	0.5294	2.941%	3.529%	3.209%
-125	+110	0.5402	2.759%	3.448%	3.065%
-130	+115	0.5506	2.593%	3.371%	2.931%
-135	+120	0.5604	2.442%	3.297%	2.806%
-140	+125	0.5699	2.304%	3.226%	2.688%
-145	+130	0.5789	2.178%	3.158%	2.578%
-150	+135	0.5876	2.062%	3.093%	2.474%
-155	+140	0.5960	1.955%	3.030%	2.377%
-160	+145	0.6040	1.856%	2.970%	2.285%
-165	+150	0.6117	1.765%	2.913%	2.198%
-170	+155	0.6190	1.681%	2.857%	2.116%
-175	+160	0.6262	1.602%	2.804%	2.039%
-180	+165	0.6330	1.529%	2.752%	1.966%
-185	+170	0.6396	1.461%	2.703%	1.897%
-190	+175	0.6460	1.397%	2.655%	1.831%
-195	+180	0.6522	1.338%	2.609%	1.769%
-200	+185	0.6581	1.282%	2.564%	1.709%

House Edge - 20-cent Money Line Bets					
f	*d*	*p*	**HAfav**	**HAdog**	**Overall HA**
-110	-110	0.5000	4.545%	4.545%	4.545%
-105	-115	0.5122	4.242%	4.762%	4.514%
-120	+100	0.5238	3.968%	4.762%	4.329%
-125	+105	0.5349	3.721%	4.651%	4.134%
-130	+110	0.5455	3.497%	4.545%	3.953%
-135	+115	0.5556	3.292%	4.444%	3.783%
-140	+120	0.5652	3.106%	4.348%	3.623%
-145	+125	0.5745	2.935%	4.255%	3.474%
-150	+130	0.5833	2.778%	4.167%	3.333%
-155	+135	0.5918	2.633%	4.082%	3.201%
-160	+140	0.6000	2.500%	4.000%	3.077%
-165	+145	0.6078	2.377%	3.922%	2.960%
-170	+150	0.6154	2.262%	3.846%	2.849%
-175	+155	0.6226	2.156%	3.774%	2.744%
-180	+160	0.6296	2.058%	3.704%	2.646%
-185	+165	0.6364	1.966%	3.636%	2.552%
-190	+170	0.6429	1.880%	3.571%	2.463%
-195	+175	0.6491	1.799%	3.509%	2.379%
-200	+180	0.6552	1.724%	3.448%	2.299%

The preceding discussion focused on the case where the money line is derived by subtracting an equal amount ($c/2$) from each part of the fair money line and the resulting line for the underdog, d, is positive. This scenario should give the reader an understanding of the mathematics of money lines and the associated house advantages.

OTHER TYPES OF SPORTS BETS

Although the point spread and money line are the two primary types of sports wagers, there are many other ways sports books offer betting opportunities. Many of these are summarized below.

Totals (Over/Under) – An over/under is a bet on the total points scored by both teams in the contest. The bettor wagers that the total will either be over or under the specified amount. Typical over/under numbers might be 36-42 in football, 190-220 in basketball, and 7-10 in baseball

Parlays – A parlay is making a single wager on the outcome of a combination of two or more contests in which all of the bettor's choices must win for the parlay to win. Parlays can be cheap to bet, with big rewards if won, but they usually lose. The payoff odds on a parlay are usually not as good as parlaying yourself on the separate games.[143] The table below shows the probabilities of winning parlays for different numbers of propositions and the house edge assuming typical payoffs.

Parlays – House Advantage

# of Propositions	*Typical Payoff*	*Probability of Winning Parlay**	*True Odds*	*House Edge*
2	13 to 5	1/4	3 to 1	10.0%
3	6 to 1	1/8	7 to 1	12.5%
4	12 to 1	1/16	15 to 1	18.75%
5	25 to 1	1/32	31 to 1	18.75%
6	50 to 1	1/64	63 to 1	20.31%
7	100 to 1	1/128	127 to 1	21.09%
8	200 to 1	1/256	255 to 1	21.48%
9	400 to 1	1/512	511 to 1	21.68%
10	800 to 1	1/1024	1023 to 1	21.78%

*Assuming a 50% probability of winning on each proposition.

143 Obviously parlaying separately yourself on the separate games is not possible if the games are played at the same (or overlapping) times.

To illustrate the mathematics in the above table, consider a three-proposition/team parlay. The probability of winning, assuming a 50% chance on each proposition is $(1/2)\times(1/2)\times(1/2)=1/8$, so for a \$50 parlay bet with three propositions, the expectation is:

$$EV = [(1/8)(\$300)] + [(7/8)(-\$50)] = -\$6.25.$$

Dividing this expectation by the amount of the wager and dropping the negative sign yields the house edge: (\$6.25/\$50) = .125, or 12.5%. Other house advantages can be computed similarly.

Teasers – A teaser is a parlay where the point spread is adjusted in the bettor's favor, with a corresponding drop in the payout.

Futures – A future is a bet on pennant, division, conference, or championships, such as the World Series or Super Bowl.

Propositions – A proposition bet can be almost anything. Who will score the first touchdown? Who will catch the most passes? Who will hit the first home run? Who will have the most rushing yards?

OTHER SPORTS BETS ISSUES

Middles – A middle can occur when two sports books have different lines or when the line moves from opening spread.[144] For example, Suppose Bookie A has Packers as 10 point favorites over Bucs while Bookie B has the Packers as 8½ point favorites over Bucs. A bettor places a \$1100 bet (to win \$1000) on the Bucs with bookie A and another bet of \$1100 (to win \$1000) on the Packers with Bookie B. If the Packers win the game by 9 points, this bettor has "middled," winning both bets for a profit of \$2000.

> Poker and American history are inseparable.
>
> John McDonald
> *Strategy in Poker, Business, and War* (1950)
>
> The game exemplifies the worst aspects of capitalism that have made our country so great.
>
> Walter Matthau
> Quoted in David Spanier, *Total Poker* (1977)

144 In point-spread betting, the spread will fluctuate up until game time to attract bettors to the side that has been under-bet and try to ensure equal amounts bets on both sides.

GAME ODDS AND PRICE SETTING

6

Unlike most other consumer purchases where the price of a product is fixed, the price of a casino game depends on the rules of the game, payoffs for winning wagers, the amount of time the player plays, and possibly the skill of the player. In this sense, the pricing in the casino gaming business is made by conscious or unintentional decisions on setting the rules for the games. This chapter explores the business and economics of game odds and pricing.

PRICE SETTING IN THE GAMING INDUSTRY

Casinos set prices on games by setting the odds. This may be accomplished through rule variations that are more or less favorable to the player, by altering the payoffs on certain wagers, or by offering a different type of game product. Regardless of the method, the house advantage determines how much it costs a player to play a particular game.

RULE VARIATIONS

Examples of games in which rule variations can affect the odds are blackjack and craps. In blackjack, the dealer hitting a soft seventeen increases the house advantage 0.2% compared to a comparable game where the dealer must stand on soft seventeen. The number of decks used, no soft doubling, and no re-splitting of pairs are other examples of rule variations in blackjack that affect the overall price of the game to the player.

The "free odds" bet in craps can vary the price of the game product by the amount of odds that can be taken. A player who bets the pass line and takes single odds is at a 0.85% disadvantage, but only a 0.61% with double odds, and 0.47% with triple odds. This means for every $100 wagered on the pass line with single odds ($50 pass line and $50 odds), on the average, the player will pay a price of about $0.85 while $100 bet on the pass line with triple odds ($25 pass line and $75 odds) will cost about $0.47. A casino allowing triple odds offers a better priced craps game than one that permits only single odds. Some casinos have offered as high as 100X odds – a player taking full 100X odds will face only a 0.02% house advantage on the combined pass line (or come) and odds wagers.

ALTERING PAYOFFS

Another way for casinos to set prices is by altering the payoffs on wagers. The field bet in craps, for example, typically offers even money on the 3, 4, 9, 10, and 11, and a 2 to 1 payoff for the 2 and 12. The house advantage for this payoff structure is 5.56%. If either the 2 or 12 (but not both) were to pay triple, the house advantage would be reduced to 2.78%. If both the 2 and 12 paid 3 to 1, the field bet would be "free," with house advantage equal to zero.

The usual casino commission on winning banker bets in baccarat is 5%. Some casinos attempt to attract and keep players by lowering this commission to 4%, or even 3% (setting the commission at only 2% would give the player the advantage). Such a reduction is equivalent to raising the payoff on this bet from 95% to 96% or 97%, with the effect of reducing the house advantage from the usual 1.06% to 0.60% (with the 4% fee) or 0.14% (3% fee).

Keno and video poker are other examples where payoffs can be easily manipulated to raise or lower the price of the game. Payoffs for various keno wagers vary widely across casinos. A full-pay Jacks-or-Better video poker machine pays 9 for 1 and 6 for 1 for a full house and flush, respectively (hence the moniker "9/6" in reference to this game), but machines paying only 8 for 1 and 5 for 1 for these hands ("8/5" machines) are common. The former returns 99.54% with perfect play while the maximum payback on the latter is 97.29%.[145] Some casinos have offered a version of Jacks-or-Better with a 7 for 1 payoff on flushes, giving the player a greater than 100% return with perfect play.

SLOT MACHINES AND PRICING

Early slot machines were mechanical and the odds of winning depended on the number of reels, number of stops on each reel, and payouts for the winning combinations. For example, if a slot machine had 3 reels with 20 stops each, and each stop had a different symbol, there would be (20 x 20 x 20) = 8,000 possible combinations. If 30% of these combinations were winners with a combined total payout of 7,500 coins, the hold percentage (house advantage) would be (8,000 – 7,500)/8,000 or 6.25%. The average price of playing this machine would be $6.25 for every $100 bet. Casino operators could adjust this price by changing the number of winning combinations, changing the symbol configurations on the reels, or changing the payout for winning combinations. With today's microchip-controlled slots, the odds of winning can be adjusted merely by altering the computer program that runs the gaming device.

OFFERING A DIFFERENT GAME PRODUCT

Variations on standard games, or completely new games altogether, are part and parcel of the product mix for many casinos. Games like Double Exposure and Spanish

145 These figures assume a basic full-pay schedule giving a 4,000-coin jackpot for a royal flush with five coins played.

21 are just twists on blackjack, and Three Card Poker, Let It Ride and Caribbean Stud are variations on poker. These differing game products offer players a wide variety of prices to pay for their gaming experience.

Casinos can compete on the price of roulette by offering the single-zero game, which costs the player about \$0.27 for every \$10 wagered, compared to the double-zero wheel with a price of about \$0.53 per \$10 bet.

OTHER FACTORS AFFECTING GAME PRICES

A factor that is not in the casino's control that affects game price is player skill in those games involving both chance and skill. In a typical six-deck game, for example, the average blackjack player gives about a 2% edge to the house but a basic strategy player is at a 0.5% disadvantage.

Game speed, or number of decisions per hour, affects the cost per hour to the player (and the expected casino win per hour), although it does not alter the basic price per unit wagered (i.e., the house advantage). In roulette, for example, a \$5-per-spin player betting at a double-zero table making 40 spins per hour can expect to pay (\$5 x 40 x .0526) = \$10.52 per hour. At a table completing 60 spins each hour, the same player would spend an average of \$15.78 per hour. The basic price in both cases is 5.26% of the amount wagered. Similarly, at a base price of 1.15%, a \$100-per-hand baccarat player will pay, on average, about \$92 per hour if playing 80 hands per hour, but it will cost this same player \$161 per hour if dealt 140 hands per hour.

MAKING SURE THE PRICE IS RIGHT: CASINO PRICING MISTAKES

Because pricing decisions in the casino require mathematical calculation, either a failure to make such calculations or errors in doing so can have negative consequences. One casino wanted to increase their baccarat play and so lowered the commission on winning banker bets to 2%. It was not long before the players made them pay for this mistake. As the following expected value calculation shows, this 2% commission gives the player a 0.32% advantage:

$$EV = (+.98)(0.4462) + (-1)(0.4586) = 0.0032.$$

Another promotion gone awry occurred in Mississippi when a casino offered 80 to 1 payoffs, instead of the usual 60 to 1, for bets on the total 4 and the total 17 in Sic Bo. With true odds of 71 to 1, the player has a huge 12.5% advantage with the altered payoff, compared to a 15.3% disadvantage with the usual payoff:

$$EV = (+80)(1/72) + (-1)(71/72) = +0.125.$$

With this advantage, a player betting \$100 per hand at 60 hands per hour can expect to win \$750 per hour. A \$500 bettor at this rate would take in \$3,750 per hour.

An Illinois riverboat reportedly lost \$200,000 in one day with a "2 to 1 Tuesdays" promotion in which blackjack naturals paid 2 to 1 (instead of the usual 3 to 2). A similar promotion with similar results occurred at a Las Vegas Strip casino. Without other compensating rule changes, paying naturals 2 to 1 can increase the player expectation enough to give the player about a 2% advantage over the house.

These examples serve to illustrate that casinos need to be careful when altering payoffs or varying rules to offer a more or less attractive game. Payoffs that are too large or rules that are too player favorable could give the players an advantage and cost the casino dearly. On the other hand, if payoffs are too low or rules too house friendly, players will not want to play. Proper mathematical analysis should be performed to ensure that payoffs and rules of the game are attractive to players and acceptable to the casino.

PRICE AND PRODUCT DEMAND

In most businesses, if the operator lowers the price of his goods or services, the volume of sales will increase, provided that his competitors also do not lower their prices. Whether the increased sales volume will justify the reduced price, however, is often unknown.

A casino manager may consider lowering the house advantage and therefore the price to the player to increase volume of play. In some cases, the motivation for this is to keep the tables full. Here the casino manager may see idle tables and come to the conclusion that he is better off filling the empty tables at worse odds because the casino already is paying for the pit bosses, dealers and others.

But, ultimately, the casino manager needs to consider the impacts of lowering the price, via a lower house advantage, on operating profits. Unfortunately, this is not a simple equation – a twenty percent decrease in price is not necessarily made up by a twenty percent increase in volume.

ODDS BET IN CRAPS

Using game pricing as a marketing tool is common. In the early 1990s, billboards along the Las Vegas highways were evidence of a price war on casino craps odds, with one casino ultimately trumping the others that offered 5X, 10X and then 100X odds by offering unlimited odds. These more extravagant promotions were eventually terminated for the reason that actual earning potential per hour based on the same player with the same average combined (pass line plus odds) bet per hand substantially decreases as the back line odds increase and the front line bets decrease.[146] This phenomenon is illustrated below.

	2X ODDS		5X ODDS		10X ODDS	
	Bet	*Win*	*Bet*	*Win*	*Bet*	*Win*
Pass Line (1.41%)	$110	$1.56	$55	$0.78	$30	$0.42
Odds (0%)	$220	$0.00	$275	$0.00	$300	$0.00
TOTAL	$330	$1.56	$330	$0.78	$330	$0.42
Avg. Bet Size*	$257		$238		$230	
**Average bet size is 2/3 of pass line amount plus 1/3 of total amount.*						

As can be seen from the table, the expected casino win on the same $330 total wager decreases from $1.56 for double odds, to $0.78 for 5X odds, to $0.42 for 10X odds. This is because with higher odds, the player can put more of the $330 on the back line odds and less on the pass line. Since the back line odds are "free" and only

146 Assuming the player takes full advantage of the odds offered.

the pass line portion of the combined amount is subject to a casino edge (1.41%), the overall casino advantage and expected win decrease.[147] The figures in this example are for one $330 wager – earning potential per hour differences are much more substantial.

SINGLE DECK BLACKJACK

Another major promotional game is single deck blackjack. In *Casino Ops Magazine*, Bill Zender took on the proposition that offering a few single deck blackjack games may help attract potential blackjack players. This may be true, he concluded, but the reduced house advantage coupled with the propensity to attract more knowledgeable players would require 30% more traffic on these games to have a positive economic effect. He also concluded that to justify the same gross gaming win, the casino would need to have about five times the volume just to break even because the house advantage is significantly reduced in single deck blackjack compared to six-deck blackjack. With average or poor players, the differences are not as extreme.

Still, the example shows the need for the casino manager to understand certain principles. The first is how simply altering rules impacts earning potential. The second is that increased volume from lowering the house advantage requires more personnel to deal, supervise and control the games. Likewise, he needs to understand the cost of complimentaries, particularly beverages, may increase with the additional volume. If a casino is not operating to full capacity, it may be better to reduce the work force than to lower house odds.

CASINO GAME PRICES

For most games, the more players at a table, the slower the pace of the game and the fewer the decisions per hour. The table on the next page shows the price per hour of some common games and wagers under typical conditions. These figures can be used as a rough guide to what a casino might expect to win in a given period of time for a given game. If casino managers know the average number of visitors and average hours gambling on a game per visitor during a 24-hour period, the average casino win per day can be calculated. For example, for a hypothetical casino with 800 visitors in a 24-hour period and each visitor gambling an average of four hours, if activity were only in double-zero roulette, the casino would net an average of $33,664.

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147 The overall casino edge on the combined pass line and odds wagers is equal to the expected casino win divided by the average bet size, which is less than $330 since one-third of the time (when a point is not established) only the pass line amount is at risk.

Prices of Games & Wagers under Typical Conditions				
Game	*Bet Amount*	*Pace*	*House Adv.**	*Price/Hr*
ROULETTE				
Double-zero	$5	40/hr	5.26%	$10.52
Single-zero	$5	40/hr	2.70%	$5.40
CRAPS				
pass	$5	30/hr	1.41%	$2.12
w/single odds	$5	30/hr	0.85%	$1.28
w/double odds	$5	30/hr	0.61%	$0.92
w/triple odds	$5	30/hr	0.47%	$0.71
Field (2x on 12)	$5	60/hr	5.56%	$16.68
Hard 4, Hard 10	$1	30/hr	11.11%	$3.33
Hard 6, Hard 8	$1	30/hr	9.09%	$2.73
Big 6, Big 8	$5	30/hr	9.09%	$13.64
Any Craps	$1	60/hr	11.11%	$6.67
Any Seven	$1	60/hr	16.67%	$10.00
Horn bet	$4	60/hr	12.50%	$30.00
BLACKJACK				
Average player	$5	60/hr	2.0%	$6.00
Poor player	$5	60/hr	4.0%	$12.00
Basic strategy, 1 deck	$5	60/hr	0.0%	$0.00
Basic strategy, 2 decks	$5	60/hr	0.3%	$0.90
Basic strategy, 6 decks	$5	60/hr	0.5%	$1.50
BACCARAT				
Banker	$100	60/hr	1.06%	$63.60
Player	$100	60/hr	1.24%	$74.40
MINI-BACCARAT				
Banker	$5	100/hr	1.06%	$5.30
Player	$5	100/hr	1.24%	$6.20
CARIBBEAN STUD				
Ante	$5	40/hr	5.22%	$10.44
Bonus	$1	40/hr	45%	$18.00
LET IT RIDE				
Ante	$5	40/hr	3.51%	$7.02
Bonus	$1	40/hr	24%	$9.60
PAI GOW POKER	$5	40/hr	2.84%	$5.68
THREE CARD POKER				
Ante/Play	$5	40/hr	3.37%	$6.74
Pair Plus	$5	40/hr	2.32%	$4.64
CASINO WAR	$5	150/hr	2.88%	$21.60
SLOTS				
5¢ machine	5¢ x 3	500/hr	8%	$6.00
25¢ machine	25¢ x 3	500/hr	6%	$22.50
$1 machine	$1 x 3	500/hr	5%	$75.00
VIDEO POKER				
Jacks or better 8/5	25¢ x 5	500/hr	2.71%	$16.94
Jacks or better 9/6	25¢ x 5	500/hr	0.46%	$2.88
KENO	$1	6/hr	27%	$1.62

See Appendix for a list of house advantages for most games and wagers ranked from lowest to highest.

VOLATILITY AND CASINO OPERATIONS 7

Understanding volatility has many useful applications to a casino setting. From the perspective of net win, most casinos will attempt to minimize volatility so that the win from the casino will remain predictable. Occasionally, however, some casinos may decide to actually go from the gaming to the gambling industry and accept wagers that are highly volatile, and offer the prospects for large profits or losses. This is a contrast between the casino grind and high rollers.

The second major reason for a good understanding of volatility is that it is a measuring tool for casino performance. In particular, is the casino performing in the manner that mathematics dictates it should?

HIGH VOLATILITY: THE CASE OF HIGH ROLLERS

The apex of volatility in the casino world is the high roller. A "high roller" is relative to the typical player at the casino. A "twenty thousand dollar player," i.e., a player capable of winning or losing $20,000 in a single visit to a casino, would be a high roller at a rural casino in Elko, Nevada. The same player at the Venetian in Las Vegas, however, would simply be a good player. This is because a $20,000 win or loss at the Venetian, which has a monthly gross gaming win of over $40 million, is statistically unimportant. In contrast, it would be statistically relevant to a casino whose monthly gross gaming revenues were less than one million dollars.

The highest of the high rollers are the "whales" – players whose gambling habits are statistically relevant to even the largest casino companies. Calling them high rollers is inadequate. They often tip more than some high rollers spend. They get virtually anything they want. Complimentary suites for an entourage of a dozen are not a problem. A side vacation to Disneyland is arranged gratis. A private jet to ferry the group around is always fueled and waiting. Free everything – food, shows, gifts of diamonds and gold. All except the gambling. Here the credit lines can be ten million dollars, only because some credit line must be established. Some bet a hundred thousand dollars a hand. They are simply called whales, because they are the biggest catch in the sea.

Only a few casinos dare deal to a whale. He can make the casino rich, or he can make the casino poor. During one trip he can beat you out of ten million dollars, the next he might lose as much.

THE TALE OF A WHALE

> No victor believes in chance.
>
> Friedrich Nietzsche
> *Aphorisms* (1878)

Casino folklore includes a story about one casino that should not have been dealing to a whale. The casino had a bankroll of only about $3,000,000. The whale hit a winning streak, laying claim to the entire bankroll. The casino, in essence, was busted. "No fear," thought the operator, "the tables must turn and the casino will win it back before the night is over." Unfortunately, the night ended early for the casino. The whale decided to cash out and fly home. A Lear jet was waiting. Ironically, the casino was paying for the jet that was going to take the casino's entire bankroll to the Land of the Rising Sun.

Out of desperation, the operator told the junket representative that he had to think of any way to keep the whale in Las Vegas. "Please stay an extra night," went on deaf ears. The whale was going to leave a winner.

Then the representative came up with the idea. He asked to accompany the whale on the jet, claiming to have to attend to business across the ocean. "No harm," thought the whale, "after all, the casino was paying for the jet." His entourage, with the representative in tow, loaded up the limousines, drove to the airport and boarded the plane.

Sitting on the runway, the representative mentioned to the whale that he heard something unusual coming from the jet's engines and had a funny feeling about flying in the plane. The whale, being superstitious, immediately wanted nothing to do with the Lear jet. "No problem," said the representative. The casino could have another jet available at daybreak. The whale could spend the night back at the hotel.

The whale had another plan. Unexpectedly, he suggested the group fly commercial as a trans-pacific flight was leaving that night from Los Angeles. It was an unusual request, because whales rarely fly commercial.

A new dilemma confronted the representative. He knew that some airlines flew every hour to Los Angeles. The whale could have taken any of these flights. Unaware of the frequency of flights, the whale allowed the representative to lead him to an airline that had only one flight to Los Angeles and a long line of persons waiting to check in. By the time the group got to the front of the line, the plane was sold out. Perhaps, suggested the junket representative, the whale could go back to the hotel and take a morning flight. He agreed, but indicated that he would not gamble. The representative suggested instead that they go to a dinner and a show. The whale saw no harm in that plan.

After a nice dinner and the show, the group made its way back to the hotel, through the casino and past the baccarat pit. The whale hesitated just long enough to get hooked. "Just a few hands before retiring for the night," he thought. A few hands turned into a few hours. Luck abandoned him and the law of large numbers became his tablemate. He soon lost back the entire $3,000,000 that he won and another $3,000,000 on top of that. The casino was saved that day. The whale, temporarily discouraged, would return.

REDUCING VOLATILITY TO REDUCE THE RISK

Casinos that deal to high rollers or even "midrange" players must understand volatility with their games. A single player, like a whale, playing a limited number of hands has a much greater chance of leaving the casino a winner than thousands of "grind" players that play the same amount of money but in smaller increments.

This volatility can hit even the largest casinos. For example, a few years back, the parent company of Caesars Palace had to publicly announce that the company was going to fall short of earnings expectations because of losses sustained in its baccarat pit, generally the domain of the high rollers.

Casinos have three options when faced with high rollers – don't deal to them, gamble with them, or develop strategies to reduce volatility created by their betting habits. One strategy utilized by an Atlantic City casino in 1990 involved a "freeze-out" game.[148]

THE FREEZE-OUT

In February 1990, another Japanese whale who had been betting at very high levels at casinos around the world, won $6 million from the Trump Plaza in Atlantic City and $19 million at the Diamond Beach Casino in Darwin, Australia. Upon returning to the Trump Plaza in May, the casino accepted this whale's challenge to play baccarat at $200,000 per hand on the condition that he do so until he was either ahead $12 million or behind $12 million. This condition assured that the whale would have to play many hands to ultimately beat the casino and thereby reduced the game volatility and risk to the casino. As it turned out, the game was terminated after 70 hours of play and the whale was behind by $9.4 million, although it is not clear who ended the challenge.[149]

One way to see how this freeze-out game reduces the casino's risk is to compute the probability that the player (or casino) will win if the game is carried to completion. Calculations using a "gambler's ruin" model (see below) show that the probability the player would be the ultimate winner in the above-described freeze-out is 18.1% compared to 81.9% for the casino. That is, the player will win such a challenge less than one in five times.

Another way to look at the effect of the freeze-out is to consider the likelihood of various possible outcomes after 70 hours worth of baccarat, the approximate duration of the whale's play. Assuming a house advantage of 1.15% and wager standard deviation of 0.94 per unit (both of these figures are about the average of the respective

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148 This and several other cases intended to highlight management decision-making considerations with regard to high rollers are discussed in William Eadington and Nigel Kent-Lemon, "Dealing to the Premium Player: Casino Marketing and Management Strategies to Cope with High Risk Situations," in Gambling and Commercial Gaming: Essays in Business, Economics, Philosophy and Science (1992).

149 Robert Johnson, "Tale of a Whale: A Gambler Wins, Loses Millions – and Duels Trump," The Wall Street Journal, June 28, 1990.

player and banker values, ties included), for a bet size of $200,000 per hand, and 80 hands per hour, the expected value and standard deviation of the total casino win after 70 hours (5,600 hands) are:

$$EV_{win} = \$200{,}000 \times 5{,}600 \times .0115 = \$12{,}880{,}000$$

and $$SD_{win} = \$200{,}000 \times \sqrt{5{,}600} \times .0115 = \$14{,}068{,}632\ .$$

With these values, standard normal distribution calculations[150] can be used to obtain the probability of a win in any specified range. These calculations show, for example, the probability that such a player will be behind by $12 million or more is 52.5%, while the probability this player will be ahead by $12 million or more is only 3.8%. There's about an 18% chance the player will be ahead at all after the 5,600 hands. The following table gives a more complete picture of the likelihood of various outcomes.

Casino Win after 5,600 Hands of Baccarat at $200,000 per Hand	
Range ($millions)	*Probability*
Less than −24	0.4%
−24 to −18	1.0%
−18 to −12	2.4%
−12 to −6	5.1%
−6 to 0	9.0%
0 to 6	13.2%
6 to 12	16.3%
12 to 18	16.7%
18 to 24	14.3%
24 to 30	10.3%
30 to 36	6.2%
36 to 42	3.1%
More than 42	1.9%

The figures in the above table show how the casino's risk is reduced when a player plays such a large number of hands (in a negative expectation game). A freeze-out game, of course, is intended to do just that.

Table Limits and Volatility

Since volatility associated with total win increases as the size of the wager increases (recall the formula for the standard deviation of the win amount is

150 These calculations can also be performed in terms of the number of units won, which has expected value 64.4 and standard deviation 70.34. At $200,000 per hand, the $12 million dollar stopping value equates to 60 units.

$SD_{win} = unit\ wager \times \sqrt{n} \times SD_{per\ unit}$), casinos can reduce volatility by limiting the size of the wagers that players can make. This is one of the reasons for table limits.[151]

LOW VOLATILITY: THE CASE OF THE GRIND JOINTS

The opposite of the casinos that deal to high rollers are the grind joints. They acquired their name by earning their casino win from small bets with large volumes. These casinos generally feature a large number of slot machines with lower denominations and low limit table games. To be successful, these casinos must have a large number of players that collectively have a large volume of small bets.

A benefit of this approach is consistency in actual win rate. For the opposite reason, that high rollers cause high volatility, low rollers create low volatility. This is because of the law of large numbers, which dictates that the more bets made, the closer the proportion of money wagered retained by the casino will be to the house advantage. Therefore, grind joints should realize actual win rates that more consistently approach the theoretical win rate than casinos that cater to premium customers. Of course, the challenge to the grind joints becomes to insure that they handle the volume of bets necessary to produce net gaming revenues sufficient to pay expenses and taxes and make a profit.

GAMBLER'S RUIN, VOLATILITY, AND THE FREEZE-OUT

The probabilities of each side winning a freeze-out played to completion can be computed using a "gambler's ruin" model.[152] The general version of this problem can be stated as follows. A gambler starts off with $\$a$ and plays against an adversary who has $\$(n-a) > 0$. The plays are independent and at each play, the gambler wins \$1 goes with probability p and loses \$1 with probability $q = 1-p$. Play continues until one of the two players go broke, so that the winning player ends up with $\$n$. What is the probability of the gambler's ultimate ruin, and what is the probability of the gambler's ultimate success? Letting $r = q/p$, these two probabilities are given by:

$$P(ruin) = \frac{r^a - r^n}{1 - r^n} \quad \text{and} \quad P(success) = \frac{1 - r^a}{1 - r^n}, \quad \text{if } p \neq q,$$ [153]

and

$$P(ruin) = \frac{n - a}{n} \quad \text{and} \quad P(success) = \frac{a}{n}, \quad \text{if } p = q.$$

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151 Table limits also ensure that the "doubling up" betting system (also called the martingale system) is doomed to fail. Players attempting this system double their bet after each loss so that when they win they will be "assured" of a one-unit profit. Unfortunately for the player, a streak of bad luck in the form of several successive losses (for a large sum of money) will spell disaster when the next bet cannot be doubled due to the table limit maximum.

152 See, for example, Richard Epstein, The Theory of Gambling and Statistical Logic (1995), Peter Griffin, Extra Stuff: Gambling Ramblings (1991), or Edward Packel, The Mathematics of Games and Gambling (1981), for further details on gambler's ruin.

153 If p<q and \$a is "large," the success probability is about (p/q)n−a. If p>q and \$n is "large," the ruin probability is about (q/p)a.

As an example, suppose a player enters a casino with \$100 and makes \$1 color bets in roulette until winning \$20 or going broke, whichever happens first. The gambler's ruin model can be applied as though the casino is the adversary who will go broke after a loss of \$20. From the gambler's point of view, the probability of winning a color bet is 18/38 and probability of losing is 20/38. Hence, $a = 100$, $n = 120$, $p = 18/38$, $q = 20/38$, and $r = 20/18$, or about 1.1111. The probability of ruin before winning \$20 is:

$$P(ruin) = \frac{(1.1111)^{100} - (1.1111)^{120}}{1 - (1.1111)^{120}} \approx 0.878.$$

This means the probability of winning \$20 before losing all \$100 is about 0.122. If the same gambler chooses to play the pass line (only) at craps, with a slightly higher $p \approx .493$, then the probability of winning \$20 before going broke increases dramatically to 0.556. For a fair game, where $p = .5$, the probability of the \$100-bankroll player winning \$20 before ruin is 0.833. These examples show that for values of p not too far from .5, a small shift in the probability of winning a single play can have a considerable impact on the probability of ruin.

DOUBLE OR NOTHING

> **Gambler's ruin . . .**
>
> This is the simple proposition that all other things being equal, the gambler with more money in a bust out game, i.e., where play continues until you win or lose all your money, is more likely to prevail over the gambler with less money. Of course, casinos not only have more money than almost all players, they also possess a house advantage to assure that all things are not equal.

A special case of the gambler's ruin problem that can be applied to the freeze-out game results when the gambler will play until either doubling the initial \$$a$ bankroll or going broke. This version can be formally stated as follows: A gambler starts with \$$a$, and on each of a series of independent wagers wins \$1 with probability p and loses \$1 with probability $q = 1-p$. The game continues until the gambler either doubles the initial \$$a$, or loses it all and is ruined.

Analogous to the example above, apply the gambler's ruin model as though the casino will go broke after the gambler doubles the \$$a$. Noting that $n = 2a$ for this "double or nothing" version, after a little algebra the following probabilities of ruin and success can be obtained from the formulas for the general model:

$$P(ruin\ before\ doubling) = \frac{q^a}{p^a + q^a},$$

and

$$P(doubling\ before\ ruin) = \frac{p^a}{p^a + q^a}.$$

These formulas assume each play results in the winning or losing of one unit[154] and that the gambler will quit if the initial capital is doubled. If a is large, then when $p>q$, the probability the (advantaged) gambler is ruined before doubling is about equal to $(q/p)^a$, the formula for the probability of ultimate ruin if the gambler plays forever (i.e., does not stop if the initial capital is doubled). If $p<q$, the probability that the (disadvantaged) gambler is ruined before doubling is about $1-(p/q)^a$, for large a. [155]

APPLYING GAMBLER'S RUIN TO THE FREEZE-OUT

In the freeze-out game described above, the whale, betting $200,000 per hand, was to play until he was either ahead $12 million or behind $12 million. Ignoring ties, a bet on player will win one unit with probability 0.4932 and lose one unit with probability with 0.5068. Thus, if always betting on the player, the whale's chances of prevailing in the freeze-out are given by:

$$P(doubling\ before\ ruin) = \frac{(.4932)^{60}}{(.4932)^{60} + (.5068)^{60}} \approx 0.164\text{, or } 16.4\%.$$

A similar calculation, adjusting for the 5% commission on winning banker bets, shows that the gambler has about a 19.9% chance of prevailing if always betting on the banker.[156] Assuming a relatively equal mix of player and banker bets, the probability that the whale would be the ultimate winner in the freeze-out, if played to completion, is about 18.1% and the casino's chances of prevailing, then, are about 81.9%, the figures cited earlier.

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154 See Peter Griffin, Extra Stuff: Gambling Ramblings (1991), for approximations in cases where the gambler may win or lose some amount other than a single unit, or for games with more than two possible outcomes.

155 How large a needs to be so that these approximations are good depends on the values of p and q. The closer the game is to being fair (i.e., the closer p and q are to .5), the larger a needs to be. For example, if p=.6, a value of a=10 yields exact and approximate ruin before doubling probabilities of .0170 and .0173, respectively. For p=.507, however, even at a=100 the exact ruin before doubling probability is .0573 versus .0608 using the approximation.

156 The values obtained from the approximate formulas assuming a is large are 19.6% and 24.9% for player and banker bets, respectively.

Gambler's Ruin – Government Style

The concept is simple enough. If the government imposes a limit on how much a person can gamble on each wager, then the potential amount that that person can lose in a given amount of time also is limited. This is the logic behind the $5 per hand bet limitation imposed in Colorado and elsewhere. The problem, however, is that the likelihood that a person will come out a winner decreases as the number of wagers increases. Take three gamblers each playing roulette, betting even money propositions and each with a $200 bankroll that they intend to play until they lose it all or double it. The first player bets $200 and has a 47.4% chance of winning $200 before losing $200. This is because on just one spin of the wheel he will either lose the entire $200, or win and double his bankroll. The second player bets $25 per hand and has a 30.1% of winning $200 before losing $200. The third player with a $5 bet limit has only a 1.5% of winning $200 before losing $200. The results are similar for other games.

COMPLIMENTARIES

8

A casino's most valuable assets are its players. The player base is the economic engine that drives revenues. An important but difficult problem for casino management is to determine the value of each player. This determination is important for many reasons. It allows the casino to develop typical player profiles and to decide game mixes that appeal to its players to assure the casino maximizes its profits. For example, if baccarat players are the most profitable, the casino should have sufficient space to meet its players' demands for baccarat tables.

Assessing a player's worth is also necessary for effective marketing programs that attempt to attract and retain players. These marketing programs often allow for targeting marketing dollars at certain segments of casino players. These can include promotions like Superbowl parties or golf tournaments for players, but most often are associated with providing free rooms, food and beverage and other complimentaries, also known simply as "comps." To be successful the promotion should both bring in more money than it costs and be the best use of the casino's money.

This chapter shows how casinos determine player-worth based on betting patterns, and how the casino can design programs for complimentaries that maximize potential revenues.

Value: Theoretical Versus Actual Win

The value of a particular player to the casino is simply how much the casino can expect to win from that player. This is often called earning potential or theoretical win.

This should not be confused with actual win. Consider two players that visit a casino for the first time. Casino management, not knowing either player, observed the first gambling for a few hours at roulette, betting $200 per spin and losing $2,000. The second also was observed playing roulette for a few hours, betting $400 per spin and winning $4,000. The two players approach the casino manager and ask for the last available ticket to a casino-sponsored concert that night. Who should get the ticket?

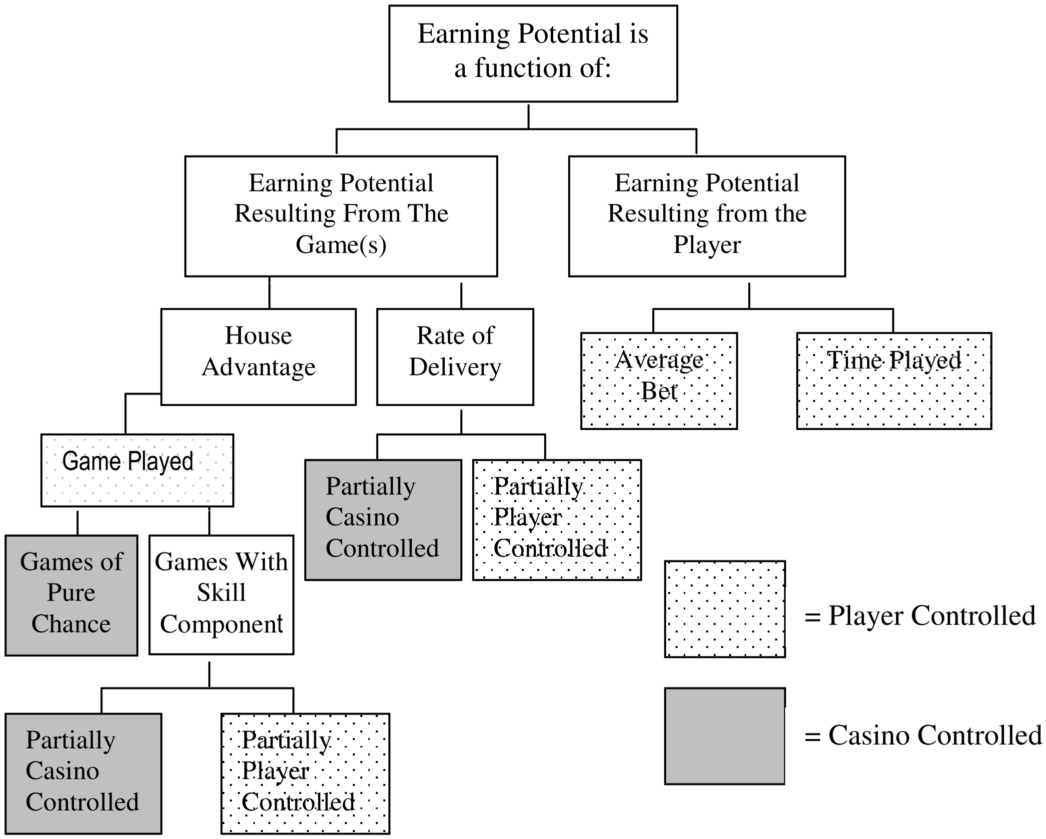
The correct answer is the player that won the $4,000, but not necessarily because he happens to have $4,000 of the casino's money. All other things being equal, although he happened to win, the casino has a better chance of winning more money from him over time due to his $400 per spin wagers compared to the other player's average of $200 per spin. While the house advantage over each player is the same over time, the total value of the $400 average bettor is greater to the casino.

As this example shows, what a casino actually wins (or loses) will often differ greatly from its theoretical win projection due to statistical fluctuations (see Chapter 1 for more on statistical fluctuations).

Casino management needs to understand the factors that go into calculating player value beyond the mere formula for determining theoretical win. Recognizing which factors go into earning potential that are under the control of the casino and which are under the control of the player is particularly important.

With this knowledge, casino management can both make adjustments in procedures that can increase the prospect of maximizing player value and make adjustments to formulas that estimate player value to assure they more accurately reflect actual player value. Both of these areas are discussed in greater detail in this chapter.

The following chart provides an expanded explanation of player value:



This chart illustrating theoretical win shows two important factors that are within the control of the casino: the house advantage and, in part, the rate of delivery (often expressed as hands or plays per hour). The player, on the other hand, determines the game he plays, his skill level in games with a skill component or with multiple betting options with different odds, average bet and time played. The player can also influence the rate of delivery by either playing slow or opting for a crowded table over a less crowded table.

CASINO CONTROLLED FACTORS

Casinos can control player value by changing the house advantage. Simply, reducing the house advantage will lower the player value of those players taking advantage of the lower house advantage. Likewise, the opposite is true. While this may seem obvious, casinos may make marketing decisions they hope will attract additional players without adequately anticipating the resulting decrease in player value. For example, a casino could decide to implement double-deck blackjack games for which the basic strategy house advantage is 0.32% compared to 0.54% for six-deck. When it does, it needs to consider that (a) the volume of play will need to increase, often substantially, to meet the theoretical win generated before the change, and (b) the individual worth of the blackjack players who take advantage of the double-deck games will be less than before the rule change. For example, all other things being equal, a blackjack player betting a total of $12,000 and playing a six-deck game is worth about the same as blackjack player betting a total of $20,000 and playing a double-deck game.

Another factor in player value that is partially casino controlled is rate of delivery, or hands per hour. This can apply to both table game play and slot machines. With table games, it refers to how the casino can increase the speed of play by dealing more hands per hour. This could involve the use of automated shuffle machines, dealer training or game procedures. With slot machines, it is the speed which the machine displays the play outcome and the time needed for the player to activate a new play.

PLAYER CONTROLLED FACTORS

Most factors that impact player value are within the control of the player. Understanding how the player controls these factors is important to the casino to assess the earning potential from the player.

Of course, all these factors are important, but an emphasis is often placed on time played. This is because volatility is reduced the longer the player is betting. Again, the casino's best friend is the law of large numbers. The number of hands a player plays is of fundamental importance to casino operations. Playing many hands takes time. Time is the casino's best ally. The more hands a person plays, the greater the casino's earning potential. This makes sense because if the house advantage on a given bet is 1.36%, the expected win on a single hundred-dollar bet is $1.36 and the expected win on 100 hundred-dollar bets is $1.36 x 100, or $136.

Moreover, as the number of hands played increases, the more likely it is that the casino's actual win from that player will approach the theoretical win and the more likely the casino will realize the full earning potential from the player. Returning to the example of the two roulette players, because the second player is betting more than the first player, over time the casino will win more from the $400 average bet player than the $200 player. Therefore, he is more important to retain as a player.

TRACKING EARNING POTENTIAL

For slot machines, tracking the exact earning potential of players is easy with the player tracking systems in use at most casinos because all factors needed to make the determination are available. Slot machine manufacturers state the house advantage of the machines in the "par" sheets. The meters on the machine record the denominations, number and total of all coins played. [157]

Determination of player value is more difficult for table games. Unlike slots, where each coin or credit played can be recorded, a player's table game betting handle, betting pattern (the particular bets placed) and, in cases where relevant, the skill of the player, are not precisely known and must be estimated through observation.

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157 The only exception is slot machines with a skill factor, such as video poker or blackjack.

For many table games, actual earning potential is the amount of each bet made by the player multiplied by the number of hands played multiplied by the house advantage for the game played as adjusted for the skill level of the player. For example, take a very good blackjack player who bets $10 per hand and plays 70 hands at blackjack. If he is playing six-deck blackjack, the house advantage is about 0.54%. The earning potential of this player is computed as follows:

$10 x 70 x 0.54% or $3.78.[158]

In other words, earning potential = $3.78.

A number of uncertainties, however, come into play in determining all of the factors in the formula. For example, can the casino accurately track the amount of every wager made by the player? Some players will not bet the same amount for every hand.[159] This can be accomplished by either sophisticated tracking systems or simple manpower that is needed to track every bet of every player. The ability to obtain perfect information on all the player-controlled factors can be expensive, either the cost of manpower or of the tracking system. In most cases, casino management has historically decided that these costs are greater than the costs of errors resulting from more rudimentary systems. Therefore, most casinos must rely on assumptions and averages to estimate theoretical win or actual earning potential of table game players.

RATING SYSTEMS AND THEORETICAL WIN

The most common estimating formula for a player's earning potential or theoretical win is:

$$Earning\ Potential = Avg.\ Bet \times Hours\ Played \times Decisions\ per\ Hour \times H.A.$$

(where H.A. is the house advantage.)

Casinos often will develop a system to measure theoretical win in a manner that is simple to administer. The system a casino uses to estimate theoretical win from players is generally referred to as a rating system. This is because under most of these systems, each player whose play is observed for purposes of estimating theoretical win is rated according to some scale. Casinos may use different methods and terminology for rating players according to the casino's theoretical win.[160] A "rated" player is one whose play will be reviewed by the casino, usually for the purposes of determining comps and to qualify players for marketing purposes.

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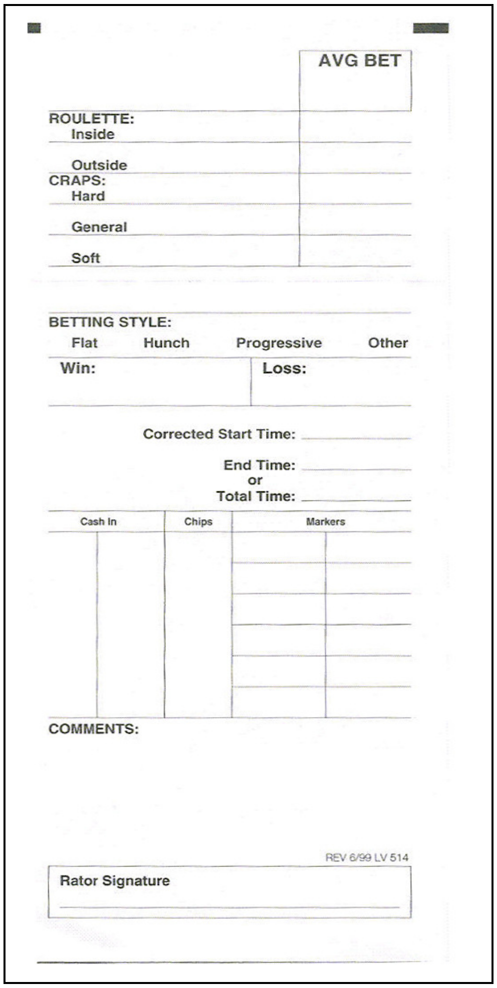
158 This level of play may qualify a player for a free buffet at many Las Vegas casinos.

159 While some table player tracking systems can accomplish this, they are not in wide use. These tracking systems commonly require a rated player to insert a player tracking card in special table slots. Microchips in his playing chips could then accurately record the amount and type of each bet made by that particular player. It also could record whether the player won or lost. From this the casino could garner both the actual win and loss from the player as well as expected value (except for adjustment based on skill level). Such a player tracking system for table games would facilitate the calculation of earning potential and player worth in much the same way as is done for slot machine play.

160 The most common method for describing rated players is by stating the casino's earning potential or theoretical win, e.g. a $20,000 player. Others may describe a rated player based on average bet, e.g. a



AVG BET

ROULETTE:
Inside
Outside

CRAPS:
Hard
General
Soft

BETTING STYLE:
Flat Hunch Progressive Other

Win: Loss:

Corrected Start Time:
End Time:
or
Total Time:

Cash In	Chips	Markers

COMMENTS:

REV 6/99 LV 514

Rator Signature

Rated players often are the lifeblood of a casino. In some casinos, 60-80% of the casino's total win comes from rated players. Moreover, often 30% or more of the rated players provide 60–80% of the rated win. This is not to downplay the importance of unrated players, often called Free Independent Players. While they provide casino revenue, another important aspect of the FIP is that they serve as a pool in which to recruit rated players.

For internal communication purposes, some casinos give a rating from 1 to 9 to rated players. A "1" player is the best, which in some casinos equates to an average bet of $700 or more. The lowest rated player is a "9," with an average bet of $75 or less in some casinos. Most often these ratings are used for marketing purpose rather than comp purposes. For example, a casino holding a New Year's Party may limit invitations to players with a rating of 3 or higher because of cost or space.

In the mid-1980's, many casinos began converting to automated (computerized) player rating systems. Using these systems, casino personnel have only to input the game played, average bet, and time played. The other factors that go into determining earning potential have already been estimated and programmed into the software. These include the house advantage and the average number of hands per hour for each game. For example, the casino may assume they will have a 2% house advantage in blackjack based on their rules in effect. This premise/assumption would only be an estimate because the skill level of the player could impact the actual house advantage by a factor as much as 4. Because of the inaccuracies of utilizing either an average skill level or game speed of all players, most systems allow for more accurate

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$200 player. This would be the average bet for a set time period that the player is expected to gamble. Most casinos expect rated players to play a minimum of 4 hours on each day of their visit. This can serve as a benchmark for an equivalent average bet the player would have made had he wagered for an average of 4 hours per night. The equivalent average bet is used to rate the player with regard to dollar levels of gambling activity and can be adjusted if the player plays more or less than 4 hours. If they gambled 8 hours in a given night, then their average wager is doubled. If they only gambled 2 hours, then their average wager is halved. If they stayed more than one night, then their minimum equivalent average bet is averaged over the course of their visit.

estimates by allowing casino personnel to enter additional data, such as player skill and speed of the game. Player skill level is usually determined by assigning a player a type of either hard (good – a player who plays perfect basic strategy), average, or soft (poor – an inexperienced or inattentive player). The casino supervisor enters game speed as slow (a full table), medium, or fast (a player playing alone). Based on the input information and preset criteria for house advantage (adjusted for the player skill level), an algorithm in a computer program produces an estimate of the casino's theoretical win for this player in actual dollar amounts. The following chart shows typical profiles and estimated house advantages (EHA) for hard, average and soft players in blackjack and craps.

	Blackjack	***EHA***	***Craps***	***EHA***
Hard Player	Perfect basic strategy	.6%	Pass line plus 10x odds only	.18%
Average Player	Typical player who fails to make correct decisions with some plays	2%	Pass Line plus odds and some proposition wagers	1.5%
Soft Player	Inexperienced or Inattentive player	3%	Bets pass line without odds, bets numbers and propositions	3%

PROBLEMS WITH ESTIMATING EXPECTED WIN FOR TABLE GAMES

Using estimates for the input factors will result in a calculated theoretical win that often does not reflect the actual theoretical win because the input estimates are merely the casino personnel's best guesses as to what the actual values are, whether it is the average bet or the speed of the game. The inaccuracies that result from using these estimates can work to the detriment of the casino and allow players to exploit the system.

AVERAGE BET

Conceptually, estimating the average amount of each bet by a player may appear simple enough. Some players, however, will periodically vary their bets throughout the course of their game play. This requires the pit personnel to observe their play and come up with a reasonable figure for the player's average bet for the time period observed. An error may occur if the pit personnel observes only a limited sample of the hands, is careless in his observations, or makes a mistake in calculation, among other things. For example, a player hoping to obtain a higher player rating (to maximize comps) may manipulate his average bet. This can include wagering larger amounts when the supervisor is watching, such as when he begins play, or when the supervisor is obviously recording ratings. The player can also hide chips so that it appears he is losing rapidly to deceive the rating supervisor into believing he is betting at the rate of his highest bet.

NUMBER OF HANDS

Casinos rely on averages to determine the number of hands played by the player. Many factors can influence how many hands are played, including the skill level of the dealers, the number of players at the table, the speed of players in making decisions, and whether a shuffler is used. For example, a person playing alone at the blackjack table may average about 200 hands per hour, while the same player playing at a full table may average only 50 hands per hour. The difference in the earning power between these two players is about 400%. Despite that the player playing alone is worth four times as much to the casino as the same player playing at a full table, many casino evaluations do not account for these differences.

Players hoping to obtain higher player ratings may use methods to play as few hands as possible in a given time period. These methods can include (a) playing on crowded tables, (b) choosing dealers who are slow or are distracted by talkative players, (c) playing slow, (d) taking many breaks, or (d) other similar tactics. The casino needs to be sensitive that sometimes the player has no choice but to play at a crowded table when the casino is busy. In this case, the casino may want to comp the player consistently between trips despite that his play is slower during busy times in the casino.

HOUSE ADVANTAGE

Casinos face another difficulty in determining how much they can expect to win from their players. In certain games, like blackjack and craps, the house advantage varies. With blackjack, the edge varies with the skill of the player, while at craps the house advantage depends on which bets a player selects. Assessing the overall advantage the house enjoys over an individual player in these games is a formidable task.

Incredibly, some casinos will use the same house advantage for determining theoretical win for every table game, regardless of whether it is blackjack (average HA ≈ 2%), craps (average HA ≈ 1.5%), baccarat (average HA ≈ 1.2%) or roulette (average HA ≈ 5.3%). While undoubtedly easier to administer, it allows for certain problems and abuses. One problem is that it overcompensates certain players at the expense of other players. For example, a good $100 per hand blackjack player with, say, a 0.6% disadvantage, has a lower value to the casino than a $100 per hand baccarat player because baccarat has a typical house advantage of 1.2%. All other things being equal, the expected return on the blackjack player will be about half that of the baccarat player. Yet, the comp privileges for each would be the same under a rating system that uses the same house advantage for each game. This has led some commentators to suggest methods whereby players can exploit deficiencies in casino rating systems to obtain more in comps than their actual expected value to the casino.

SKILL LEVEL

As mentioned above, house advantage also varies from player to player when any degree of skill is involved. In one episode of the old television series *Vegas*, a person was described as an expert baccarat player. Yet, despite the James Bond mystique that the general public perceives with the game, players only bet with the player or the banker (ignoring the tie bet), each with roughly the same house advantage. Whether the player is Sean Connery or Bonzo the monkey really doesn't make much difference.

In other games, however, the difference in earning potential between good and bad players is substantial. For example, consider two players. The first is a high roller who plays a $100 per hand, is a very bad blackjack player, and likes to play alone. He plays six hours per day for three days. The second player is also a high roller who plays a $100 per hand, but is a very good blackjack player. He plays eight hours per day, but always plays with five friends at the same table. Using a traditional equation with preset average values for advantage and game speed to compute expected return, the good blackjack player would be rated higher than the bad blackjack player. From a mathematical perspective, however, the bad blackjack player has a higher expected value.

Expected Win	
	Traditional Ratings Formula
Good Blackjack Player	$100 x 8 hrs x 90 hands/hr x 2% = $1,440
Bad Blackjack Player	$100 x 6 hrs x 90 hands/hr x 2% = $1,080
	Actual Expected Returns
Good Blackjack Player Playing at a Slow Table	$100 x 8 hrs x 50 hands/hr x 0.6% = $240
Bad Blackjack Player Playing Alone	$100 x 6 hrs x 200 hands/hr x 3% = $3,600

This exercise illustrates the limitations of rating systems. Casino management may want to further reward those players who, because they are unskilled or who play alone, increase the expected return to the casino.

Likewise, management needs to be sensitive that some players will attempt to use or abuse the comp policies by exploiting deficiencies in the rating systems. If this becomes a problem, they may need to modify the casino rating system or allow for manual overrides of the system to decrease player ratings for certain players.

COMPLIMENTARIES

Casinos determine players' values for competitive reasons. Casinos can compete for players in many ways. Sometimes the competition involves creating a casino environment that potential players find desirable. This has resulted in creations such as the Bellagio and Venetian Resorts in Las Vegas. In other cases, casinos compete based on the odds of the particular game. For example, casinos may offer slot machines with a smaller house advantage or more player-friendly rules in blackjack.

Complimentaries also are an important competitive tool. Since the mid-1950s, complimentaries, known simply as comps, have been an important marketing tool for the casinos. Comps can serve several purposes. First, the casino can use them as a means of giving a player a discount based on the volume of their play. In essence, these volume players get a portion of their expected losses back in the form of comps. The second purpose is simply to compete in the market place. If competing casinos are giving comps to a certain segment of players, a casino must provide substantially equivalent or better comps to compete for the same market segment. Third, a casino can use comps to inspire loyalty and, therefore, repeat business from players. As the following chart indicates, New Jersey casinos, like their counterparts in Nevada and elsewhere, make extensive use of comps.

New Jersey Comps – Six Months[161]	
A C Hilton	$49,978,000
Bally's Park Place	63,363,000
Caesars	65,517,000
Claridge	24,913,000
Harrah's Marina	47,014,000
Resorts	34,216,000
Sands	34,557,000
Showboat	48,303,000
Tropicana	56,153,000
Trump Marina	34,444,000
Trump Plaza	52,898,000
Trump Taj Mahal	71,078,000
Industry Total	$582,434,000

Types of Comps

Comps can include anything from free drinks to payment of every conceivable expense including room, food and beverage ("RFB") and airfare. Typically, comps are divided between "soft" and "hard" comps. Soft comps are simply those where the cost of the comp to the casino is less than the retail price. An example of a 100% soft comp is a line pass to a popular buffet. This does not cost the casino anything. Other typical soft comps would include rooms, food, beverages and in-house casino shows. Note, however, that the average comp check in a casino restaurant is typically 1.5 to 3 times the bill of a paying customer. Moreover, certain factors can multiply the cost of comps, such as the number of people, length of stay, transportation costs and special events. In contrast, hard comps are those whose cost to the casino equals the retail price, usually because the casino has to purchase them from unaffiliated third parties. Typical hard comps include airfare, tickets to outside events such as other casino shows, and off-premise restaurants.

What is a hard comp for one casino may be a soft comp for another. For example, a casino that owns a golf course is providing a soft comp for a round of golf where another casino might have to pay full retail to comp its players for golf.

161 For six months ending June 30, 2000.

The value of soft comps can change even within a casino environment. For example, suppose the casino has a show that runs 50% full on weeknights and 100% full on weekends. Providing show comps during the week has almost no incremental cost to the casino as the comped players are basically occupying empty seats. During the weekend, however, the comped players would be displacing persons who would have otherwise paid full retail.

ESTABLISHING A COMP POLICY

Like most other aspects of casino life in its early days, comps were decided based on no other criteria than the casino executive's best discretion. With discretion came both good and bad decisions and abuses. As junkets became popular, the only criterion for RFB (room, food and beverage comps) and airfare was putting money on deposit. Often junketeers were not playing enough to justify their comps or, in some cases, not gambling at all. Abuse also came from casino executives who traded comps with others for their own benefit as opposed to the casino's bottom line.

The first attempt to bring accountability to comps was to require the player to place a certain amount of money as "front money" and to draw markers to that amount. The false assumption was, again, the misunderstanding of hold percentage. The assumption was that the casino would retain the typical hold percentage on the player's play. Therefore, if the typical hold percentage was 15%, the casino would hold on average $1,500 on $10,000 in markers. This should justify comps of about $300-450, or 20-30% of the "expected win."[162] Clearly, casinos needed a better system for the granting of complimentaries.

The concept behind comps is to reward players with benefits that are equivalent to their level of play. Logically, high rollers should receive the best comps, including room, food, beverage and airfare, while low limit players might earn only food in the less expensive restaurants and beverages. This concept is called equivalency. The casino should set what equivalent rate of comps a player should receive based on player's value to the casino. A Las Vegas casino may reward a player with comps worth 20%-50% of their expected theoretical loss.

Like other marketing tools, complimentaries are most successful when they yield the greatest return for each marketing dollar spent. Therefore, like other casino decisions, the mathematics of complimentaries must be understood to establish comp policy and evaluate the success of the comp program.

Comp policy is simply setting forth what the casino requires of a player to receive a complimentary. These policies are usually a function of both a profit and loss analysis and competitive conditions. Profit and loss analysis considers each dollar of net gaming revenues and decides how much of it can be dedicated to complimentaries while achieving target profitability. The casino must look at each dollar of net gaming revenue and determine both fixed and variable costs in generating that dollar. The remaining amount is available for comps. If the casino decides that giving out comps will not increase net gaming revenue, then the casino should retain this entire amount.

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162 As discussed in Chapter 2, several factors may impact hold percentage, the most important in this case being time played.

A liberal comp policy will tend to generate the highest gross revenue for the casino but may not maximize net revenues. For example, a 50% comp policy dictates that 50% of all gross revenue is returned to the players in the form of comps. If the casino's gross revenue is $100, but the fixed non-comp expenses are 50%, then the remaining $50 available for comps is given back to the player and the casino has no profit. But, as a marketing tool, comps should generate greater net revenue for the casino. Clearly, giving back to the player all of remaining amounts might maximize gross revenue, but will leave the casino with no profit. The goal, therefore, is to maximize net revenues, which are gross revenues less expenses and comps. The following chart, used for illustrative purposes, shows that reducing the comp policy will reduce gross revenue, but finding the right levels will maximize casino profit.

Casino Profit

*Comp Policy**	*Gross Revenue*	*Non-Comp Expenses*	*Available Revenue*	*Comp Amount*	*Casino Profit*
50%	$100	$45	$55	$50	$5
40%	$80	$38	$42	$32	$10
30%	$60	$30	$30	$18	$12
20%	$40	$20	$20	$8	$12
10%	$20	$15	$5	$2	$3
None	$10	$10	$0	$0	$0

* *Percent of Theoretical Win*

In this case, the ideal comp policy would be between 20% and 30%. The above example assumes that gross revenues will decrease in proportion to the decreases in comp policy. In a competitive environment, this does not occur. Instead, all other things being equal, if a casino gives back less than its competitors in comps, it will experience a more dramatic decrease in rated players. Likewise, if it can give greater comps, it will experience a more dramatic increase in rated players.

Typical Comp Schedule			
Comp	*Cost of Comp*	*Theoretical Win That Casino May Set To Justify Comp*	*Blackjack Equivalent That Casino May Set For Player to Earn Comp*[163]
Parking	Undefined	0	No qualifying required
Cocktails	$1	$5	All players
Buffets	$4	$20	Unrated players @ $10/hand for 1.5 hours ($21 theoretical win)
Coffee Shop Meal	$10	$50	Unrated players @ $25/hand for 1.5 hours ($52.50 theoretical win)
Room	$60	$300	$25/hand for 8 hours ($280 theoretical win)
RFB	$200	$1,000	$100/hand for 8 hours ($1120 theoretical win)
RFB & Airfare	$600	$3,000	$300/hand for 8 hours ($3380 theoretical win)
RFB – Suite & Airfare	$1000	$5,000	$500/hand for 8 hours ($5600 theoretical win)
RFB – Penthouse & Private Jet	$10,000	$50,000	$5,000/hand for 8 hours ($56000 theoretical win)

Once the casino establishes a comp policy and criteria, it implements the comp program. The casino then must consistently analyze and modify the comp policy and procedure in light of its estimate of comp potential and the player's expectation of what he wants from comps. For example, the casino may underestimate the cost of giving out a restaurant comp. This can occur because a "comp" player at a restaurant usually spends more than a paying customer. How much more may be initially underestimated. This may require the casino to increase the criteria for obtaining such a comp. Likewise, the expectation of the player may cause the casino to re-evaluate its policies. For example, a casino competing for foreign players may need to modify its comp policies if its competitors begin to offer better comps. This may force the casino to increase the comp benefits as a percentage of the theoretical win or to abandon the market segment for a particular type of player.

163 Assuming a 2% house advantage and 70 hands per hour.

Actual Slot Comp Schedules Year 2000 For Dollar Slot Players Playing $20,000 Coin-in Per Day		
Casino	*Room*	*Food*
Sam's Town	Suite	RFB
Orleans	Suite	RFB
4 Queens	Suite	2 Steak Dinners
Caesars	Room	Limited RFB
Mandalay Bay	Room	Limited RFB
Rio	Room	$50
Station Casinos	None	2 Steak Dinners
Fiesta	None	1 Steak Dinner
Reserve	None	$40

For all the romance, glimmer, and glamour of Las Vegas, the city is built upon a foundation of pure mathematics. Each establishment, from the billion-dollar Bellagio down to the Howard Johnson's Hotel and Casino, operates on one simple principle: the theory of probability.

Andy Bellin
Poker Nation (2002)

REBATES AND DISCOUNTS

9

Except for management at a few casinos throughout the world, this chapter on rebates and cash discounts will have little application. These matters are generally reserved for high rollers, those players whose gambling volumes are so high that they can demand cash discounts. But "cash back" programs are beginning to take many forms, even in slot clubs for everyday players. Rebates and cash discounts are similar to comps. Instead of receiving a meal or a hotel room, the benefit to playing is a monetary return not provided to most other players.

Comps may also come in the form of discounts or rebates to premium players. Casinos may employ one of several programs. Three typical types of rebates and discounts are the "dead chip" program, rebates on theoretical loss, and discounts on actual losses. A short explanation of each follows.

REBATES ON THEORETICAL LOSS

Probably the most popular form of rebates is based on theoretical loss. In principle, this is a straightforward method to comp valued players. It is a program that can be easily adapted to both table and slot machine play.

> **Down on the first floor, I finally made my way to the slot booth. Club Metro awards one point for every dollar played at any machine. Four hundred points earn $1 cash back, for a rebate of .25 percent.**
>
> - *Strictly Slots Magazine*

Slot play can be tracked through the use of player slot club cards. Once the player inserts his or her card into the slot machine, it tracks every coin the player plays. From this, the casino can pay back a small percentage of each coin played.

Quarter Players Playing $5000 Coin-in Per Day[164]		
Casino	*Cash Back*	*Effective Rebate**
Sam's Town	$4	4% (.08% of coin-in)
Caesars	$8	8% (.16%)
Orleans	$10	10% (.2%)
Mandalay Bay	$15	15% (.3%)
4 Queens	$25	25% (.5%)
**(Assuming a theoretical win of $100 at 2% House Advantage)*		

As explained previously, accurately determining a player's theoretical win (earning potential) for table games is not possible without tracking every bet made by the player. In the case of high rollers, however, the only efficient use of manpower may be to have a casino employee physically observe and record every single bet placed by the premium player so that theoretical win can be accurately assessed. This is because the amount of the rebate to high rollers can be significant. For example, if a casino is willing to pay back 50% of theoretical loss, then the player would receive 50% of the house advantage for each $1 wagered over the course of play. Thus, if the edge were 1.2%, then 0.6% of each dollar wagered would be paid back to the player, regardless of whether the player won or lost. The casino would retain the advantage at 50% of the usual value, or 0.6%. When the player is a million dollar player, this amount becomes worth tracking by hand.

DEAD CHIP PROGRAMS[165]

Dead chips are non-negotiable gaming chips that can be used for gambling but cannot be exchanged for regular chips or cash. In a typical program the player purchases a certain amount of non-negotiable chips at a discounted rate, usually through a voucher called a chip warrant, and uses these chips for wagers just like normal chips. Once the player has used all the non-negotiables, the remaining regular chips from winning wagers may be redeemed for cash. Some casinos, particularly in Asia and Australia, use dead chip programs as a marketing strategy to attract higher-end players. The general idea is to give players a bonus or refund on non-negotiable gaming chips in return for a commitment to a certain level of play. These programs can be effective, but because playing with dead chips alters the game advantage, a casino must be careful in setting appropriate bonus amounts. Awarding a bonus or refund can sometimes result in an effective house advantage that is too low to offset operating costs, and may even result in the player having the edge.

In this section we provide the general mathematical framework for assessing the effective house advantage under dead chip programs for a variety of games and

164 Source: Vegasplayer.com

165 This section is adapted from "The Mathematics and Marketing of Dead Chip Programmes: Finding and Keeping the Edge," by Robert C. Hannum and Sudhir H. Kale, International Gambling Studies (2004), Vol. 4., No. 1, 33-45 (Taylor & Francis, Ltd, www.tandf.co.uk/journals).

wagers. More specifically, we: (1) briefly review the general background for dead chip programs and marketing to high-rollers, with particular emphasis on the relevant mathematics; (2) explain the underlying math behind dead chip programs; and (3) discuss the resulting marketing and management implications. The analysis suggests that many casinos have not adequately factored in the house advantage when using dead chip programs and the discussion presented here offers a framework for effective use of these programs.

BACKGROUND

Casino executives frequently perceive that premium or high-end players account for a disproportionately large share of revenues and profits. Bill Eadington – a respected gaming scholar – has noted that just ten high-rollers may account for as much as one-third of baccarat revenues for Las Vegas's major casinos.[166] Others contend that premium players are essential for a casino to produce cash flows commensurate with the large capital outlays that go into building a mega-resort.[167] This long-standing perception of high-rollers being saviors of a casino establishment has recently been questioned by some.[168] Very often it is the lack of understanding of the mathematical underpinnings behind various high-roller programs that sustains the illusion of premium players as the casino industry's most prized segment.

> **It should not be forgotten that gambling is a multi-million pound business because people enjoy it.**
>
> *U.K. Home Office Report*
> February 1978

Until recently, little was known about the precise effect of dead chips on the statistical advantage for different games and casinos often opted for ad hoc procedures in determining the amount of the bonus or rebate they are willing to award in a given situation. Further, casinos have been reluctant to use non-negotiables for games other than baccarat, a traditional favorite among high-rollers and the game for which casinos have some comfort when it comes to the use of dead chip programs. Due to increasing competition for high-end players, these programs are receiving more attention and the need for understanding their impact on the casino's bottom line has never been greater.

Although a few authors have addressed casino dead chip programs, the mathematical understanding of such programs is limited. Prior to 2004, the scattered results published in this area are limited – one article, in a discussion of the marketing and management issues on the risks of dealing to high-end players, provides an approximate formula for the effective house advantage in baccarat when using a dead

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166 Innis, M., Healy, T. and Keeley, B., "Hunting for Whales", Far Eastern Economic Review (2003), December 4, pp. 57-58.

167 See, e.g., Binkley, C., "Reversal of Fortune", Wall Street Journal (2001), September 7, pp. A1, A8.

168 See, e.g., MacDonald, A., "Dealing with High-Rollers: A Roller Coaster Ride?" www.urbino.net (2001), posted March 18; Lucas, A.F., Kilby, J. and Santos, J., "Assessing the Profitability of Premium Players", Cornell Hotel and Restaurant Administration Quarterly (2002), August, pp. 65-78; Watson, L. and Kale, S.H., "Know When to Hold Them: Applying the Customer Lifetime Value Concept to Casino Table Gaming", International Gambling Studies (2003), Vol.3, No.1, pp. 89-101.

chip discount,[169] and others illustrate the potential impact of these programs on casino revenues using examples for baccarat and roulette.[170] In each of these presentations, the mathematical results are limited to simple wagers involving a single winning outcome and for a single round of dead chip purchases, with examples from only baccarat or roulette. Until recently, there was no generally applicable framework that allows for calculation of the effective statistical advantage resulting from the use of a dead chip program with possibly multiple rounds of purchases under a variety of games and wager scenarios. This framework is provided below.

> **There is no such thing as luck. It is all mathematics. There are three kinds of cards – good cards, bad cards and indifferent cards. You must play them according to what they are. That is not a contradiction. You may have luck for an hour or two, even a day or two, even a week. But what people call luck is merely an established fact seen through the spectacles of after events.**
>
> Nico Zographos (*d.* 1953)
> Renowned 1920s and 30s baccarat gambler, head of Greek syndicate which operated a famous no-limit game.

On a wider plane, there has been some general discussion on the impact of high-rollers on a casino's bottom line from the mathematical angle. One author[171] cites examples of several companies blaming the lower than expected table game hold for their poor quarterly or half-yearly results. The author suggests these poor results are brought about by dealing high limit baccarat to a select few (predominantly Asian) customers. Another group of authors,[172] in an analysis of the profitability of premium players, conclude that gaming executives should be aware that high-rollers can and do win, as premium play is often characterized by risk and volatility with a potentially small contribution margin. The slight profit margins combined with the volatility of the game can produce extreme results within a given financial quarter. These same authors go on to demonstrate that a player betting $3,000 per hand of baccarat for nine hours contributes a mere $280 toward a casino's fixed costs (assuming the theoretical house advantage holds through the duration of play). In another study,[173] aggregate data from the Australian gaming industry was used to estimate the risk and profitability associated with casinos' major market segments. The authors of this study conclude that at a time when many casinos are busy building or refurbishing multi-million dollar facilities to cater exclusively to

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169 Eadington, W.R. and Kent-Lemon, N. "Dealing to the Premium Player: Casino Marketing and Management Strategies to Cope with High Risk Situations", in Gambling and Commercial Gaming: Essays in Business, Economics, Philosophy and Science (1992), Eadington, W.R. and Cornelius, J.A. (eds.), pp. 65-80.

170 See MacDonald, A., "Non-Negotiable Chips: Practical Procedures and Relative Costs of Non-Negotiable Play on Table Games", www.urbino.net (1996), Casinos 2000, Article 11; Kilby, J. and Fox, J., Casino Operations Management (1998).

171 MacDonald, A., "Dealing with High-Rollers: A Roller Coaster Ride?" www.urbino.net (2001), posted March 18.

172 Lucas, A.F., Kilby, J. and Santos, J., "Assessing the Profitability of Premium Players", Cornell Hotel and Restaurant Administration Quarterly (2002), August, pp. 65-78.

173 Watson, L. and Kale, S.H., "Know When to Hold Them: Applying the Customer Lifetime Value Concept to Casino Table Gaming", International Gambling Studies (2003), Vol.3, No.1, pp. 89-101.

'mobile customers' (high-rollers), casinos do not necessarily have to engage in one-upmanship to woo the fickle-minded high-rollers. In catering to premium players, the expense for every player should be justified by the return it will bring[174] and it clear that a sound understanding based on mathematical probabilities is required for a casino to effectively tailor its marketing programs toward high-rollers.

EFFECTIVE HOUSE ADVANTAGE

This section provides the mathematical details necessary to determine the effective house advantage associated with dead chip programs under a variety of scenarios.

DEAD CHIP PROGRAMS

> **Life is a gamble at terrible odds – if it was a bet, you wouldn't take it.**
>
> Tom Stoppard
> *Rosencrantz and Guildenstern are Dead*
> (1967)

Dead chip programs are usually targeted toward 'junkets.' A junket is a group of players who travel to a casino specifically for the purpose of gaming. The travel is prearranged through a junket representative and the players' costs associated with traveling to and staying at the casino are absorbed by the casino.[175] The general process of a typical dead chip program is as follows.[176] A junket group arrives at a casino and players deposit front monies with the casino in exchange for chip purchase vouchers, also known as cheque credits, or chip warrants. The vouchers are then used to purchase non-negotiable chips which the players use for wagers during the course of play. Losing wagers are taken by the dealer while winning wagers are paid in regular ('live') chips and the original wager remains as non-negotiable chips. Once a player has exhausted his (most junket players are men) non-negotiable chips, he takes his live chip winnings and exchanges these for more non-negotiable chip vouchers at the casino cage. The value of this exchange is recorded and added to the player's/group's non-negotiable chip purchase voucher schedule. When the junket group departs, the non-negotiable chip purchase voucher is totaled and all remaining non-negotiable chip purchase vouchers are deducted as well as any non-negotiable chips held. This provides a value for the turnover – the cumulative amount of money wagered – upon which a commission is paid to the junket group operators. All chip purchase vouchers and chips are converted to cash or deducted from personal cheques held with a new cheque value written out, which may or may not include commission. The commission payment, then, is calculated essentially on the loss of non-negotiable chips over the period.

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174 Bowen, J.T. and Makens, J.C., "Promoting to the Premium Player: An Evolutionary Process', in The Business of Gaming: Economic and Management Issues (1999), Eadington, W.R. and Cornelius, J.A. (eds.), pp. 229-239.

175 Kilby, J. and Fox, J., Casino Operations Management (1998), p. 339.

176 See, e.g., MacDonald, A., "Non-Negotiable Chips: Practical Procedures and Relative Costs of Non-Negotiable Play on Table Games", www.urbino.net (1996), Casinos 2000, Article 11.

A few of the larger casinos in jurisdictions outside the Asia-Pacific region have recently implemented or are considering implementing dead chip programs for premium players. A version that surfaced in Las Vegas gives players a 'bonus' of additional free non-negotiable chips at the time of purchase.[177] A player wagering at baccarat, for example, may receive $103,000 in non-negotiable chips for his $100,000 in cash, a 3% dead chip bonus. Subsequent purchases may carry a reduced bonus. Thus, once the $103,000 in non-negotiable chips is gone, the player may receive $102,000 in non-negotiable chips for the second purchase of $100,000 in cash (2% bonus). For the third and all subsequent purchases of $100,000 in cash, the player may be given $101,000 in non-negotiables (1% bonus). Note that the player receives the entire buy-in amount in dead chips, and not, for example, $100,000 in negotiable chips and $3,000 in dead chips. Although it is obvious the effect of such a bonus program is to reduce the advantage the casino holds over the player, the precise amount of the effect is not immediately evident.

> **It seemed to me that calculating your chances really rather little, and certainly isn't as important as some gamblers make it out to be. They sit there with sheets of graph paper before them, mark every stroke, reckon, compute the odds, calculate, and finally place their bets. They lose, exactly as we simple mortals who play without calculating anything.**
>
> Fyodor Dostoevsky
> *The Gambler* (1867)

The reduction in house advantage associated with the dead chip bonus model is a function of the average length of time a chip will be wagered before lost, which in turn depends on the probability of losing the wager. The logic can be seen clearly with a simple coin-tossing experiment. Consider a wager on the outcome heads in a series of tosses of an honest coin. With heads and tails equally likely, a bet on heads will lose every other toss on average; thus the average number of tosses before the wager is lost is two. If the coin is biased so that the probability of heads is equal to .90, then the average number of tosses before a wager on heads is lost is ten. Generally, if P_L is the probability of losing for each of a series of independent wagers, then $1/P_L$ is the average number of trials (wagers) until a bet is lost. This result can be viewed in terms of the well-known geometric probability model.[178] If p represents the probability of success on each of a series of independent Bernoulli trials and X = the number of trials until the first success occurs, then the expected value of X is equal to $1/p$. In double-zero roulette, for example, a straight-up single number bet will last on average 1/(37/38) = 1.027 trials and an even-money wager will last on average 1/(20/38) = 1.90 trials. Thus, the ratio of actual turnover to non-negotiable losses is a function of the probabilities associated with the individual game and particular wager placed. For roulette this ratio is close to one on single number straight-up wagers, close to one-and-a-half for dozens and columns bets, and close to two for even-money wagers.

177 Kilby, J. and Fox, J., Casino Operations Management (1998).
178 See, e.g., Freund, J.E., Introduction to Probability (1973).

This helps explain why in a game like baccarat, casino operators are willing to pay commissions – usually 1.5% to 2.0% – that are greater than the house advantage (which is around 1.2% to 1.8%) on non-negotiable chips. This is because for baccarat, the effective house advantage on non-negotiable chips can be viewed as being twice that of the normal value due to the ratio of actual versus non-negotiable turnover being about 2:1 (slightly more for banker bets; slightly less for player bets).

BASIC DEAD CHIP MATHEMATICS

The following example may help in understanding the formal mathematical results. Suppose a player pays \$100,000 on buy-in and receives \$101,500 worth of non-negotiable chips, reflecting a 1.5% bonus. Assume further this player wagers these non-negotiables, one chip per wager, only on the baccarat banker, with probabilities of winning and losing equal to P_W = .458597 and P_L = .446247 respectively (probability of tie is .095156). On average, it will take 227,452.7 wagers (101,500/.446246) before all the non-negotiable chips are lost, after which time the player will have received 99,093.7 live chips for wagers won (.458597 x 227,452.7 x .95). Thus the casino profit is \$100,000 – \$99,093.7 = \$906.30, equating to an effective house advantage of 0.40% (906.3/227,452.7 = .0040). This reasoning generalizes to the following result.

<u>Fundamental Theorem of Dead Chips</u>: Suppose a player receives $X(1+B)$ non-negotiable chips in exchange for X dollars, so that B represents the dead chip bonus. Suppose further the player wagers these non-negotiable chips until lost, one per wager on a series of independent wagers each with probability of winning w units equal to P_W and probability of losing one unit equal to P_L. It is assumed that all wagers are made with non-negotiable chips; losing wagers are taken by the dealer, and winning wagers are paid in regular chips with the original wager remaining as a non-negotiable chip.

(1) Then the effective house advantage is given by

$$HA^* = HA - \frac{B}{1+B} P_L \,.$$

(2) If $B > \dfrac{HA}{wP_W}$ then $HA^* < 0$.

<u>Proof</u>: To prove (1), note that on average a non-negotiable chip unit will be wagered $1/P_L$ times before it is lost, and therefore the average number of units wagered with $X(1+B)$ non-negotiable chips before all are lost is $X(1+B)/P_L$. Thus the player's expected win is $[X(1+B)/P_L]wP_W$ units and the house retains $X - [X(1+B)/P_L]wP_W$ units. The effective house advantage, then, is

$$HA^* = \frac{X - [X(1+B)/P_L]wP_W}{X(1+B)/P_L} = \frac{P_L - wP_W - BwP_W}{1+B} \,.$$

Using the fact that $HA = P_L - wP_W$, this becomes

$$HA^* = \frac{HA - BwP_W}{1+B} = \frac{HA - B(P_L - HA)}{1+B} = HA - \frac{B}{1+B} P_L .$$

This proves (1); the proof of (2) follows easily.

Note that (2) gives the maximum bonus possible without giving the player the advantage. The preceding fundamental result is illustrated in the two tables below with examples for wagers in baccarat, craps, and roulette. Margin of error figures for selected numbers of trials are included in the tables to shed light on volatility and permit calculation of 95% confidence limits for the actual win percentage.

Effective House Advantages* – Baccarat and Craps

	Baccarat		**Craps**	
	Banker[1]	***Player***[1]	***Pass Line***	***Don't Pass***[1]
House Advantage	1.17%	1.36%	1.41%	1.40%
Margin of Error[2]				
n=10,000	±1.95%	±2.00%	±2.00%	±2.00%
n=100,000	±0.62%	±0.63%	±0.63%	±0.63%
n=1,000,000	±0.19%	±0.20%	±0.20%	±0.20%
Bonus	***HA****	***HA****	***HA****	***HA****
0.00%	1.17%	1.36%	1.41%	1.40%
0.25%	1.05%	1.24%	1.29%	1.28%
0.50%	0.92%	1.11%	1.16%	1.15%
0.75%	0.80%	0.99%	1.04%	1.03%
1.00%	0.68%	0.86%	0.91%	0.90%
1.25%	0.56%	0.74%	0.79%	0.78%
1.50%	0.44%	0.62%	0.66%	0.65%
1.75%	0.32%	0.49%	0.54%	0.53%
2.00%	0.20%	0.37%	0.42%	0.41%
2.25%	0.08%	0.25%	0.30%	0.29%
2.50%	-0.03%	0.13%	0.18%	0.17%
2.75%	-0.15%	0.01%	0.06%	0.05%
3.00%	-0.27%	-0.11%	-0.06%	-0.07%
Max Bonus	2.43%	2.77%	2.87%	2.85%

1 Figures given excluding ties.

2 The maximum difference (two standard deviations) between the actual and theoretical win percentage, with 95% confidence, after the given number of trials. For example, there is a 95% probability that after 100,000 (equal size) baccarat banker bets the actual casino win percentage will be within ±0.62% of the house advantage. For normal baccarat play, this means there is 95% confidence that the actual win rate for 100,000 banker bets will be between 0.55% and 1.79%.

* Effective house advantage is given by *HA**. For example, a 1.75% dead chip bonus given to

a player betting on the baccarat banker will effectively lower the house advantage from the normal 1.17% (excluding ties) to 0.32%. The maximum bonus percentage that can be given before the player has the advantage on this bet is 2.43%.

Effective House Advantages* – Roulette

	Single-Zero Roulette			Double-Zero Roulette		
	Evens[1,2]	*Dozens*	*Single #*	*Evens*[1]	*Dozens*	*Single #*
House Advantage	2.70%	2.70%	2.70%	5.26%	5.26%	5.26%
Margin of Error[3]						
n=10,000	±2.00%	±2.81%	±11.68%	±2.00%	±2.79%	±11.53%
n=100,000	±0.63%	±0.89%	±3.69%	±0.63%	±0.88%	±3.64%
n=1,000,000	±0.20%	±0.28%	±1.17%	±0.20%	±0.28%	±1.15%
Bonus	***HA****	***HA****	***HA****	***HA****	***HA****	***HA****
0.00%	2.70%	2.70%	2.70%	5.26%	5.26%	5.26%
0.25%	2.57%	2.53%	2.46%	5.13%	5.09%	5.02%
0.50%	2.45%	2.37%	2.22%	5.00%	4.92%	4.78%
0.75%	2.32%	2.20%	1.98%	4.87%	4.75%	4.54%
1.00%	2.19%	2.03%	1.74%	4.74%	4.59%	4.30%
1.25%	2.07%	1.87%	1.50%	4.61%	4.42%	4.06%
1.50%	1.94%	1.70%	1.26%	4.49%	4.25%	3.82%
1.75%	1.82%	1.54%	1.03%	4.36%	4.09%	3.59%
2.00%	1.70%	1.38%	0.79%	4.23%	3.92%	3.35%
2.25%	1.57%	1.22%	0.56%	4.11%	3.76%	3.12%
2.50%	1.45%	1.05%	0.33%	3.98%	3.59%	2.89%
2.75%	1.33%	0.89%	0.10%	3.85%	3.43%	2.66%
3.00%	1.21%	0.73%	-0.13%	3.73%	3.27%	2.43%
Max Bonus	5.56%	4.17%	2.86%	11.11%	8.33%	5.71%

1 Any of the even-money wagers (covering 18 numbers).
2 No *en prison*.
3 The maximum difference (two standard deviations) between actual and theoretical win percentage, with 95% confidence, after the given number of trials. For example, there is a 95% probability that after 1,000,000 (equal size) even-money wagers in single-zero roulette, the actual casino win percentage will be within ±0.20% of the house advantage. For normal roulette play, this means there is 95% confidence that the actual win rate for 1,000,000 even-money wagers on the single-zero wheel will be between 2.5% and 2.9%.
* Effective house advantage is given by HA*. For example, a 2.50% dead chip bonus given to a player betting on even-money wagers in single-zero roulette will effectively lower the house advantage from the normal 2.70% to 1.45%. The maximum bonus percentage that can be given before the player has the advantage on this bet is 5.56%.

Multiple Rounds, Multiple Winning Outcomes, Mixtures of Wagers

The fundamental theorem can be extended to results for multiple rounds of dead chip purchases, wagers involving multiple winning outcomes (for example, slots, keno, video poker, and certain bets in craps), and mixtures of wagers. These extended results are given below. Proofs are analogous to that of the preceding fundamental theorem and are omitted. In the results that follow, it is assumed that all wagers are made with non-negotiable chips, losing wagers are taken by the dealer, and winning wagers are paid in regular chips with the original wager remaining as a non-negotiable chip.

Multiple Rounds. Suppose a player purchases k rounds of non-negotiable chips, exchanging X_i dollars for $X_i(1+B_i)$ non-negotiable chips on round i, for $i = 1, 2, \ldots k$. Thus $B_1, B_2, \ldots B_k$ represent the dead chip bonuses on each of these purchases. Suppose further the player wagers these non-negotiable chips until lost, one per wager on a series of independent wagers each with probability of winning w units equal to P_W and probability of losing one unit equal to P_L. Then the effective house advantage after k rounds is given by

$$HA^* = HA - \frac{\sum_{i=1}^{k} X_i B_i}{\sum_{i=1}^{k} X_i (1 + B_i)} P_L .$$

Note that if the purchase amounts (X_i) on each round are equal, then the above result reduces to

$$HA^* = HA - \frac{\overline{B}}{1 + \overline{B}} P_L ,$$

where $\overline{B} = (\sum B_i) / k$.

Multiple Winning Outcomes. Assume the same framework as in the fundamental theorem, except that each wager has (m+1) possible outcomes (payoffs) $w_1, w_2, \ldots, w_m$, and –1, with probabilities $P_1, P_2, \ldots P_m$, and P_L, respectively, with $w_i \geq 0$ for $i = 1, 2, \ldots m$, and normal house advantage given by $HA = P_L - \sum_{i=1}^{m} w_i P_i$.

(1) Then the effective house advantage is given by

$$HA^* = HA - \frac{B}{1 + B} P_L .$$

(2) If $B > \frac{HA}{P_L - HA}$, then $HA^* < 0$.

If you wanna make money in a casino, own one.

Steve Wynn
Quoted by David Spanier in *All Right, Okay, You Win* (1992).

The table below provides an example of the multiple winning outcomes result using m = 4 and shows the details for three dead bonus percentages: 1.5%, 4.0%, and 8.7%. In this example, with a relatively large probability of losing and normal statistical advantage equal to 6.0%, the maximum bonus without giving the player the advantage is 8.70% [from result (2) above].

Multiple Winning Outcomes*							
Wager		**Bonus = 1.5%**		**Bonus = 4.0%**		**Bonus = 8.7%**[1]	
Pay	*Prob*	**Wagers**[2]	**Win**[2]	**Wagers**[2]	**Win**[2]	**Wagers**[2]	**Win**[2]
-1	0.750	101,500	0	104,000	0	108,696	0
0	0.100	13,533	0	13,867	0	14,493	0
1	0.100	13,533	13,533	13,867	13,867	14,493	14,493
2	0.045	6,090	12,180	6,240	12,480	6,522	13,043
100	0.005	677	67,667	693	69,333	725	72,464
HA = 6.00%		135,333	93,380	138,667	95,680	144,928	100,000
		House Win	6,620	House Win	4,320	House Win	0
			HA* = 4.89%		**HA* = 3.12%**		**HA* = 0.00%**

1 Maximum bonus without giving player the edge is 8.695652174% (used in calculations).

2 Player wagers made and live chips won on average assuming 100,000 unit buy-in.

* Effective house advantage is given by HA*. For example, a 4.0% dead chip bonus given to a player betting on this multiple outcome wager (two left columns) will effectively lower the house advantage from the normal 6.00% to 3.12%. Assuming a 100,000 unit buy-in, this player will make on average 138,667 wagers with the 104,000 dead chips received, winning 95,680 negotiable chips, resulting in a casino win of 4,320 units: 4,320/138,667 = .0312.

Thus far we have assumed dead chips are used on a single type of wager (or independent wagers with the same payoff probability distribution). The next result shows how to compute the effective statistical advantage when dead chips are used for a mixture of different bets.

<u>Mixtures of Wagers</u>. Suppose a player receives $X(1+B)$ non-negotiable chips in exchange for X dollars, so that B represents the dead chip bonus. Suppose further the player wagers a fraction f_i of these non-negotiable chips on wager i, for $i = 1, 2, \ldots s$, one chip per independent wager, until lost. Let P_{W_i} and P_{L_i} represent the probabilities of winning and losing, and w_i the winning payoff associated with wager i. Then the effective house advantage is given by

$$HA^* = \frac{1-(1+B)\sum f_i\left(\frac{w_i P_{W_i}}{P_{L_i}}\right)}{(1+B)\sum\left(\frac{f_i}{P_{L_i}}\right)} = \sum_{i=1}^{s} r_i \cdot HA_i^*,$$

where $r_i = \frac{f_i / P_{L_i}}{\sum_{j=1}^{s} f_j / P_{L_j}}$ and HA_i^* is the effective house advantage for wager i.

The table below illustrates the calculation of effective statistical advantage for several mixtures of roulette and baccarat wagers.

Mixtures of Bets*

Roulette (Single-Zero) Mix – 1.5% Bonus

	P_L	P_W	w	HA	f_i	r_i	HA_i*
Evens[1]	0.5135	0.4865	1	2.70%	0.200	0.294	1.94%
Dozens	0.6757	0.3243	2	2.70%	0.200	0.223	1.70%
Corner	0.8919	0.1081	8	2.70%	0.200	0.169	1.38%
Split	0.9459	0.0541	17	2.70%	0.200	0.159	1.30%
Single #	0.9730	0.0270	35	2.70%	0.200	0.155	1.26%
							HA* = 1.59%

Baccarat[2] – 50% Banker, 50% Player – 1.5% Bonus

	P_L	P_W	w	HA	f_i	r_i	HA_i*
Banker	0.4932	0.5068	0.95	1.17%	0.500	0.507	0.44%
Player	0.5068	0.4932	1	1.36%	0.500	0.493	0.62%
							HA* = 0.53%

Baccarat[2] – 75% Banker, 25% Player – 1.5% Bonus

	P_L	P_W	w	HA	f_i	r_i	HA_i*
Banker	0.4932	0.5068	0.95	1.17%	0.750	0.755	0.44%
Player	0.5068	0.4932	1	1.36%	0.250	0.245	0.62%
							HA* = 0.48%

Baccarat[2] & Roulette (Single-Zero) Mix – 1.5% Bonus

	P_L	P_W	w	HA	f_i	r_i	HA_i*
Banker	0.4932	0.5068	0.95	1.17%	0.250	0.291	0.44%
Player	0.5068	0.4932	1	1.36%	0.250	0.283	0.62%
Evens[1]	0.5135	0.4865	1	2.70%	0.250	0.279	1.94%
Single #	0.9730	0.0270	35	2.70%	0.250	0.147	1.26%
							HA* = 1.03%

1 Any of the even-money wagers (covering 18 numbers); no *en prison*.

2 Figures given excluding ties.

* Effective house advantage is given by HA*. For example, given a dead chip bonus of 1.5%, a player betting equal proportions of dead chips on baccarat banker, baccarat player, roulette even-money and roulette single number wagers will result in an effective house advantage of 2.10%.

DEAD CHIP COMMISSION AFTER PLAY

In some programs, the dead chip bonus is awarded as a commission in cash after the termination of play rather than at the time of the buy-in. In this situation, the effective house advantage computation is straightforward, and can be shown to be $HA^* = HA - BP_L$.

Note that giving the dead chip bonus as a commission at the end of play will always result in a lower effective house advantage than giving the bonus at the time of buy-in, all else equal. The difference between the effective house advantage under the commission model and the original model is easily derived, and is given by:

$$\Delta = \frac{B^2}{1+B} P_L .$$

MARKETING AND MANAGEMENT IMPLICATIONS

Casino management and marketing strategies should be based on mathematically sound business decisions with a mechanism to assess the long term theoretical win or loss to the business. The preceding mathematical framework and results given in the tables provide casino management with reference values for appropriate dead bonus percentages and upper limits for these bonuses that could be made available to players of baccarat, craps and roulette. As with all expected value calculations, these figures apply in the long run and as such can serve as the basis for sound business decisions. It would be unrealistic, however, to presume that short term results will conform precisely to the theoretical advantage. As one expert rightly points out,[179]

> Baccarat is a game of chance and so fortunes can, and do, pass both ways across the gaming tables. The house advantage on baccarat averages a very low 1.3% to make this close to a true even money bet every time cards are drawn for a new hand. An interesting statistic is that 14,981,640 hands need to be played for there to be a 95% confidence interval of results falling between a win rate of 1.25% and 1.35%. If you multiply this number by a bet of $100,000 you require $1,498,164,503,243 in turnover for win rate "certainty." An astounding number in anyone's view!

Although the laws of probability warrant that the actual win percentage will converge on the theoretical win percentage, management needs to be concerned about potentially large deviations from the expectation that may occur in the short run. The margin of error figures in the preceding tables provide upper bounds, with 95% confidence, on the size of this deviation for several numbers of trials (hands) and can be used to obtain 95% confidence limits for the actual win rate. About baccarat volatility, however, the level of turnover required for the actual win percentage to fall within a small level of tolerance from the mean can be astronomical, so analysts and senior management need to have very strong stomachs if involved in baccarat high end play.[180] Even if the house advantage holds through the duration of play, the casino will still have to foot the bill for room, food and beverage, airfare allowance and gaming taxes.

Thanks to the higher house advantage in roulette, a casino is on more solid ground when it offers dead chip bonus to roulette players. In fact the maximum bonus in roulette for a player betting 'evens' (any of the even-money bets, such as red or

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179 MacDonald, A., "Dealing with High-Rollers: A Roller Coaster Ride?" www.urbino.net (2001), posted March 18.

180 MacDonald, A., "Game Volatility at Baccarat", Gaming Research and Review Journal (2001), Vol.6, No.1, pp. 73-77.

black) could be twice as high as a baccarat player betting on the banker. Having thus determined the house advantage and the resulting bonus a casino could safely live with for the various games, bonuses should be based on specific games as well as a punter's total buy-in for dead chips. Of course, the house advantage can be even higher in other games – keno and some slots, for example – but these are not usually the games of choice for high-rollers.

The situation gets most uncertain from a casino's standpoint when dead chip players are allowed to play a variety of games. Since house advantage is a statistical function of the particular game played and particular wagers made, establishing a realistic bonus for a player betting on both blackjack and craps becomes an extremely unwieldy proposition. Unless the precise mixture of games to be played can be controlled in some way, a casino may need to be more conservative if allowing a mixture, perhaps using as the basis for the dead chip bonus the game that would result in the lowest bonus percentage. Otherwise, it makes sense to design dead chip bonus programs under the condition that the chips could be used for play in only one specified game. Note too, that for games that involve an element of skill, the house advantage will vary depending on the player. In such situations, calculations assuming optimal strategy will provide a baseline reference for skilled players, but the actual house advantage will be higher for those players who do not use optimal strategy. Of course, for many games, including baccarat, the game most associated with dead chip program play, skill is not an issue.

Another implication of the mathematics of dead chips relates to the wisdom of providing the dead chip bonus at the time of buy-in as opposed to at the termination of play. Giving a dead chip bonus at the end of play will always result in a lower effective house advantage than giving the same bonus at the time of buy-in. Given slim operating margins, casinos are better off giving commission at the time of buy-in. Although many players may be unaware of the mathematical intricacies surrounding this choice, more savvy players and junket representatives will seek out dead chip programs offering commission at the end of play rather than at buy-in.

Besides considering the win rate volatility in deciding commissions, other factors such as volume, bet-limits, incentives and credit risk also need to be factored in. In setting betting limits associated with high-rollers, one should pay attention to capital reserves of the company, the company's propensity for risk, fixed and variable expenses associated with the operation, the total handle, and so on. Not too many companies (or their shareholders) would be thrilled in the event of being forced to utilize their capital reserves to pay for table losses or fixed expenses. Incentives accrued to high-rollers should be accurately estimated in order to arrive at the bonus figure to award on dead chips. Incentives typically total somewhere between 50% and 70% of theoretical win. This allows little room for bonuses or commissions. Bear in mind that dealing with high net-worth individuals means that you are often negotiating with very influential businesspeople who demand (and usually get) the best of everything. Costs associated with airfare, hotel suites, golf course fees, and food and beverages do add up, and offering these as comps cuts significantly into the casino's theoretical win. Furthermore, the credit risk associated with high-rollers needs to be factored in when determining the bonus amount. As the industry saying goes, 'You have to win the money twice.' Since most high-rollers play on credit, casinos have to win the money over the tables *and* they also have to win by getting paid, which may be neither very prompt nor certain.

The mathematics underlying dead chip programs sheds further light on the profitability and risk associated with high-end play. In offering incentives to high-rollers, casinos may sacrifice a substantial portion of their house advantage, thus creating a significant dent in profitability. Some authors[181] have suggested that instead of continuing to offer high incentives primarily to premium players, casinos may wish to consider marketing strategies and programs that target the $25 to $200 bettors. These mid-level punters constitute the backbone of the casino gaming business; offering rebate and discount programs will lower the price of the game for these players and, at the same time, have a positive impact on casino revenues and customer retention.

Although it is natural to view programs such as the dead chip offerings in terms of a player's single visit to the casino, it would not be difficult to extend the analysis to apply to multiple visits. With the current emphasis on customer relationship management (CRM), it is advisable to consider player profitability over the entire length of a player's relationship with the casino. Given the volatility associated with short-term play of the high-end segment, it would make sense to make every attempt toward enhancing the lifetime value of high-rollers. At a time when casinos all over the world fiercely compete for a share of the high-rollers' wallets, engendering loyalty among high-rollers is a formidable task.

Dead chip programs have been used by casinos for some time – particularly in the Asia-Pacific region – but the mathematics underlying these programs has not in the past been well-understood. Comprehending the effect these programs have on game advantage is crucial to the casino's ability to offer an attractive dead chip package while still conducting sound business. As the popularity of dead chip programs grows, so does the need for a clear understanding of the precise impact of this marketing initiative on the mathematical edge. Recent history of several casinos ending up on the losing side of the table vis-à-vis high-rollers underscores the importance of thoroughly understanding the mathematical realities behind incentive programs. Many casino executives assume that the high-roller segment is inherently profitable. Combine that assumption with a misunderstanding of gaming mathematics and you have the makings of a downward profit spiral.

Bear in mind that the mathematics only guarantees the actual win rate will be close to the house advantage after a very large number of trials. In the short term, deviations from the theoretical win will inevitably occur, making dealing to high-rollers an inherently risky proposition for most casinos. Decisions on commission should therefore be made in conjunction with decisions on bet limits, other incentives and credit risk. All these decisions should be formulated against the backdrop of a casino's financial reserves, its tolerance for risk, shareholder expectations and expected turnover volumes. As one author notes, 'This (high-roller segment) can be

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181 See, e.g., Watson, L. and Kale, S.H., "Know When to Hold Them: Applying the Customer Lifetime Value Concept to Casino Table Gaming", International Gambling Studies (2003), Vol.3, No.1, pp. 89-101.

the sexy, sharp end of the business but at the end of the day it is low margin and high risk. Lots of risk!'[182]

A Caveat

Stewart Ethier notes that the effective house advantage as defined above is the ratio of the player's expected loss in live chips to the expected amount bet in dead chips. Concerned about the different units of measurement in the numerator and denominator, Ethier suggests a possible alternative measure of the effective house advantage in which the denominator of the HA^* as defined above is multiplied by the expected value in live chips of a dead chip. This quantity is the expected value of $w \times (X_{Geometric(P_L)} - 1)$, or wP_W / P_L, or $1-(HA/P_L)$, where $X_{Geometric(P_L)}$ is a geometric random variable with parameter P_L. Ethier's proposed alternative, which results in increasing the effective house advantage (vis-à-vis the definition of HA^* above), then, is

$$HA^*_{alt} = \frac{HA - [B/(1+B)]P_L}{1-(HA/P_L)}.$$

There are benefits to each of the two formulations of the effective house advantage. HA^*_{alt} can easily be used to determine when the dead chip program is better for the player than no program at all. That is, it can answer the question, when is $HA^*_{alt} < HA$? Using the definition of HA^*_{alt} above and working through the math, the answer to this question is when

$$B > \frac{(HA/P_L)^2}{1-(HA/P_L)^2}.$$

For example, with double-zero roulette, *B* must be larger than 1.01% for the dead chip program to be helpful.

HA^* on the other hand, enjoys the properties that (1) $HA^* < 1$ always, and (2) $HA^* = HA$ when *B* = 0. That is, the effective house advantage as originally defined will never be greater than 100%, and when *B* is equal to zero (no bonus) it reduces to the ordinary house advantage.

A Real Live Dead Chip Program – Casino Cash

An Australian version of a dead chip program is called "Casino Cash."[183] This program rewards players with "free play" chips (i.e., dead chips) that can be used to play any casino game. If the player wins, he will be paid in live chips. Players get immediate reward for every hour of play at each table game. To simplify the system, comparable expected wins from each table game based on minimum bet levels are placed in each of four levels.

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182 MacDonald, A., "Dealing with High-Rollers: A roller Coaster Ride?" www.urbino.net (2001), posted March 18.

183 Andrew MacDonald, "Designing a Loyalty Program", Casino Executive 56-57, October 2000.

	Roulette	*Blackjack*	*Big Wheel*	*Sic Bo*	*Baccarat*	*Caribbean Stud*	*Craps*	*Pai Gow*
L1	$1	$5	$2	$2	---	---	---	---
L2	$2	$15	$5	$5	$5	$5	$5	---
L3	$5	$25	---	---	$25	$25	---	$25
L4	$10	$50	---	---	$50	---	---	$50

For every hour of play, the player is rewarded with "casino cash" that is immediately accessible. To encourage play during off-peak times, double bonus cash can be awarded. To encourage longer play, a random cash award is granted for playing for three or more hours.

DISCOUNTS ON ACTUAL LOSSES

Another type of discount program is based on observed player loss over the course of their play. Casinos sometimes offer rebates on actual losses to premium players to encourage loyalty and promote further business from these players.

A major reason for this type of system is to meet player expectations. Ratings based on theoretical win may be difficult for some players to understand, especially when the player has lost his entire bankroll but receives comps based on a significantly lower theoretical win. For example, suppose the player actually loses $8,000 when his theoretical loss suggests he should only have lost $1,000. In such a case, limiting his comps to $200 (20% of theoretical loss) may be inconsistent with the player's expectations given his more substantial actual losses.

Providing rebates on player losses typically requires the player to agree to gamble a specified number of hours at a minimum amount per hand to qualify. If he meets these minimum requirements, then he is entitled to a discount against losses, which can be as high as 20%.

Because the percentage of actual loss rebated is not the same as the percentage of theoretical loss, a program giving rebates on actual losses can be risky. These programs require the casino to take advantage of the law of large numbers since otherwise, refunding 20% of losses for a short period of play can result in a negative expectation to the casino. The following baccarat example shows the expected values of the banker and player bets, assuming the casino rebates 20% of the gambler's losses after every hand:

$$EV(Banker) = (+0.95)(0.4585974) + (-0.8)(0.4462466) = +0.0787.$$

$$EV(Player) = (+1)(0.4462466) + (-0.8)(0.4585974) = +0.0794.$$

Thus, the player would enjoy a 7.9% advantage if the 20% rebate were applied to every single hand. Requiring the gambler to play for a long period of time, however, before settling with a 20% rebate on the aggregate loss will allow the casino to retain an advantage.

Taking the baccarat example to the other extreme, if a player played for a sufficiently long time so that the law of large numbers virtually ensured the house win of 1.15% (the average of the usual advantages on banker and player bets), a 20%

rebate on aggregate losses would still give the casino a net win of 0.92%, the same house advantage realized with a 20% rebate on theoretical loss.

The example above shows how the casino can be at risk if a rebate on loss policy is offered to a player who only plays for a limited period of time. The casino may be engaging in a net negative expectation game unless the player is required to play a certain number of hands. How long, then, should a casino require a player to participate in a given game before the likelihood of an expected loss is sufficient to justify the giving of a certain rebate on the player's actual losses? Or, turning this around, for a given duration of play, what percentage rebate on loss can the casino give and still retain the advantage? This question is addressed in an excellent paper by Andrew MacDonald,[184] who argues that rebate on loss policies are often haphazard processes and should change to policies/standards based on mathematically-sound business decisions, in much the same way that paying complimentaries as a percentage of drop or credit line has evolved to basing these on calculations of theoretical casino win. MacDonald's approach is to determine the percentage rebate on actual loss that is equivalent to a specified percentage rebate on theoretical loss for a given number of hands played.[185] The procedure outlined below is adapted from MacDonald's system.

> **Chance favors only the prepared mind.**
>
> Louis Pasteur
> Quoted in James H. Austin, *Chase, Chance, and Creativity* (1977)

The percentage rebate on actual loss equivalent to a specified refund on theoretical loss is a function of the house advantage, the number of hands played, the wager standard deviation, and the desired theoretical loss equivalency. The calculation also entails the use of the *Unit Normal Linear Loss Integral* (*UNLLI*), a mathematical relative of the standard normal probability distribution.

Roughly speaking, the values of the *UNLLI* function give the sum of the products of all values of a standard normal random variable's positive distance above a number "*a*," multiplied by their corresponding probabilities. The formal mathematical specification of this function is:

$$UNLLI(a) = \int_a^{\infty} (z-a) f(z) dz \,,$$

where $f(z)$ is the standard normal probability density function.[186] A table of values of this function is given below.

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184 Andrew MacDonald, "Player Loss: How to Deal with Actual Loss in the Casino Industry," in The Business of Gaming: Economical Management Issues (1999).

185 See also Peter Griffin, "Should You Gamble if They Give Back Half Your Losses," in Extra Stuff: Gambling Ramblings, (1991).

186 An alternative specification of the UNLLI function that is particularly useful for computing UNLLI values in a spreadsheet, is: $UNLLI(a) = \frac{e^{-a^2/2}}{\sqrt{2\pi}} - a(1 - Z(a))$, where Z(a) is the cumulative standard normal distribution function.

UNIT NORMAL LINEAR LOSS INTEGRAL (UNLLI)					
Z	**.00**	**.02**	**.04**	**.06**	**.08**
0.0	.3989	.3890	.3793	.3697	.3602
0.1	.3509	.3418	.3329	.3240	.3154
0.2	.3069	.2986	.2904	.2824	.2745
0.3	.2668	.2592	.2518	.2445	.2374
0.4	.2304	.2236	.2169	.2104	.2040
0.5	.1978	.1917	.1857	.1799	.1742
0.6	.1687	.1633	.1580	.1528	.1478
0.7	.1429	.1381	.1334	.1289	.1245
0.8	.1202	.1160	.1120	.1080	.1042
0.9	.1004	.0968	.0933	.0899	.0865
1.0	.0833	.0802	.0772	.0742	.0714
1.1	.0686	.0659	.0634	.0609	.0584
1.2	.0561	.0538	.0517	.0495	.0475
1.3	.0455	.0436	.0418	.0400	.0383
1.4	.0367	.0351	.0336	.0321	.0307
1.5	.0293	.0280	.0267	.0255	.0244
1.6	.0232	.0222	.0211	.0201	.0192
1.7	.0183	.0174	.0166	.0158	.0150
1.8	.0143	.0136	.0129	.0123	.0116
1.9	.0111	.0105	.0010	.0094	.0090
2.0	.0085	.0080	.0076	.0072	.0068
2.1	.0065	.0061	.0058	.0055	.0052
2.2	.0049	.0046	.0044	.0041	.0039
2.3	.0037	.0035	.0033	.0031	.0029
2.4	.0027	.0026	.0024	.0023	.0021
2.5	.0020	.0019	.0018	.0017	.0016

Table entries are values of *UNLLI(a)* = $\int_a^\infty (z-a)\cdot f(z)dz$, where $f(z)$ is the standard normal probability density function. For example, *UNLLI*(1.46) = .0321.

COMPUTING REBATE ON ACTUAL LOSS

Assuming an even-money payoff and equal bets, the following formula may be used to determine the necessary percentage rebate on actual loss:

$$EQUIV.\ REBATE = \frac{N \times HA \times LE}{(N \times HA) + [UNLLI(z) \times \sqrt{N} \times SD]}$$

where N = the number of hands,
HA = house advantage,
SD = wager standard deviation,
LE = theoretical loss equivalency,
$z = (N \times HA)/(\sqrt{N} \times SD)$,
UNLLI(z) = unit normal linear loss integral for z (see table).

Note that the expression $N \times HA$ appearing in the above formula represents the player's expected loss over the series of N hands and that $\sqrt{N} \times SD$ represents the standard deviation associated with this expected loss (i.e., SD is the standard deviation for a single wager while $\sqrt{N} \times SD$ is the series standard deviation – see Chapter 1 for further details on wager and series standard deviations). Thus the equivalent rebate formula can be rewritten in terms of these quantities as:

$$EQUIV.\ REBATE = \frac{Expected\ \ Loss \times Loss\ Equivalency}{Expected\ \ Loss + [UNLLI(z) \times Series\ SD]}$$

where $Expected\ \ Loss = N \times HA$,
$Series\ SD = \sqrt{N} \times SD$,
z = *Expected Loss/Series SD.*

Assuming the house advantage is not too large (a reasonable assumption for most casino wagers), the wager standard deviation for an even money bet is close to 1, and so the series standard deviation for N hands is close to $\sqrt{N}$.

As an example, suppose a casino wishes to give a rebate on actual loss in baccarat that would be equivalent to a 50% rebate on theoretical loss. Further assume the following values: HA = 1.27% (the average of banker and player edges ignoring ties), wager standard deviation equal to 1 (approximately correct), and N = 1,200 decisions (non-tied hands).[187] Then the *Series SD* = $\sqrt{1{,}200}$ = 34.641, z = (1,200 x .0127)/34.641 = 0.440, and *UNLLI(z)* = *UNLLI*(0.440) = .2170.

The equivalent rebate on actual loss is:

$$EQUIV.\ REBATE = \frac{1{,}200 \times .0127 \times .50}{(1{,}200 \times .0127) + [.2170 \times 34.641]} = 0.3348.$$

Thus, after 1,200 (non-tied) hands, a refund of 33.5% of actual player losses would be equivalent to a 50% refund on theoretical loss.

MacDonald gives five reasons why it is worthwhile for casinos to use an analysis such as that given above to implement a rebate on loss policy. First, this method provides a mechanism by which any existing rebate on loss policy can be analyzed to assess the long-term or theoretical cost to the business. Second, the method provides a

187 Since about 9.5% of baccarat hands end in ties, about 1,326 total hands would produce, on average, 1,200 non-tied decisions. Thus, using n=1,326 and HA = 1.15% - the average of banker and player edges when ties are included – would yield similar results.

means to structure variable percentage rebates on loss which can be both attractive and combined with rebates on turnover or the provision of other comps. Third, the method provides a challenge to incorporate a rebate on loss element into the standard calculation of premium player comps, and in so doing, addresses the dissatisfaction of players who incur a substantial loss but are not entitled under a theoretical loss-based to the complimentaries they believe they should be. Fourth, the method is not as complicated as it appears, as most player rating systems are, or can be, computerized. Fifth, players will accept this method just as they accept other player-rating methods – largely on trust – and this method has the advantage that it can legitimately be praised as quantified, objective, and equitable.

MIXING SYSTEMS

By tracking both the minimum equivalent bet and the observed loss, the casino can give comps when the player loses a substantial amount of money in a short period. In this case, because play is shortened by a run of bad luck, the player's theoretical loss would be far less than their actual loss. The observed loss could still be used to reward the player for his play and meet his expectations.

Most casinos will track both theoretical win and observed losses. The reason for this is simply that some players will lose most if not all of their bankroll quickly either because they are extremely bad players or they experience an incredible streak of bad fortune. Although the house advantage represents the average outcome in any game, there is a bell-shaped (normal) distribution of outcomes centered on the expected house win. Some players will end up at the lower end of the house win distribution with a large player win. Others, perhaps merely "unlucky," will be at the upper end, with a resulting large player loss and house win. Some casinos will grant comps or rebates if the player loses all or more (often 80% or more) of their credit line or front money deposit even if they do not meet the criteria that the casino sets for minimum theoretical loss.

REGULATORY ISSUES

10

As one of the most heavily regulated industries and one built on the mathematics of games, mathematical issues often arise in the regulatory process. While regulation often attempts to achieve many goals, most regulatory systems share common objectives: keep the games fair and honest and assure that players are paid if they win. Fairness and honesty are two different concepts. A casino can be honest, but not fair. Honesty refers to whether the casino offers games whose chance elements are random. For example, a slot machine is honest if the outcome of each play is not predetermined in the casino's favor. Fairness is very much a mathematically based consideration. How much of each dollar wagered should the casino be able to keep?

In economic terms, the regulators attempt to set the maximum price a casino can charge players for the gambling experience. The converse of maximum price regulation is minimum price regulation, that is, the lowest amount a casino can charge. As discussed in Chapter 5, price in the casino is a function of the odds the casino offers to the player. Therefore, regulatory price controls are a function of controlling the game odds.

Regulations related to honesty are generally twofold. The first is to assure that the games produce random results and players have an equal opportunity to win each prize offered on any given play. Moreover, these results should conform to the approved or represented odds of the game being offered. These regulations can range from standards for random number generators in gaming devices to methodologies to test whether the games are performing as mathematically expected. This chapter covers these regulatory issues.

REGULATIONS RELATING TO FAIRNESS

FAIRNESS AND REGULATORY PRICE CONTROLS

INTRODUCTION

Fairness refers to the games being designed so that they do not take unreasonable advantage of the player. For example, some slot machines in traditional casinos are designed to pay back on average 95% of all wagers accepted. The five-percent retained is the casino's profit. This is a reasonable amount for the casino to retain to pay for its capital costs, operating expenses and a fair profit. Setting the machines to retain 40%, however, may not be reasonable.

In most industries, regulatory price controls mean the government sets prices for services. For example, a state public service commission may set basic telephone rates at $9.00 per month. In the casino industry, government sets rates by dictating odds of the games. For example, a jurisdiction may require casinos to offer no more than double odds on craps, and their roulette table must have 0 and 00. These regulations set prices in the casino industry; so do requirements that a casino use multiple decks in blackjack. With slot machines, these standards are usually defined by setting a minimum payout, i.e., prohibiting the casino from theoretically retaining more than a set amount of each dollar wagered. For example, many jurisdictions provide that gaming devices must have a minimum payout, such as 70%. This means that the maximum average price that the casino operator can charge is $.30 for each dollar played.

RATIONALES FOR PRICE CONTROLS

INADEQUATE INFORMATION

Three rationales are argued as the basis for setting maximum price payouts. The first rationale is that players generally are incapable of obtaining information about the true price of the casino games. Because of this, they cannot make rational decisions on whether to gamble or not, or to comparison shop. In contrast, casinos that have perfect knowledge of pricing can exploit the player.

In most retail transactions, consumers provide the best protection in assuring the transaction is fair. This comes because consumers can view the product or services and compare the cost relative to other identical, similar or substitute products. A key here is that consumers have knowledge of price and other market information. In the casino industry, however, consumer knowledge about pricing ranges from sophisticated to oblivious. Moreover, even sophisticated gamblers may have difficulty in appreciating the price of certain bets. Table games are most susceptible to determining pricing. Assuming random results, players can determine the odds of the game based on its rules and method of play. Even unsophisticated players can buy any of a myriad of books that explain both odds and optimum play. In many gaming devices, however, the player is incapable of determining the odds of winning and the frequency of payouts. For example, a video slot machine can have a virtually unlimited number of stops on each reel. The number of such stops is a matter of programming the software that drives the gaming device.

A regulatory response to the lack of consumer information about pricing is to set price in an attempt to assure that the games are fair to the player.

FAIRNESS AND MONOPOLIES

A second rationale for price controls applies to monopoly markets. If the casino exists in a monopoly (or oligopoly) market, setting maximum prices may be necessary to prevent the monopolist from using its market power to exploit players.

Government ordinarily does not regulate prices in competitive economies. The reason is simple. Suppose that, in a perfect competitive environment, the competitive price for homogenous product like milk is $1.00 a gallon. The competitor who attempts to charge a higher price for milk will not sell its product. If casino players have prefect knowledge of the odds in every casino game, then they can make decisions on where to gamble based on the information.

In contrast, a monopoly casino can still charge monopoly prices. It would do this by increasing table minimum bets or minimum denominations of its gaming devices. This would raise the average cost that a player would pay to play in the casino. To compensate for both factors, a jurisdiction needs to set maximum theoretical wins on gaming devices and table games based on minimum bets for these games as well as the mix of devices per denomination. A monopoly casino also can raise the cost of non-casino products or services. For example, it could charge high parking rates, admission to the casino, or raise the price of food and beverages.

From a policy perspective, minimum price setting in the casino industry is rarely, if ever, justifiable. On one hand, the casino industry can use it to fix prices. In the worst situation, minimum price setting may harm the jurisdiction's ability to compete with other jurisdictions.

Price setting disrupts a competitive market. Often minimum price setting helps established firms because it eliminates competitive pricing. If the industry's market is only locals, price setting may be good for casinos that do not have to compete based on price, and the price set is above market price. This is bad for players, however, which will not get the lowest prices.

If the market is national or regional, minimum price setting may harm both the players and the casinos. A problem in these jurisdictions is that serious players do not like the rigidity of fixed odds. If the cost of traveling is lower than the "price" differences, players will seek better deals at casinos that compete based on odds. Players only benefit from price setting in systems, such as monopolies, where the casino could and would give worse odds if the law did not mandate lower odds.

A similar consideration is regulatory flexibility in adapting to market conditions. Does the casino have business discretion to change game rules and procedures, game mix, and to introduce new games? If it does, the casino industry can adopt new trends and technologies very quickly. For example, in a flexible system, when a new game becomes popular, the casino offers it for play without the need for a laborious approval process.

PLAYER PROTECTION

A third rationale for price controls is to protect the players from losing too much money. In a jurisdiction that adopts to protect the public in all gambling transactions, regulators may attempt to set high minimum payouts to protect the gambler because it would prevent the player from losing too much money during the time that the casino can open in a given day.

Colorado

- The slot machine game program must theoretically pay out at least 80.0 percent and no more than 100.0 percent of the amount wagered. The theoretical payout percentage is determined using standard methods of probability theory. When applied to games whose outcome is determined in whole or in part by skill, the 100.0 percent theoretical payout shall be computed using the optimal play strategy for compliance of the given game tested and the 80.0 percent theoretical payout will be computed using the lowest manufacturer's expected return for the game program.
- The slot machine game program must have a probability of obtaining the maximum advertised single payout better than 1 in 17 million. A multi-link progressive system slot machine game program must have a probability of obtaining the maximum advertised payout better than 1 in 50 million.

Fairness and Product Differentiation

In markets where the government or other factors do not create a monopoly or oligopoly, whether a market is competitive is usually based on four factors that contribute to a competitive economy. The first factor is whether the products supplied by competitors are identical. Economists refer to identical products as being homogenous. If the products are homogenous, players can base their decisions to buy solely on price. The second factor is whether any seller is so large that it can effect the decisions of other sellers by increasing or decreasing output. In a competitive economy, no seller can effect the output decisions of other sellers. The third factor is whether sellers have the same access to input, such as raw materials or workers. The fourth factor is whether all buyers have accurate and immediate information on pricing and output.

Until the middle of the 1960s, casinos offered basically the same product: casino gaming. The games offered by the various casinos were similar. In this homogenous environment, casinos competed for players based primarily on price. This situation was similar to other industries. For example, if consumers are informed of price differences between milk, milk producers must compete on the basis of price. If one producer sells its milk at $1 per gallon, it sells all of its milk before another producer can sell its milk at $1.10. To compete, the second producer would have to lower its prices to $1.00. The casino industry in the 1960s was similar, except that the nature of its pricing was different.

Almost every casino game can be modified to change the odds to the player. In blackjack, the casino can offer variations, such as single versus multiple decks; allowing players to surrender for half their bet; allowing double down on any two cards; and other modifications that affect the theoretical house advantage. In roulette, the odds are drastically changed if the casino has one or two "zeros" on the wheel. The odds of winning at gaming devices can change by simply changing the symbols on each reel, or in this age of microchips, simply changing the computer program that runs the gaming device.

> **South Dakota/Wisconsin**
>
> At least 80 percent and no more than 95% of the amount wagered. … maximum payout probability greater than 1 in 17,000,000.

If the modern casino industry were homogenous, sophisticated gamblers could make informed pricing decisions on which casino offers the most competitive prices on almost all games offered. The product offered by the casinos, however, is no longer homogenous. The casino product has evolved in the past thirty years. Rather than simply paying for the opportunity to gamble, the player is buying a "casino experience." Some players are willing to pay more to play in one casino as opposed to another. This could be because the favored casino is more opulent, has childcare facilities, is a resort, or a variety of other factors.

The idea of creating a "casino experience" was both a response to player demand and an attempt by casino owners to differentiate their product from competitors. Since computer-controlled gaming devices have become the largest source of casino revenue, product differentiation became less evident. Regardless of which casino a player visits in a market, they all have basically the same gaming devices and games. Even where some differences exist, casinos have greater difficulty advertising them and most players, particularly tourists or casual gamblers, have greater difficulty in understanding them. For example, they may not understand the pricing differences between allowing and not allowing the surrender rule in blackjack, but they do know they want to visit a particular casino that has a unique shopping mall or a live action pirate show.

A casino experience extends beyond the games offered to include casino ambiance, service, and amenities, such as rooms, food, pools, entertainment, and other attractions. More frequently, players are paying for a casino experience product package. Not only are the packages less homogenous, but the pricing of the package also is more complex because it includes the cost of gaming and other costs, such as room, food, beverage, and entertainment. Despite this, the pricing of casino games remains an important factor in the pricing of the casino experience, particularly for premium and experienced players.

WHAT IS FAIR?

While good policy reasons may exist in some jurisdictions to set maximum price, the government may have difficulty in fairly setting standards. Where the government decides to set the minimum rate, it should consider the maximum permissible bet, the maximum average price (or minimum payout), and hours of operation.

Most jurisdictions attempt to set maximum prices by mandating the maximum theoretical win the casino can derive from a game. The word "fair" is subject to consideration of many factors, many of which are difficult, if not impossible, to define in regulations.

Mississippi

All gaming devices submitted for approval must theoretically pay out a mathematically demonstrable percentage of all amounts wagered, which must not be less than eighty percent (80%) or greater than one hundred percent (100%) for each wager available for play on the device.

Regulations that set maximum price restrictions tend to be overly general. For example, regulations can require casinos to pay back 80% on slot machines. This approach has practical problems. These types of required paybacks may or may not be fair to the player depending on the minimum wager. If an 80% payback applies to a nickel gaming device, the average cost per hour for the player to play this machine with one coin-in and 325 pulls per hour is $3.25. This may be fair to the player, but unacceptable to the casino operator if the cost of maintaining a gaming device exceeds $3.25. As a result, the casino may not offer 5-cent slot machines if it is limited to a 20% advantage. In contrast, if the player is playing a dollar machine at a 20% advantage with three coins in, the average price is $195 per hour.[188] This is not a fair advantage over the player,

If a jurisdiction wishes to set prices, it must consider the gaming device denomination or table limits in deciding what is fair to both the player and the casino. The result of placing broad maximum price restrictions that apply regardless of denomination of game or whether the game is a table or gaming device, risks being unfair to both the player and casino. For example, such pricing requirements may actually increase the price of playing if the casino must eliminate low denomination machines or table games that become unprofitable to offer to players because of maximum price restrictions. This can be partially addressed by having different maximum pricing for different denominations.

Other factors influence price. These include the pace of the game and, in many cases, the player's skill level. Therefore, on table games or other games, the government may want to consider the average number of plays per hour and allow the casino to have higher theoretical win in the slower games.

What Fairness Standards Should Apply?

Most frequently, regulators set prices in the gaming industry by requiring the casinos to theoretically pay out a mathematically demonstrable percentage of all amounts wagered, e.g., not less than eighty percent (80%). This approach presents some problems.

Using theoretical wins will provide an average price per play, but it does not provide meaningful comparisons between different types of games or an easy standard for calculating costs. For example, a casino may charge more "per hand" for blackjack than a gaming device but make more per hour on a gaming device because the player can play two to six times more plays per hour.

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188 Concededly, individual players often play for the jackpot, which is substantially higher in a dollar machine. Where players are only gambling for entertainment, pay for that service, and know the cost of playing nickel versus dollar machines, most players would choose the less expensive nickel machines and not play the other gaming devices.

Some regulators may realize all the practical problems with trying to set maximum price by dictating theoretical win and look for some other standard. The most obvious is to review the average "hold" from the game as experienced over a trial period or other time period. Attempting to set minimum prices by dictating the maximum "hold" for casino games raises other problems.

As noted earlier, hold is important to the casino operator. It gives the operator a rough estimate of how the table games are performing. It is not, however, an accurate tool to test the fairness of the games. Casino operators use "hold" for casino games only because no better method of evaluating performance is available. Unlike gaming devices, no method exists to calculate handle on casino games. Thus, casinos rely on hold percentages to give them basic information on how the games are performing.

Regulators, however, should not rely on hold as a substitution for theoretical win percentage in deciding whether the game meets minimum standards. Many factors other than theoretical win can influence hold percentage, such as game interest, dealer conduct, and theft.

Nevada

All gaming devices must theoretically pay out a mathematically demonstrable percentage of all amounts wagered, which must not be less than 75 percent for each wager available for play on the device.

Gaming devices such as video poker that may be affected by a player's skill must meet this standard using a mode of play that will provide the greatest return to the player over time.

For gaming devices that are representative of live gambling games, the mathematical probability of a symbol or other element appearing in a game outcome must be equal to the mathematical probability of that symbol or element occurring in the live gambling game.

For other gaming devices, the mathematical probability of a symbol appearing in a position in any game outcome must be constant.

A better standard is average price per hour. Price per hour is a function of average price per play multiplied by the average number of plays per hour.

A second set of problems concern the inherent difficulty of setting maximum prices on table games. Governments can easily set maximum theoretical wins based on the denomination of the game when dealing with gaming devices. It is far more difficult, however, when dealing with table games. Except in jurisdictions with maximum bet restrictions, table games often allow a range of bets, e.g., from $5 to $100. Thus, one player at the table may be betting $5 per play while another player is betting $500 per play (five different crap bets at $100 per bet).

Another practical consideration is how to treat a game, such as craps, that allows for different bets. Some craps bets have low house advantages, while others are high. Should the government ban the bets with a house advantage over the maximum amount or consider a higher average of all bets on the table?

Mississippi

The third largest gambling state in the United States has regulations regarding gaming devices almost identical to that of Nevada. The minimum theoretical payout is 80% (and a 100% maximum is explicitly stated) and the random selection process must meet 99 percent confidence limits using a standard chi-squared test for goodness of fit.

If you set a maximum theoretical win of 5%, does the typical craps game qualify? An informed player can play the game with a theoretical win of less than 1%. But, uninformed players might make some bets, such as a "hard way," bet that have a theoretical win of over 5%.

Equally problematic are games involving elements of skill. The odds on blackjack are radically different depending upon the skill level of the player, which may range from a slight advantage for a skilled card counter, to a substantial disadvantage for a new player with no knowledge of even basic strategy. For example, a typical blackjack player pays about 1.5% more than a player using what is known as "basic strategy," i.e., making the decision in every instance with the highest expected return for the player. In setting the maximum theoretical win in these circumstances, should the regulators consider the most skilled or least skilled players?

MAXIMUM BET RESTRICTIONS

In some states, the government seeks to protect players from losing too much money in any given session. Some make mathematical sense. For example, a maximum loss restriction can assure that players do not continue to chase their losses, an activity that over time will assure that they intensify their losses. A problem with maximum loss restrictions is enforcement. One state attempted to do this by requiring players to buy script and use it to gamble. The casino could only issue a maximum amount of script to each player in a given session. The problem was that the script became more valuable as a black market item than its face value. This placed problem gamblers at an even greater disadvantage in their attempts to chase losses.

Another method is to put maximum bet limits. Until recently, South Dakota had a maximum bet of $5.[189] While theoretically this limits the potential loss that a player may sustain, it may also lower the casino's volatility and assure that more players end up with a net loss.

RESTRICTING MINIMUM PRICE

In the casino industry, minimum price setting would require a casino to theoretically retain at least a set amount of each wager made. For example, in South Dakota, the payout on gaming devices cannot exceed 95% of the amount wagered. Therefore, the casino must charge the player a minimum average price to play a gaming device of 5¢ for every dollar played. The law precludes a casino from lowering the price below the minimum to benefit the player. With casino games, like

189 In April 2001, this was raised to $100.

blackjack, it would prevent the casino from applying rules to the game such that players' theoretical disadvantage is at least a certain percentage.

Four rationales for minimum price setting are common. First, it may prevent firms from using persistent price discrimination to maximize monopoly profits, or where such pricing allows large firms to reduce competition. Congress expressed the latter concern when it adopted the Robinson-Patman Act in 1936, which makes it unlawful "to discriminate in price between different purchasers of commodities of like grade and quality . . . where the effect . . . may be substantially to lessen competition or tend to create a monopoly . . ."[190] In passing the Act, Congress was concerned with manufacturers giving quantity discounts to large retailers. This resulted in the large retailers selling comparable products at lower prices than small "mom and pop" businesses. In this sense, the Robinson-Patman Act is anti-competitive. If a manufacturer must sell at a higher price to the large retailer, the public will ultimately pay more for the same goods. Congress, however, felt that preservation of small businesses justified this cost.

The policy behind the Robinson-Patman Act, regardless of its much-debated merit, has no application to pricing by casinos. The Robinson-Patman Act focuses on discrimination in the cost of goods to the retailer, not in the prices that the retailer charges consumers. A casino's major product is not a tangible good purchased from a manufacturer/wholesaler, but a service. This service is to provide a gaming opportunity to players. While the cost of this service can be affected by the cost of equipment, the Robinson-Patman Act concerns would be directed at discrimination in the cost of that equipment and not the player's price to participate in gaming.

A second concern with price discrimination is its use by monopolies or a cartel to maximize its profits. When a monopolist sets a single price for its goods or services, it will chose the price where its marginal cost, i.e., the cost of producing one additional unit, equals its marginal revenue, i.e., the revenue derived from producing one additional unit. This price will exceed what the price would be in a competitive market where the price is equal to marginal cost.

A monopolist, however, is not making all the money possible if it sets a single price because some players would pay more than the monopoly price if no other alternative existed. The monopolist would prefer to sell the goods to these persons at the higher price. Similarly, some players would be willing to pay more than the competitive price, but less than the monopoly price.

A monopolist could make money by selling the product at a profit to these persons if it did not have to lower the price to other players. The monopolist can capture these extra profits only by perfect price discrimination, i.e., to sell the product to each player at the maximum price that the player is willing to pay. A monopolist with perfect price discrimination captures from the consumer the equivalent of consumer surplus.

Whether price discrimination, as opposed to monopoly pricing in general, is a proper subject for government intervention is a subject of considerable debate because, with perfect price discrimination, output levels are consistent with output levels in a competitive market. Others disagree, noting that price discrimination is

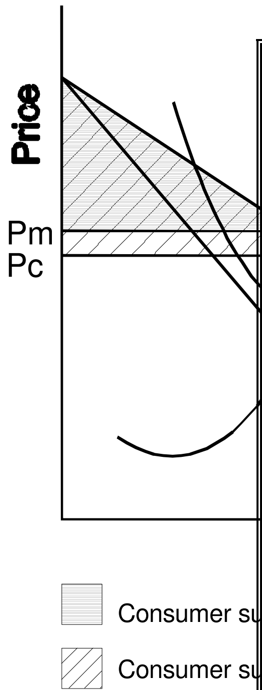
190 15 U.S.C.A. § 13A.

never perfect and results in output losses, that accomplishing price discrimination is expensive, and that price-discriminating monopolists will spend more to protect their monopoly because they are making substantially more profit.

Envisioning a casino market with economic conditions that would allow persistent price discrimination is difficult. Assuming a single demand curve, a casino can discriminate based on price when it charges different room rates for different days of the week and special days of the year, or offers different incentives (price lowering mechanisms, such as coupons, rolls of nickels, etc.) for players on “fun buses” and not to walk-in players.

Creating a monopoly in the casino industry without government assistance is difficult. Without government-created barriers to entry, competing firms can quickly enter a gaming market if demand exceeds supply, and existing casinos can exert no coercion to keep players from dealing with its competitors.

In situations where a casino has significant market share because of government regulation, the government might want to encourage price discrimination if the government’s goal is to maximize government revenues through gaming taxes. Because price discrimination maximizes profits to the monopoly casino, it also will benefit the government that bases taxes on gaming revenues. Price discrimination also assists in maximizing output that could help achieve other governmental goals, such as creating jobs. Moreover, if most casino players are tourists, the government may not be concerned that the market suffers from inefficiencies and that price discrimination by a casino monopoly results in a substantial transfer of wealth from the player to the casino and the government.



Colorado

Randomness means the unpredictability and absence of pattern in the outcome of an event or sequence of events. Events in slot machines are occurrences of elements or particular combinations of elements that are available on the particular slot machine. A random event has a given set of possible outcomes, each with a given probability of occurrence. The set of these probabilities is called the distribution. Two events are independent if the outcome of one has no influence over the outcome of the other. The outcome of one event cannot affect the distribution of another event if the two events are independent. The random number generator in a slot machine must produce game plays that are random and independent.

Requirements for randomness testing in Colorado state that selection process is considered random if the following specifications are met:

(1) … satisfies at least 99 percent confidence level using standard chi-squared analysis. Chi-squared analysis is the sum of the squares of the difference between the expected result and the observed result.

(2) … satisfies at least 99 percent confidence level using the Median Runs Test or any similar pattern checking statistic. The Median Runs Test is a mathematical statistic that determines the existence of recurring patterns within a set of data.

(3) … is independently chosen without reference to any other event produced during that play. This test is the correlation test. Each pair … at least 99 percent confidence level using standard correlation analysis.

(4) … is independently chosen without reference to the same event in the previous game or games. This test is the serial correlation test. The event is considered random if it meets at least 99 percent confidence level using standard serial correlation analysis.

(5) The random number generator and random selection process must be impervious to influences from outside …

Having said all this, creating near perfect price discrimination in the casino industry is impossible. A casino has difficulty setting different odds on the same game or gaming device for different players. In contrast, a casino can vary the odds between different denomination games and gaming devices.

Casinos also can price discriminate by giving rebates on every wager or discounts on losses. One method of rebates is to give the player more in face value than he bought, e.g., $105 in chips for every $100. These chips can be non-negotiable and the player would be required to gamble them at particular games, such as baccarat or in the race book. By doing this, the casino is effectively lowering the odds to the players who have a greater chance of winning than those players who do not receive rebates. Discounts on losses allow casino players, usually credit players, to have to repay on a portion of the total money lost at the casino. For example, if the player loses $10,000 but has prearranged a 10% discount, he need only repay $9,000. This also effectively changes the odds of the games.

A third argument for setting minimum price is that it prevents "predatory" pricing. Here, large and well-financed casinos would initially lower odds below the point where the casino could make a profit. By doing so, it would capture most of the market share. Its competitors would go out of business either because they do not lower prices and lose players, or do lower prices but cannot make a profit. Because the larger casino can operate at a loss longer than the smaller casino, the small casino will first go out of business. After a large casino drives competition out of the market, it can then raise prices to levels above the competitive price. For this to occur, however, you would need an uncommon market situation where one competitor is so much better financed than the other competitors that it can afford to lose significant money until its competitors close. Moreover, the market must be such that new competitors cannot easily enter the market after the large casino increases its prices and begins making super-normal profits. This is an unlikely circumstance in the casino industry.

A fourth argument is that setting the price discrimination at 100% is reasonable because anything over that amount would result in a loss. In such circumstances, the state would lose tax revenues because the losses from the gaming devices would offset gains from other devices. This logic is defective in most circumstances because it ignores that most casino operators are rational businessmen who will not intentionally lose money to save on taxes unless the tax savings are greater than the revenue losses. The only logical reason that a casino operator would operate with theoretical payouts over 100% is as a marketing tool (i.e., a "loss leader"). The idea is that a player will come into a casino only by the attraction of the loss leader, and once there, buy other products. For example, in Nevada, many casinos will pay out more in winnings on bingo that it collects from the players. The reason that they continue to offer bingo or do not raise the price is because bingo players are generally slot players. The corresponding increase in slot revenues justifies the losses in bingo.

REGULATIONS RELATING TO HONESTY

PROPER STANDARDS FOR DETERMINING A GAME'S HONESTY

Casino games come in two primary forms, those controlled by a computer and those that are not. Methods to determine the honesty of non-computer games depend on the equipment used to determine the random elements of the game. For example, precise instruments are used to assure that all of the sides of a pair of dice are the same length and width and are parallel. Likewise, roulette tables can be inspected to assure that they are perfectly balanced.

Traditional methods to assure the honesty of games must give way when a computer determines the game outcome.

ASSURING RANDOMNESS

Computer-controlled gaming devices attempt to create an absolute game of chance.[191] For example, the State of Washington considers a device to be an absolute game of chance if it meets a 99% confidence limit on standard tests for randomness in its dealings for each hand.[192]

Typical legislation requires that:

The random number selection process must conform to an acceptable random order of occurrence and uniformity of distribution; the field of numbers must be mixed after each game by using a random number generator; after the field of numbers has been mixed and before the start of the game, the field of numbers is to be frozen with all numbers used for play taken in order from top of the frozen field.[193]

Nevada

All gaming devices manufactured and distributed for use in Nevada must meet certain mathematical standards for randomness in choosing the game's outcome. Regardless of the method, the greatest regulatory concern is that the device selects the cards or symbols within acceptance levels of randomness.

"Randomness" is defined as the observed unpredictability and absence of pattern in a set of elements or events that have definite probabilities of occurrence.

Nevada's requirements state that a gaming device must use a random selection process to determine the game outcome of each play of a game. This process must ensure that each possible permutation or combination of symbols, cards or numbers is available for random selection at the outset of each play. No pattern of results, based on the previous game's outcome, the amount wagered, or the style or method of play, should be detectable. Nevada requires that the random selection process "must meet 95 percent confidence limits using a standard chi-squared test for goodness of fit."

Different gaming devices meet the various standards for random selection of cards on a poker machine or symbols on a reel-type slot machine in different ways. For example, on one video poker device, each time a player initiates the game; the device's software electronically shuffles the deck in its memory enough times that the hand is random. If the game is played with one deck, seven shuffles are random; two decks, nine shuffles; and six decks, twelve

191 Cabot, Casino Gaming: Policy, Economics and Regulation, 377 (1997) Each time a play is initiated, a new randomly-shuffled deck of cards is used in video gaming devices.

192 Id.

193 Montana Administrative Rules, 23.16.1906 (1990).

shuffles.[194] This is better than traditional casino games because most casinos only shuffle two to six decks between three to four times at the end of a complete shoe.[195]

Other video poker devices will, through their software, generate a string of random numbers from a pool of 52 numbers for each play. Each number will be assigned a card in a normal 52-card deck. Often the string is only ten numbers long, corresponding to the maximum number of cards needed to play a hand of video poker. In reel-type slot machines, the computer generates random numbers that correspond to a stop on the reels. A separate random number is generated for each reel.

Regardless of the method, the greatest regulatory concern is that the device selects the cards or symbols within acceptance levels of randomness. Software specifications attempt to create a fair and random game.[196] Software for video gaming devices is the internal direction that dictates the game's play.[197] The software is usually located on EPROMS[198] in the logic board. EPROMS contain the information that controls the video gaming devices' play.

At the heart of all computer-controlled gaming devices is the random number generator ("RNG"). A RNG is merely an algorithm written into the computer software that is found on the EPROM. The algorithm is a mathematical formula that produces numbers. The newest devices often use a 32-bit random number generator. This allows a device to generate numbers between zero and two to the 32nd power, or, in whole numbers, to 4,294,967,296 different sequences of random numbers.[199] Depending on the design parameters, each sequence may correspond to a group of symbols on a reel-type slot machine or to a group of cards on a video poker machine.[200] While the design of the algorithm may differ between programmers, the results generated by the algorithm should meet three primary tests. It should:

- ♣ Meet **minimum confidence levels** that the numbers generated are random;
- ♣ **Run continuously**; and
- ♣ Have **proper seeding**.

MINIMUM CONFIDENCE LEVELS

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194 See generally, Computer: "Seven Times and It's a Good Shuffle", United Press International, Jan. 9, 1990. See also "In Shuffling Cards, 7 is Winning Number", N.Y. Times, Jan. 9, 1990, at 1, col. 1.

195 Clark, The Dictionary of Gambling, 207 (1987). A "shoe" is the housing for the cards used during casino play. Id. A shoe can hold multiple decks at a time, and is mainly used for blackjack. id.

196 Cabot, Casino Gaming: Policy, Economics and Regulation, 517 (1997)

197 Id. at 534 (1997)

198 An "EPROM" (Electronic Programmable Read-Only Memory) is a computer chip which governs the operation of a slot machine. Id.

199 Dwight and Louise Crevelt, Slot Machine Mania, 15 (1988).

200 Id.

Because a computer program relies on an algorithm to generate numbers, the numbers generated can never be completely random. Similarly, at a table game, the shuffle of the deck can never be completely random because the dealer will split the deck in about the same place, and shuffle the cards in about the same way for each shuffle. Thus, the beginning placement of each card will influence the final placement of each card in the deck. A computer program can do a better job of randomly arranging a deck than can a human dealer, but because the RNG is based on a computer program, it can never be completely random. Instead, the numbers generated must appear random to a high degree of confidence.

> **South Dakota/Wisconsin**
>
> … at least 95% confidence level using the standard chi-squared analysis. Chi-squared analysis is the sum of squares of the difference between the expected result and the observed result. … runs test for patterns or any similar pattern testing statistic (at least 95% confidence level)
> … standard correlation analysis for reel positions independently chosen (at least 95% confidence level)
> …standard serial correlation analysis reel positions re: previous game (at least 95% confidence level)

Some tests that are typically performed as part of the random number generation test are the Chi-Squared Analysis, the Reel-to-Reel Correlation Test, and the Serial Correlation Test. For example, South Dakota requires the devices to have a random selection process that satisfies a 95% confidence level using standard chi-squared runs and serial correlation tests.[201]

> **Iowa/Louisiana**
>
> The game in each device shall be random and shall be tested to at least a 99% certainty using a standard correlation analysis. Correlation test or analysis is when each card, or number position is chosen independently without regard to any other card or number drawn within that game play.

Continuous Generation of Numbers

A second general requirement for an RNG is that it runs continuously. This means that while the player is playing the device or while it sits idle, the RNG is continuously producing new numbers. Because the microprocessor moves so rapidly, choosing a new random number every millisecond, even if a player knows the sequence of numbers that were about to appear, he would have almost no chance of being able to stop the microprocessor on a specific number.

All RNGs need human interaction to stop processing pseudo-random numbers, and produce a number or set of numbers for the play.[202] A player provides what is known as "player-instigated action" that signals the microprocessor to produce the number or set of numbers. The nature of this action will vary between devices and

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201 D. Gromer & A. Cabot, South Dakota, in International Casino Law (3d Ed.) (Cabot, et al., eds.,1999).

202 Dwight and Louise Crevelt, supra note 144, at 16-17.

manufacturers. It could be the acceptance of coins or tokens by the device, the pull of the handle, or some other pre-defined time. This accomplishes many objectives. Foremost, it assures that persons cannot string together plays of the device to derive patterns. It also prevents a person from observing the outcome of games on one device, finding a second device with the same RNG, and waiting until the outcomes are in synch and playing the second device knowing the outcome.

RANDOM SEEDING

The third primary test is random seeding. This means that each device, when initially activated or reset, will start with a different or random number. In a reset, this can be accomplished by having the RNG store the last-known value, and begin at that point when reactivated.

NEAR-MISS PROGRAMS

The existence of "near-miss" programs was brought to prominence in a disciplinary action by the Nevada regulators against a licensed manufacturer. In its complaint,[203] the regulators stated that all gaming devices manufactured and distributed for use in Nevada must meet certain mathematical standards for randomness in choosing the game's outcome. The regulators alleged that the manufacturer's devices did not meet the standard for randomness because it contained a "near miss" feature. In other words, the manufacturer programmed its machines so that certain jackpot symbols appeared, in losing combinations, more frequently than they would through normal random selection.

Most slot machines independently select the final position of each reel, and then decide whether the combination of reels is a win or a loss. The manufacturer's machines first decided whether the game result was a specific winning combination (for example, three 7's) or a loss; and then, if a loss, decided the positions of the reels from one of a group of tables of losing combinations.

As a result, certain losing combinations that included jackpot symbols appeared more frequently than other losing combinations that did not include jackpot symbols. The effect was to give the player the illusion that he was nearly winning almost every time he lost. This was the origin of the label "near miss."

Regulators brought a disciplinary action that included a demand that the manufacturer replace or modify all "near-miss" machines. In response to the regulators' complaint, the manufacturer admitted the allegations concerning how the machines decided the game outcomes, but denied any wrongdoing. The manufacturer denied that the "near-miss" feature represented an unacceptable manner of play. The manufacturer argued that no law or regulation prohibited its game process, or required a process of first independently selecting the position of each reel and then deciding if the game outcome is a win or a loss.

The manufacturer said the regulators' standards for approval of new gaming devices were internal standards that did not rise to the level of law or regulation which were constantly changing without notice to manufacturers. At that time, Nevada did

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203 State Gaming Control Board v. Universal Distributing of Nevada, Inc., No. 88-4 (Nev. Gaming Comm'n) (May 5, 1988).

not specifically prohibit a “near-miss” feature, nor did it have written minimum standards for gaming devices.

The manufacturer’s arguments in support of the suitability of the near-miss feature were threefold.[204] First, it argued that the “near-miss” feature did not decide whether the player won or lost. Instead, the feature only selected the losing combinations after the microprocessor had randomly determined that the player had lost. Second, the manufacturer contended that other manufacturers got similar results by using reel strips with more jackpot symbols on one reel than on other reels. Third, the manufacturer argued that the “near-miss” devices were the most successful and popular devices, and that it had not received a single player complaint.

After receiving extensive testimony from several experts in statistics, the regulators decided that a randomly occurring “near miss,” such as that on a traditional reel strip machine, was acceptable. The regulators decided that the “near miss” was not a randomly occurring “near miss,” and that the manipulation of a losing combination to alter the result shown to the player was unacceptable.

The final issue was whether a gaming device was random if it manipulated the losing combination to alter the result shown to the player. The regulators decided that such a device was not random. The regulators ordered the retrofitting of all the manufacturer’s “near-miss” machines.

Near-miss programs also can be programmed in video poker machines. They design this program to display an “almost” winning combination (near win) when the player has lost that game.[205] After a player receives the hand and discards unwanted cards, the program begins. Before revealing the player’s final hand, the program determines if that hand is a loss. If it is a loss, the program searches the frozen field for a combination close to a winning hand and displays those cards.[206] The result is that they lead the player, although losing that game, to believe that he or she almost won, and that a winning combination may soon appear.[207] In reality, while a winning combination may be statistically unlikely, the possibility of winning or coming close to winning often induces that player to play again. Many players set a limit upon how much they will play, but after seeing the near-miss, they often play more then they would have otherwise. [208]

To curb the programming of ghost and near-miss programs, governments often institute regulations that rely on Government Test Laboratories and the integrity of the manufacturers.[209] To combat ghost programs, regulators run various tests on the device’s software before its licensing, and recall the devices for periodic retesting.

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204 In the Matter of the Petition of Universal Co., Ltd. and Universal Distributing of Nevada, Inc. for a Declaratory Ruling, No. 88-8 (Nev. Gaming Comm'n) (Oct. 13, 1988).

205 The game determines if the player has lost. If so, the game displays a hand close to a winning combination. Id.

206 Id.

207 Id. The program induces the player to believe that a jackpot may appear within the next few tries. Id.

208 Id.

209 Cabot, Casino Gaming: Policy, Economics and Regulation, 385 (1997)

As technology enhances, the test laboratories must work with the manufacturers, and depend on their honest submission of fair games. Id.

Regulators also limit and monitor access to the devices since persons with access to the logic board can install ghost programs after inspection.[210] Regulators can also limit the type of medium on which software can be stored in the machine, such as chips with permanent read-only memory, or hard drives using sophisticated data encryption. These requirements make it more difficult to install ghost programs after the devices have been inspected and approved by regulators. Ghost programs are typically not outlawed by name or description, but tangentially through detailed and specific legislation that dictates how proper software should run. Any intentional variations from these narrow legal parameters may carry criminal sanctions.[211] However, regulators increasingly rely on the manufacturers to submit legitimate software because, as computer technology grows more complex, finding hidden programs is harder for game-testing technicians.[212] If a game is found to contain deceptive software, the manufacturer may be liable for criminal sanctions similar to those imposed for ghost program violations.

Because the risks of being caught with deceptive software carry great penalties, jurisdictions with legalized video gambling have been successful in preventing deceptive programs.[213] If a manufacturer is caught submitting a deceptive program, not only would it risk criminal sanctions, but the government would stop business relations with that manufacturer.[214] If an operator were caught trying to install a deceptive program, he would also risk criminal sanctions and the loss of license. Because the legalization of video gambling will generate a substantial income to the manufacturer and operator, risking involvement with the use of deceptive gaming practices is illogical.[215]

CHEATING

Cheating in casino gambling has three broad categories. The first is in simply predetermining the outcome of the game. This would occur, for example, where the deck in a card game has the cards in a predetermined order or a skilled dice player can "slide" the dice so that a desired combination will appear.

The second category is "past posting," or in other words, placing or removing a bet after the outcome has already been determined. A skilled cheat can increase or decrease his bet after the game ends. For example, a cheat with a losing hand in blackjack can "pinch the bet" by palming one chip in the stack wagered. If the cheat adds a chip after learning that he has a winning hand, he pressed or past-posted the bet. Both pressing and past posting are illegal.

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210 Id. A logic board is the brain of the game. Id. It contains all the EPROM's and electrical components necessary to run the game.

211 Standards On Operating Coin-Operated Video Gaming Devices, appx. III, 14 and appx. I, 17 (N. Am. Gaming Regulators Ass'n. 1990).

212 Id.

213 Cabot, Casino Gaming: Policy, Economics and Regulation, 386 (1997)

214 Id.

215 Id.

The third category is to obtain knowledge unavailable to other players that gives that player a theoretical advantage over other players or the house. A good example of this is shining in blackjack. Shining most often occurs in blackjack. Shining is the method of learning the dealer's hole card using any reflective device. This could include a metallic cigarette lighter, a facet on a ring, a polished fingernail, or a small mirror. If a person knows the dealer's hole card, he has an advantage over the casino through having additional information to figure out whether to "double down," i.e., relinquishing his hand and only losing half his bet, or determining whether to hit or stand. The statistical advantage gained by the player that can learn the dealer's hole card is substantial. Knowledge of the dealer's hole card on every hand can give a skilled player who would otherwise play about even or at a slight disadvantage more than a 9% advantage.[216]

To cover these possibilities, most casino states have laws that prohibit a person from placing, increasing, or decreasing a bet or to decide course of play after acquiring knowledge, not available to all players, of the outcome of the game or any event that affects the outcome of the game.

The idea that the player can acquire an advantage by learning the dealer's hole card was the point of a game introduced in the 1980's at the Vegas World Casino in Las Vegas. The game was called "Double Exposure Blackjack" and allowed the players to see the dealer's hole card before having to make a decision on his own hand. The gimmick, however, was that the alteration in other rules more than compensated for the player's advantage in seeing the dealer's hole card. Rule modifications used to offset the player advantage have varied, but include the dealer winning tied hands (except possibly a player's blackjack), paying even money instead of the normal 3 to 2 for blackjacks, and restrictions on doubling down and resplitting. Depending on the combination of rules in effect and the number of decks used, Double Exposure Blackjack typically gives the casino a 0.3% - 2.0% advantage.

The biggest and first crap game is mentioned in Greek mythology. Zeus, Poseidon, and Hades rolled dice for shares of the Universe. Poseidon won the Oceans. Hades won the Underworld. Zeus won the Heavens and is suspected of having used loaded dice.

Mario Puzo
Inside Las Vegas (1976)

ADVANTAGE PLAY

Advantage play is a broad term to describe a situation in which a player through some method of play can acquire an advantage over the casino in the context of a gambling contract.[217] In other words, advantage play is where the player can overcome the mathematical advantage that is built into every house-banked casino game. Advantage play is more than a hypothetical question. Some experts believe that

216 The exact figure depends on the number of decks and the rules in effect. For a single deck game with typical rules the advantage is 9.9%. See Richard A. Epstein, The Theory of Gambling and Statistical Logic (1995).

217 Dustin Marks, Cheating at Blackjack and Advantage Play 101 (1994).

advantage players take as much as three (3) percent of all moneys wagered in commercial casinos.[218]

Casino-style gambling[219] involves a contract, which is simply a promise or set of promises, between the casino and the player.[220] The law provides punishment for the breach of such promises.[221] In the context of casino play, the casino promises to pay the patron a predetermined sum if certain conditions are met. In turn, the patron agrees to pay the sum of the wager. Take slot machine play as an example. After the patron deposits the correct number of coins or tokens and activates the device by pushing the play button or pulling the handle, the major condition to payment of a jackpot is that the reels must come to rest aligned on a winning combination or payline.

Most gambling contracts, however, have two unusual aspects. The first unusual aspect is that the major condition to the contractual obligations of the casino and the player is determined in whole or part on the outcome of chance, typically a "random" event.[222]

relationships between the government, the advantage player and the casino in three contexts. The first context is whether the advantage player should receive any protection form the government to allow him to ply his skills without interference or exclusion by the casinos. In other words, should the advantage player have a right of access to the casino as opposed to the casino's right to exclude the advantage player?

The second context is how contract law should be interpreted in dealing with disputes between casinos and players involving advantage play. Here, the primary decision is whether the courts will require or not require the casino to pay alleged winnings to an advantage player.

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218 Rod Smith, Advantage Gambling: Official request 'guidance,' Las Vegas Review Journal, July 10, 2003, http://www.reviewjournal.com/lvrj_home/2003/Jul-10-Thu-2003/business/21692188.html.

219 Gambling involves any activity in which a person places a bet or wager. Generally, a bet or wager occurs when a person risks something of value on the outcome of an uncertain event (1) in which the bettor does not exercise any control; or (2) which is determined predominately by chance. ________________

Courts often oversimplify this definition by concluding that gambling is any activity that involves a prize, consideration, and chance. ________________Without all three elements, the activity is not gambling. Chance is a difficult element to define. By its dictionary definition, chance means "An apparent absence of cause." Webster's New Twentieth Century Dictionary 301 (2d ed. 1979). The difficulty, however, comes in determining when chance predominates a particular activity. To assess whether an activity meets this element, one needs to envision a continuum with pure skill on one end and pure chance on the other. The prevailing rule in the United States is that the element of chance is met if chance predominates, even if the activity requires some skill. See e.g., State v. Hahn, 553 N.W.2d 292, 293 (Wis. Ct. App. 1996). Some games, like a traditional slot machine, meet this requirement since they contain pure chance. Other games, like chess, that contain no chance are not gambling. Between these are many games that contain both chance and skill, such as blackjack (probably gambling) and backgammon (probably not gambling). Yet, once one enters this gray area, he or she is at risk because a court ruling could be upheld regardless of the trial court's decision. In these cases, a court would have to decide whether skill or chance predominates a particular activity. In the only case involving backgammon, however, a New Jersey court held that it was illegal gambling. Boardwalk Regency Corp. v. Attorney Gen. of N.J., 457 A.2d 847, 852 (N.J. Super. Ct. Law Div. 1982). In that case, the court did not apply the majority predominance test, but based its decision on the fact that backgammon contained some chance. To further complicate the analysis, courts in different states can apply the same test to the same activity with conflicting results.

220 See infra notes ___ and accompanying text.

221 See infra notes ___ and accompanying text.

222 See footnote 5 and accompanying text.

The third context is whether the government should criminalize certain forms of advantage play.

Not all advantage play shares similar elements. All advantage play can be placed in one of five categories based on the following factors:

- Is the advantage play consistent with the defined rules of the game?
- Does the advantage player use information readily available to all players, as opposed to attempting to acquire information not readily available to all players, that would provide an advantage in determining or predicting what was intended to be a random event?
- Does the advantage player attempt to take advantage of known errors by the casino?
- Does the advantage player attempt to alter the random event that serves as the basis for the game result?

In light of these factors, the first category of advantage play is when the player uses superior skill in analyzing the game data that are available to all players and where both the players and casinos contemplate the use of such data as part of the contractual relationship. This data is typically covered by the basic rules of the game. This type of advantage play is only applicable to casino games involving some skill. This would include blackjack and video poker.[223]

One court noted that this type of advantage play involves a "highly skilled player who analyzes the statistical probabilities associated with [a casino game] and, based on those probabilities, develops playing strategies which may afford him an advantage over the casino."[224] The best and most prevalent example of this would be the card counter in blackjack who has acquired skill in analyzing the cards played and can then determine when the player acquires a theoretical advantage over the casino.[225] In Blackjack, the basic rules provide that the player shall initially receive two cards and, based on the value of those cards, make decisions as to whether to hit, stand or take other actions.[226] Therefore, the game rules anticipate that the player will use the data.

The second category of advantage play is when the player uses superior skill in analyzing the data that are available to all players but such data are not part of the basic rules of the game but can impact outcome. An example of this is shuffle tracking, which will be discussed later, where the player predicts the order of the cards in the deck based on how the dealer shuffled the deck. All the factors necessary to conduct shuffle tracking are available to all players, but the shuffle tracker is attempting to defeat the random event that defines the game. The basic contract between the player and the casino contemplates that the shuffle will be random. Therefore, a player who tracks the shuffle is gaining an advantage outside the basic rules of the game.

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223 An exception to this is progressive slot machines. The play of these machines involves no skill. Because the progressive meter and the potential payout change, however, so does the player expectation. The skill involved in progressive slots is calculating the player expectation and frequency of the payout.

224 Bartolo v. Boardwalk Regency Hotel Casino, Inc., 449 A. 2d 1339, 1343 N.J. Super. Ct. Law Div. (1982).

225 Although the overall house advantage in a typical blackjack game is positive, the advantage shifts back and forth between player and house as cards are removed from play. Simply put, card counting systems identify those hands on which the player has the advantage; in these situations, the card counter then increases the bet size.

226 Jim Kilby and Jim Fox, Casino Operations Management, 139 (1998).

The third category of advantage play is when the player takes advantage of the casino's mistakes. This could include errors made by the casino in posting its terms and conditions or by taking advantage of malfunctioning gaming devices that either play too much or too often. An example of this would be a person that trolls the casino floor at a grand opening seeking slot machines that are mislabeled as to the amount of the jackpot where the error favors the player.

The fourth category of advantage play is where the player acquires knowledge, not typically or readily available to other players, that provides an advantage in determining or predicting what was intended to be a random event. An example would be a blackjack player that is able to learn the dealer's hole card before having to make a decision on how to play his hand.[227] This category of advantage play also is outside the defined rules of play.

The fifth category of advantage play is where the player alters the random event to his favor. Examples of this would be where the player tries to manipulate the dice at the craps table so that they result in a combination that favors the bets placed by the player. Whether this category of advantage play is cheating has been the subject of inconsistent court decisions.[228]

Category One Advantage Play: Using Superior Skill In Analyzing Factors Within the Game Rules That Are Available To All Players

Card Counting

Advantage play is often associated with card counting in the game of blackjack.

The edge gained by a card-counter depends on several factors, including the number of decks in play, the penetration (how far into the pack cards are dealt before reshuffling), the rules of the game, the bet spread, and the system used. A skilled counter will typically have a 0.5% to 1.5% advantage over the house.[229] More decks and shallow penetration (not dealing very far into the pack before reshuffling) tend to decrease the counter's advantage.

Video Slot Players

Video poker as played in casinos is a game of mixed skill and chance. In a typical video poker game, the player randomly draws five cards from a deck of 52 cards and has the option of keeping or discarding up to all of those cards. For each card discarded, the player will randomly draw an additional card from the cards remaining in the original 52 card deck. The combination of cards remaining in the player's hand is compared to a pay chart. If the player has any of the required combinations, he is paid according to the pay chart. The skill element involved in video poker is the determination of which cards to hold or discard.

Some casinos offer video poker machines that when played at optimum skill pay back more on average than they would collect. The casinos rely on players not being

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227 Marks, supra note 3 at 113.

228 See notes 77-82 , infra, and accompanying text.

229 Martin Millman, in A Statistical Analysis of Casino Blackjack, American Mathematical Monthly, (1983), estimated that with four decks the counter could gain 1.35% and with six decks 0.91%.

able to play at optimum skill to insure a profit.[230] Despite this, some players can play at optimum skill and have an advantage albeit slight (about .5%) over the casino.

PROGRESSIVE SLOTS

Progressive slots are slot machines in which one or more of the payouts increase by a set amount for each coin or credit played that does not result in the player winning that payout.[231] For example, suppose a progressive slot machine has a top payout of $100 and 5 cents of every dollar bet increases the progressive jackpot. If the player plays the machine for $1 and does not win, the progressive jackpot will increase to $100.05. The progressive jackpot will increase in this way until it is won and it will then be reset to its original starting point. If the progressive jackpot increases to a certain point without being won, then the theoretical payout results in a positive player advantage.

This slight statistical advantage is what attracts the professional slot teams. These are organized and financed teams of professional slot players that attempt to exploit those progressive slots that have a positive player expectation. Only certain progressive slot machines will meet the slot team's criteria. First, the number of slot machines that are linked to the progressive jackpot must be manageable. The team needs to monopolize all the slot machines to avoid the risk that a non-member will win the jackpot. Second, the statistical frequency of hitting the jackpot must be consistent with the slot team's bankroll. No slot team has an unlimited bankroll. Based on probabilities, the team needs to have enough cash on hand to play all the linked machines with the progressive jackpot until one team member hits the progressive jackpot. The slot team can use the Poisson probability distribution to determine the probability of hitting the jackpot over the course of a specified number of plays. The Poisson distribution (named after the 19th century French mathematician Simeon Poisson) is commonly used to calculate the probability of rare events. As applied to gaming devices, the probability of X jackpot hits in n plays can be computed using the following formula:

$$P(X) = \frac{e^{-\mu}\mu^{X}}{X!},$$

where μ is the average number of jackpot hits over the course of the n plays (or the number of cycles in the n plays), and e is the mathematical constant approximately equal to 2.71828. The value of μ will depend on the particular game and the number of plays, and can be computed by dividing the number of plays in question by the number of plays in a cycle. To illustrate, consider a 3-reel, 96 stop per reel slot machine with a cycle of 884,736 plays ($96 \times 96 \times 96$) and one jackpot combination. This machine will average one jackpot for every 884,736 plays. The probability the jackpot will not be hit over the course of one cycle (μ=1) is:

$$P(0) = \frac{e^{-1} \times 1^{0}}{0!} = 0.3679\,.$$

230 Becky Yerak, Greektown bans 30 winners: Skilled video poker players proved unprofitable for casino, Detroit news, November 25, 2002, http://www.detnews.com/2002/business/0211/25/a01-19565.htm.

231 See generally, David Sklansky, Getting the Best of It 199 (1997).

This implies that 63.21% (subtracting $P(0)$ from one) of the time there will be at least one jackpot over the course of a cycle.[232] The probability of no jackpot over the course of any other fixed number of plays can be obtained by replacing the value of μ with the appropriate average as described above.[233] The following table shows the probability of no jackpot being hit on this example machine for a variety of numbers of plays.

Probability of No Jackpot: 884,736-play Cycle Slot Machine			
Number of Plays	*Avg. Number of Jackpots (μ)*	*Probability of Hitting Jackpot*	*Probability of No Jackpot*
1 million	1.130	67.7057%	32.2943%
2 million	2.261	89.5708%	10.4292%
3 million	3.391	96.6320%	3.3680%
4 million	4.521	98.9123%	1.0877%
5 million	5.651	99.6487%	0.3513%
6 million	6.782	99.8866%	0.1134%
7 million	7.912	99.9634%	0.0366%
8 million	9.042	99.9882%	0.0118%
9 million	10.173	99.9962%	0.0038%
10 million	11.303	99.9988%	0.0012%

The probabilities in the table above refer to the behavior of an individual slot machine for which the probability of hitting the jackpot on a single play is a little better than one in a million (1/884,736 = .00000113). Over the course of one-million plays on this machine, the probability of hitting the jackpot is 67.7%, with this probability increasing as the number of plays increases. If several machines are considered, the probability that at least one hits the jackpot during the course of a fixed number of plays will be greater than the probability that one specific machine will hit during this same number of plays.[234] For example, the probability that at least

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232 This probability of hitting the jackpot at least once (or not at all) over the course of one cycle is the same for any single-jackpot machine.

233 If reasonable estimates are available for the average number of plays per hour for a given machine, calculating these probabilities over a given time is possible. Suppose, for example, an 884,736-cycle slot machine had not yielded a jackpot in three years. Constant play might produce about 400 decisions per hour, so assuming the machine was played about 50% of the time an average of 200 decisions per hour is reasonable. This activity level translates into a little over 5 million plays for a three-year period. The Poisson formula shows the probability of no jackpot over this many plays to be 0.00263, or about 1 in 380. If the machine was played constantly at 400 plays per hour for three years, there's only a 1 in 145,000 chance of no jackpot during this time. At an average of 100 plays per hour it would be surprising but possible to see no jackpot in three years: the probability is 0.051 (about 1 in 20).

234 Put another way, for a given device the probability of an extreme outcome, such as no jackpot for a long period of time or several in a short period, is quite small. With many machines on the casino floor, however, there's a good chance that at least one of them will exhibit extreme behavior. To see why, consider the example machine in the table above in which the probability of observing no jackpot over the course of three-million plays of this machine is .03368. For a collection of 100 of these machines, the probability they all produce at least one jackpot when each is played three-million times is (1-.03368)100

one machine in a carousel of ten of these same type will hit the jackpot during one-million plays on each is (assuming independence) .99998766.[235]

The previous table shows a progressive slot machine that has a cycle of 884,736 and a probability of a little better than one in one million of hitting the progressive jackpot on any given play. If the probability of hitting the progressive jackpot is one in ten million (as might be the case on a different machine), the slot team would need an astronomical bankroll to have reasonable assurances that it would hit the jackpot before running out of money. Therefore, slot teams avoid the "mega" jackpot progressive that have very infrequent hits or winners. To minimize their potential exposure, they concentrate on progressive carousels that feature lower jackpots with a higher frequency of payouts.

CATEGORY TWO ADVANTAGE PLAY: USING SUPERIOR SKILL IN ANALYZING FACTORS OUTSIDE THE GAME RULES THAT ARE AVAILABLE TO ALL PLAYERS

SHUFFLE TRACKING

Many commercial gambling games rely on proper randomization to ensure fairness and maintain the desired house advantage or players' win/loss rate. Gaming regulators, for example, acknowledge the importance of randomization in gaming devices such as video poker, slot machines, and shuffle machines by requiring that they meet minimum confidence levels on standard statistical tests. Indeed, the greatest regulatory concern is that the device selects the cards or symbols within acceptable levels of randomness.[236]

In card games, the random aspect of the game is the shuffle of the cards by the casino. This is done either by a mechanical shuffler or by hand. Mechanical shufflers are usually reviewed and approved by regulatory authorities to assure that the outcome of every shuffle will appear random. Non-machine card shuffles are not typically subject to such regulatory scrutiny. A nonrandom shuffling process may alter the odds of a game, if players can use a predictable pattern inherent in the shuffle process as an advantage play.[237]

Shuffle tracking may allow a player to follow segments of cards through a shuffle and by so doing know roughly when these segments will appear during ensuing rounds of play. If such tracking can be employed in a situation when slugs of favorable or unfavorable cards have been identified in the pre-shuffle pack (for example, in blackjack where multi-deck shoe games are quite common), a player

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= .0325, and so the probability at least one of these yields no jackpot over three-million plays is .9675, or 96.75%. With a collection of 200 such machines, this probability rises to 99.9%, and with 300 devices, this probability is 99.997%. Even though there is a small probability that an individual device will exhibit a certain extreme behavior, when a large enough collection is considered, there's a good chance this behavior will appear in at least one of the collection.

235 Assuming independence, one million plays on each of ten machines is equivalent (probabilistically) to ten-million plays on one machine.

236 Cabot, supra note 9 at 388.

237 The theory of card shuffling has been studied by numerous authors using mathematical models. See e.g., Aldous & Diaconis, Shuffling Cards and Stopping Times, 93 American Mathematical Monthly (1986) 333-48; Bayer & Diaconis, Trailing the Dovetail Shuffle to Its Lair, 2 Annals of Applied Probability 294-313 (1992); Donner & Uppulini, A Markov Chain Structure for Riffle Shuffling, 18 SIAM J. of Applied Mathematics 191-209 (1970); Griffin, The Theory of Blackjack (1996).

could then utilize knowledge of the slug locations during the play of the next shoe by adjusting strategy and bet sizes accordingly. Such shuffle-tracking techniques are not possible, of course, if the shuffle is random.[238]

Given the absence of empirical study in this area, Bob Hannum, one of the authors of this article conducted research into shuffle tracking theory. Using standard casino playing cards, two professional dealers shuffled a variety of six-deck, two-deck, and single-decks. Cards were ordered from 1 to *n* prior to each shuffle iteration and the post-shuffle position recorded for each card. Thus for each iteration of each shuffle procedure, data consists of the permutation array $\pi(1), \pi(2), ..., \pi(n))$, where $\pi(i)$ is the pre-shuffle position of the card whose post-shuffle position is *i*.

Six-deck shuffles were chosen to reflect procedures in use at selected major Las Vegas Strip casinos in early 1997. Thirty shuffle iterations were obtained representing six different shuffle procedures (see below for further details). While the data here reflects shuffles executed according to these procedures, inconsistencies exist in dealers' adherence to the house procedure on live games.[239]

A mathematically random shuffle would tend to produce graphs with no apparent pattern. Inspection of the six-deck shuffle plots exposes the striking nonrandom nature of procedures A and F.[240] For procedure A[241], the similarity of the displays across the twelve iterations suggests that this shuffle is quite predictable. A similar comment can be made for procedure F,[242] though the small sample size of two iterations advises some caution. Given the nature, strength, and consistency of the patterns of these two procedures, knowledge of the order of the cards before a shuffle could aid in knowing the order, at least in terms of segments of cards, after a shuffle. The graphs for procedures B,[243] C,[244] D,[245] and E[246] reveal a much less obvious level of orderliness.

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238 Evidence of non-random shuffling as it relates to gambling games can be found in Thorp, Nonrandom Shuffling with Applications to the Game of Faro, 68 J. of the Am. Statistical Assn 844 (1973); Emanuel & Sutrick, Nonrandom Shuffling Strategies in Blackjack, 17 Communications in Statistics 2954 (1988); Gwynn & Snyder, Man vs. Computer: Does Casino Blackjack Differ from Computer-Simulated Blackjack? 8 Blackjack Forum 6 (1988); Wong, Professional Blackjack (1994).

239 For example, a dealer may cut the right pile when the procedure calls for cutting the left pile, or may vary the order and pairing of grabs from left and right piles as in procedure F. The primary six-deck data does not include any "plugging" of cutoffs (unused cards) into the discards as is sometimes now done for multi-deck games. To shed light on this matter, we have included the results of a set of shuffles for one procedure executed using this plugging process.

240 See appendix for plots of pre- vs. post-shuffle position for each shuffle iteration of the six-deck procedures.

241 Procedure A (Avg. time = 55.6 sec., s=12.4, for 7 shuffles)

1. Break to 2 piles.
2. Cut R pile in half.
3. With grabs from each pile: 1riffle, remove bottom third to top of deck, 1riffle, to cut.

Note: L=left, R=right, C=center, RSR=riffle-strip-riffle, RRSR= riffle-strip-riffle-riffle.

242 Procedure F (Avg. time = 90.9 sec., s=11.6, for 10 shuffles)

1. Break into 2 piles.
2. Cut L pile only and Strip ½ of L pile
3. Break L into set of 3 piles (L1, L2, L3) and R into set of 3 piles (R1, R2, R3).
4. Crisscross ½-deck grabs from L-set and R-set (for our sample we paired L1&R2, L3&R1, L2&R3) and RSR to 1 pile, to cut.

243 Procedure B (Avg. time = 66.4 sec., s=14.8, for 7 shuffles)

1. Break into 2 piles.

Although these visual representations make clear the nonrandomness of shuffle procedures A and F, a detailed statistical analysis will show that all six procedures fail on most tests for mathematical randomness.[247]

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2. Cut R pile only.
3. Start a 3rd pile center by grabs from L and R (about ¾-deck each) and 1 riffle.
4. Crisscross with grabs from L and C, R and C, etc., 1 riffle until finished to 1 pile.
5. Break into 2 piles.
6. Cut left pile only.
7. Simple grabs from L and R, 1 riffle (4-5 times), to cut.

244 Procedure C (no times available)

1. Break into 2 piles.
2. Grabs from each strip-riffle into one pile.
3. Repeat 1 and 2.
4. Break into 2 piles.

Grabs from each pile, 1 riffle into 1 pile, to cut.

245 Procedure D (Avg. time = 66.4 sec., s=6.7, for 8 shuffles)

1. Break into 2 piles
2. Start a 3rd pile center by grabs from L and R (about ¾-deck each) and 1 riffle.
3. Crisscross with grabs from L and C, R and C, etc., 1 riffle each until finished to 1 pile.
4. Break into 2 piles.
5. Simple grabs from L and R, 1 riffle (4-5 times), to cut.

246 Procedure E (Avg. time = 99.2 sec., s=15.2, for 6 shuffles)

1. Break into 3 piles.
2. Strip-1 riffle each pile (entire) once.
3. Grabs from each of the three piles, then RRSR into 1 pile, to cut.

247 See Appendix B.

Summary Statistics – Six-deck Shuffles					
Shuffle Procedure	*Concordance Coeff. W*[1248]	*Table* χ^2 *Value*[2249]	*K-W Test Statistic*[1250]	*Rising Seq. Avg. R*	*Avg. Gap Size* (χ^2 *Value*)
A (m=12)	0.932**	4,752.8**	229.7**	25.6**	23.3 (152,469**)
B (m=4)	0.798**	211.6**	72.3**	84.5**	82.9 (18,693**)
C (m=4)	0.497*	94.8*	20.6*	62.5**	60.4 (12,035**)
D (m=4)	0.563**	164.5**	46.0**	78.8**	77.0 (19,354**)
E (m=4)	0.776**	586.6**	103.5**	86.5**	84.9 9,557**)
F (m=2)	0.752**	513.7**	85.9**	64.5**	62.4 (15,393**)[3251]

* $p < .05$
** $p < .01$

Given casinos have fixed shuffle procedures, when one such procedure results in a non-random shuffle (particularly as non-random as those in the empirical study cited above) an advantage player could observe, then simulate and study this procedure to determine where specific clumps of cards from the pre-shuffle deck will likely be located in the post-shuffle deck. Once this "mapping" from the pre- to post-shuffle decks has been accomplished, a player can utilize this knowledge as a method of advantage play by predicting when favorable and unfavorable cards will be dealt after the (non-random) shuffle and then varying bet size and strategy accordingly. This "shuffle tracking" method of exploiting non-random shuffles to identify clumps of favorable or unfavorable cards in the post-shuffle deck has been refined to advantage play techniques such as ace prediction, ace tracking, sequence tracking, and key carding.[252]

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248 Based on ½-deck segments.

249 Based on 1-deck segments.

250 Based on ½-deck segments.

251 Cells paired due to small expected frequencies.

252 For an excellent discussion of shuffle-tracking techniques, particularly ace tracking, and the effects of non-random shuffling on the game of blackjack, see e.g., David McDowell, Blackjack Ace Prediction: The Art of Advanced Location Strategies for the Casino Game of Twenty-One (2004). Snyder, The Blackjack Shuffle Tracker's Cookbook (2003).

CATEGORY THREE ADVANTAGE PLAY: TAKING ADVANTAGE OF THE CASINO'S MISTAKES.

MISTAKES IN POSTING ITS TERMS AND CONDITIONS OR TAKING ADVANTAGE OF MALFUNCTIONING MACHINES

Some advantage players create the advantage by intentionally exploiting mistakes by the casino or its employees. A good example is the advantage player or team of players that are present on opening nights of any new casino. They understand that errors are most likely to occur when a casino is in the mist of training employees and deploying several hundred or thousand new gaming devices. In an unreported case, a newspaper described the exploits of one such player that began on the opening night of a casino as follows:

> He noticed a bank of slot machines where the payouts for the $100 machines and the $1 machines had been mistakenly reversed. Over the course of several hours, he won $27,000 in cash and comps by collecting $100 machine payouts from $1 machines.[253]

CATEGORY FOUR ADVANTAGE PLAY: ACQUIRING KNOWLEDGE NOT AVAILABLE TO OTHER PLAYERS THAT PROVIDES AN ADVANTAGE IN DETERMINING OR PREDICTING WHAT WAS INTENDED TO BE A RANDOM EVENT.

HOLE-CARDING

Hole-carding is a technique used primarily in blackjack to learn the value of the dealer's hole card before the player needs to make a decision on how to play his or her hand.[254] The advantage derived from hole-carding can be more substantial than card counting.[255]

Most hole-carding is done intentionally. One surveillance expert described a typical team that took advantage of a weak dealer.[256] In that case, the team consisted of two people. One person was in the "third base side," meaning the seat nearest to the dealer's right hand, and the other in the first spot, nearest the dealer's left hand. "Not only was the third base player slouching in his seat, his signals to the other player were simple. The hole-carder on the third-base side of the game had a stock of green ($25 chips) and red ($5 chips). To signal stand, he touched the red (stand/stop), to signal hit, he touched the green (hit/go) and for insurance he tapped the green stack with a chip pinched between his fingers to represent an 'I' (insurance). The dealer later was proven to be a weak dealer and not in the group…. "[257]

In the case of the player sitting on the third base side, slouching, he was right "74 percent of the time at reading the hole card."[258] If a person knows the dealer's hole

253 Rod Smith, Civil Liberties: Disadvantaged: Casinos, police officials often intimidate legal patrons, lawyers say, Las Vegas Review Journal (July 6, 2003).

254 Marks, supra note 3 at 113.

255 Id.

256 Douglas Florence, Advantage Play: Blackjack, Global Gaming Business, July 2004, at 18.

257 Id.

258 Id.

card, the player has an advantage over the casino through having additional information to figure out whether to double his bet, surrender a hand (thus, only losing half a bet), or determining whether to hit or stand.[259] The impact that "hole-carding" has on the house advantage will vary depending on the advantage player's proclivity to correctly identify the hole card and to what advantage he uses the information. If the player knows the dealer's hole card every time and played every hand to maximum advantage, it would result in a 10% advantage over the house.[260] The typical advantage player, however, would not play to maximum advantage because hitting a 19 against the dealer's 20 would be too suspicious and draw inquiry.

"Shining" is a form of hole-carding where the player learns the dealer's hole card by using any reflective device. This could include a metallic cigarette lighter, a facet on a ring, a polished fingernail, or a small mirror.

CATEGORY FIVE ADVANTAGE PLAY: ALTERING THE RANDOM EVENT TO THE PLAYER'S FAVOR.

While the differences between cheating and advantage play may appear simple, various courts have struggled with the distinction particularly in dealing with certain forms of categories two, three, four and five advantage play. In each case, the ultimate question becomes how far a player can go to exploit a casino's errors, omissions or other vulnerabilities before such activity should be a crime. Category five advantage play, however, would appear to clearly meet all the elements of casino cheating, but have still created conflicting court decisions.

SLOT HANDLE MANIPULATION

While no longer a relevant concern to the casino industry because of improved technology, slot handle manipulation was the basis for significant litigation.

The "cherry squeeze" or "handle popping" involved the handle manipulation of mechanical or electro-mechanical slot machines to control the reel alignment. The facts of one case revealed that a player pulled the slot machine handle down two-thirds with his right hand, and then gently hit the handle with his left hand. This activity did not damage the machine. By using this technique, the manipulator stuck two reels on the payout line while the other one was spinning, thus setting the machine up for a jackpot. According to a 1986 Nevada Gaming Control Board memorandum, one suspect admitted removing more than $60,000 per year using this method. The memorandum revealed that there were about 40 other such cheats in Northern Nevada alone. The cheaters were careful not to draw attention to themselves, and rarely emptied the contents of the machine.

In a series of cases involving slot handle manipulation, the Nevada Supreme Court began down a very curious path. The court held that "handle popping" was not covered by the criminal statute because "[t]he physical

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259 To cover these possibilities, statutes often state that a person may not place, increase, or decrease a bet or decide course of play after acquiring knowledge, not available to all players, of the outcome of the game or any event that affects the outcome of the game. See Nev. Rev. Stat. 465.070(2).

260 Stafford Wong, Basic Blackjack (1992).

characteristics and potential pay offs of slot machines are not altered by handle manipulators."[261]

DICE SLIDING

The Nevada Supreme Court did not follow the same reasoning as slot manipulation in a case involving dice sliding. In craps, players are given the opportunity to "toss" two dice that will result in numbers from 2 to 12. This is the random event upon which the gaming contracts between the casino and the players at that table will be decided. In dice sliding, the skilled player is able to slide one or both dice across the table rather than tossing them and allowing them to roll.[262] Thus, a predetermined "roll" may be chosen and the outcome of the game manipulated by the slider.

In *Skipper v. State*,[263] a dice slider challenged his cheating conviction on the same ground as that raised by the handle popping slot players in the earlier handle popping cases. Claiming that the criminal statutes[264] were unconstitutionally vague, Skipper argued that the statutes failed to alert the average dice player that dice sliding constituted criminal conduct.

The Nevada Supreme Court rejected the argument finding that the rules of craps clearly require a roll of the dice (the random event that is the basis of the contract). As the dice do not roll when they are slid, dice sliding was found to be a clear violation of the rules of the game providing adequate notice to the average player that sliding violates the anti-cheating statutes.

Again, the basis of the crime of dice sliding is that the person *intentionally* slides the dice to impact the random element of the game. If a player improperly throws the dice so that they slide instead of being rolled, the condition of a random event would not be met and the dice would be thrown again. The thrower, however, has committed no crime.[265]

Dice sliding should be contrasted with dice setting. The latter is a process by the shooter or thrower in a dice game to exert some control over the numbers that the dice will show after they have been tossed.[266] The concept is that the casino issues rules that the thrower arranges the dice in their hand before throwing them. By controlling (and standardizing) the motion of the throw, dice setters claim to be able alter the random selection of results to favor certain dice combinations that will give them an advantage over the casino.

Dice setting is different than dice sliding because the dice setter is working within the prescribed rules of play to attempt to influence the roll. The casino prescribed the rule with the belief that regardless of what proper method that the advantage player employs, it will not impact the random-aspect of the throw. In other words, the casino considers dice setting to be more based on superstition than science.

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261 Lyons v. State, 775 P.2d 219, 222 (Nev. 1989).

262 Marks, supra note 3 at 198 (1994).

263 879 P.2d 732, 734 (Nev. 1994).

264 Nev. Rev. Stat. 465.070(7), 465.083

265 In Lyons v. State, 775 P. 2d 219 (1989), the defendant challenged a Nevada statute that defined cheating to include "alter[ing] the selection of criteria which determine: (a) The result of a game; or (b) The amount or frequency of payment in a game…."

266 http://www.dicesetter.com/setting/setting.html

CURRENT CASINO RESPONSES TO ADVANTAGE PLAY

BARRING THE ADVANTAGE PLAYER

While some forms of advantage play such as card counting are “not considered cheating, nor is it illegal,”[267] Nevada casinos and casinos in most other states routinely exclude suspected advantage players from gaming. These casinos may exclude advantage players at their discretion.[268] For example, no Nevada statute requires casinos to admit suspected card counters.[269]

Allowing a casino to bar advantage players and others has a common law origin. “At common law, proprietors of privately-owned places of entertainment and amusement were not obligated to serve the general public.”[270]

In many states, the major gambling establishments are racetracks. Cases involving exclusion of patrons from racetracks have provided substantial support for this common law principle.[271]

CHANGING THE RULES OF PLAY

Casinos may counter some forms of advantage play simply by changing the rules of the play of the game. These countermeasures are often statistically designed to reduce or eliminate the player advantage.

BLACKJACK

Any advantage created through card counting, of course, is negated when the deck is shuffled. Possible casino countermeasures against card counting include (1) preferential or at-will shuffling,[272] (2) using multiple decks,[273] (3) changing maximum bet or restricting the bet size to the table minimum for new players to the game,[274] (4) having shills occupy all other seats at table, (5) limiting players to a single wager,[275] (6) prohibiting players from joining game in midshoe,[276] (7) short cut – placing cut

267 Uston v. Hilton Hotels Corp., 448 F. Supp. 116, 118 N.I. (D. Nev. 1978).

268 Id. at 119 holding (exclusion did not violate federal civil rights laws).

269 Id.

270 Perry Z. Binder Note, Arbitrary Exclusions of "Undesirable" Racetrack and Casino Patrons: The Courts Illusory Perception of Common Law Public/Private Distinctions, 32 Buff. L. Rev. 699 (1982).

271 See, e.g., Phillips v. Graham, 427 N.E.2d 550 (Ill. 1981); James v. Churchill Downs, Inc., 620 S.W.2d 323 (Ky. Ct. App. 1981); Tropical Park, Inc. v. Jock, 374 So.2d 639 (Fla. Dist. Ct. App. 1979); Nation v. Apache Greyhound Park, Inc., 579 P.2d 580 (Az. Ct. App. 1978); Presti v. N.Y. Racing Assn, Inc., 363 N.Y.S.2d 24 (1975); Watkins v. Oaklawn Jockey Club, 86 F. Supp 1006 (W.D. Ark. 1949), aff’d, 183 F. 2d 440 (8th N.Y. App. Div. Cir. 1950).

272 See Bill Zender’s Card-counting for the Casino Executive for further details on casino countermeasures.

273 See Urbino is a website hosted by longtime casino executive Andrew MacDonald. www.urbino.net/v1/pages/ar3_4.htm

274 Id.

275 New Jersey Casino Commission Regulation 19:47-2.14 specifically authorizes this countermeasure. It provides:

> A player may only wager on one box at a Blackjack table unless the casino licensee, in its discretion, permits the player to wager on additional boxes.

276 New Jersey Casino Commission Regulation 19:47 2.5(a) provides:

card further from back,[277] and (8) using an automatic shuffler or a continuous shuffling shoe.[278]

"The shuffle-at-will occurs when the dealer is instructed to reshuffle prior to the cut-card manner because the casino card-counting team has determined that the shoe is player-favorable and card-counters are suspected to be playing at a given table."[279]

Technology also has evolved that could allow casinos to achieve advantages over the advantage player. One example is the use of "smart" gaming tables. These tables can use a variety of technologies to track wagers made, cards dealt, and payouts. For example, one smart table can track every card that is dealt out of a shoe in the game of blackjack. This table potentially could be used by a casino to call for a reshuffle anytime the deck favors the advantage player.

Progressive Slot Teams

Two strategies are available for slot teams that play progressive slot machines with a positive player expectation. The first strategy is for the casino to retain the discretion to limit players to playing one machine. Thus, when a slot team tries to monopolize a carousel, they will need more team members and incur greater expense in doing so.

The second strategy is to reduce the increment with which each play of the slot machine will increase the progressive jackpot. If, for example, five cents of every dollar normally goes to fund the progressive, this can be reduced to one cent.[280] This substantially increases the risk to the slot team organizer because he will not receive as much of his money back in the form of the jackpot. Therefore, the jackpot has to be higher before positive player expectation justifies risking the team bankroll.

> Immediately prior to commencement of play, after any round of play as may be determined by the casino licensee and after each shoe of cards is dealt, the dealer shall shuffle the cards so that they are randomly intermixed.

277 Some casinos may use two cut cards. The first at about 50% penetration and the second at about 85%. If the casino suspects that a card counter is playing, it will initiate the shuffle at the first cut card. www.urbino.net/v1/pages/ar3_4.htm.

278 Florence, supra note 67 at 18.

279 Doug Grant, Inc. v. Greate Bay Casino Corp., 3 F. Supp. 2d 518, 525 (D. N.J. 1998). New Jersey Casino Commission Regulations specifically permit shuffling at will. New Jersey Casino Commission Regulation 19:47 2.3 provides:

> After the cards have been shuffled . . . , a casino licensee may, in its discretion, prohibit any person, whether seated at the gaming table or not, who does not make a wager on a given round of play from placing a wager on the next round of play and any subsequent round of play at that gaming table unless the casino licensee chooses to permit the player to begin wagering or until a reshuffle of the cards has occurred.

280 Of course, the casino cannot misrepresent what portion of each pull goes toward the progressive jackpot.

APPENDIX

TABLE 1. STANDARD NORMAL PROBABILITY TABLE

STANDARD NORMAL PROBABILITY TABLE (Z TABLE)

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997

Entries represent the cumulative probability less than or equal to z. For example, P(Z less than or equal to 1.72) = .9573.

TABLE 2. HOUSE ADVANTAGES – LOWEST TO HIGHEST

House Advantages for Major Casino Wagers		
Game	***Bet***	***HA****
Blackjack	Card-Counting	–1.00%
Video Poker	Selected Machines	–0.50%
Craps	Don't Pass/Don't Come w/10X Odds	0.12%
Craps	Pass/Come w/10X Odds	0.18%
Craps	Don't Pass/Don't Come w/5X Odds	0.23%
Craps	Don't Pass/Don't Come w/3/4/5X Odds	0.27%
Craps	Pass/Come w/5X Odds	0.33%
Craps	Don't Pass/Don't Come w/3X Odds	0.34%
Craps	Pass/Come w/3/4/5X Odds	0.37%
Craps	Don't Pass/Don't Come w/2X Odds	0.45%
Craps	Pass/Come w/3X Odds	0.47%
Blackjack	Basic Strategy	0.50%
Craps	Pass/Come w/2X Odds	0.61%
Craps	Don't Pass/Don't Come w/1X Odds	0.68%
Craps	Pass/Come w/1X Odds	0.85%
Baccarat	Banker (5% Commission)	1.06%
Baccarat	Player	1.24%
Craps	Don't Pass/Don't Come	1.36%
Craps	Pass/Come	1.41%
Craps	Place 6 or 8	1.52%
Craps	Don't Place 6 or 8	1.82%
Blackjack	Average Player	2.00%
Three Card Poker	Pair Plus	2.32%
Craps	Lay 4 or 10	2.44%
Craps	Don't Place 5 or 9	2.50%
Pai Gow Poker	Skilled Player (Non-Banker)	2.54%
Roulette	Single-Zero	2.70%
Craps	Field (2 or 12 Pays Triple)	2.78%
Sic Bo	Big/Small	2.78%
Red Dog	Basic Bet (Six Decks)	2.80%
Pai Gow Poker	Average Player (Non-Banker)	2.84%
Casino War	Basic Bet	2.88%
Craps	Don't Place 4 or 10	3.03%
Craps	Lay 5 or 9	3.23%
Three Card Poker	Ante	3.37%
Let It Ride	Basic Bet	3.51%
Blackjack	Poor Player	4.00%
Craps	Lay 6 or 8	4.00%
Craps	Place 5 or 9	4.00%
Slots	Dollar Slots (Good)	4.00%
Sports Betting	Bet $11 to Win $10	4.55%

Craps	Buy (Any)	4.76%
Slots	Quarter Slots (Good)	5.00%
Caribbean Stud	Ante	5.22%
Roulette	Double-Zero (Except Five-Number)	5.26%
Craps	Field (2 And 12 Pay Double)	5.56%
Slots	Dollar Slots (Average)	6.00%
Craps	Place 4 or 10	6.67%
Sic Bo	One of a Kind	7.87%
Roulette	Double-Zero, Five-Number Bet	7.89%
Slots	Quarter Slots (Average)	8.00%
Craps	Big 6, Big 8	9.09%
Craps	Hard 6, Hard 8	9.09%
Sic Bo	7, 14	9.72%
Craps	Hard 4, Hard 10	11.11%
Craps	Any Craps	11.11%
Craps	Three, Eleven (15-1)	11.11%
Craps	C&E	11.11%
Craps	Hop Bet – Easy (15-1)	11.11%
Craps	Horn High 3, Horn High 11 (30-1 & 15-1)	12.22%
Craps	Horn Bet (30-1 & 15-1)	12.50%
Sic Bo	8, 13	12.50%
Sic Bo	10, 11	12.50%
Craps	Horn High 2, Horn High 12 (30-1 & 15-1)	12.78%
Craps	Two, Twelve (30-1)	13.89%
Craps	Hop Bet – Hard (30-1)	13.89%
Sic Bo	Any Three of a Kind	13.89%
Sic Bo	5, 16	13.89%
Sic Bo	4, 17	15.28%
Sic Bo	Three of a Kind	16.20%
Craps	Any Seven	16.67%
Craps	Two, Twelve (29-1)	16.67%
Craps	Three, Eleven (14-1)	16.67%
Craps	Horn High – Any (29-1 & 14-1)	16.67%
Craps	Hop Bet – Easy (14-1)	16.67%
Craps	Hop Bet – Hard (29-1)	16.67%
Sic Bo	Two-Dice Combination	16.67%
Sic Bo	6, 15	16.67%
Sic Bo	Two of a Kind	18.52%
Sic Bo	9, 12	18.98%
Big Six Wheel	Average	19.84%
Keno	Typical	27.00%

**House Advantages under typical conditions, expressed “per hand” and including ties, where appropriate. Optimal strategy assumed unless otherwise noted.*

TABLE 3. HOUSE ADVANTAGES – BY GAME

House Advantages for Major Casino Wagers		
Game	***Bet***	***HA****
Baccarat	Banker (5% Commission)	1.06%
Baccarat	Player	1.24%
Big Six Wheel	Average	19.84%
Blackjack	Card-Counting	−1.00%
Blackjack	Basic Strategy	0.50%
Blackjack	Average Player	2.00%
Blackjack	Poor Player	4.00%
Caribbean Stud	Ante	5.22%
Casino War	Basic Bet	2.88%
Craps	Pass/Come	1.41%
Craps	Pass/Come w/1X Odds	0.85%
Craps	Pass/Come w/2X Odds	0.61%
Craps	Pass/Come w/3X Odds	0.47%
Craps	Pass/Come w/5X Odds	0.33%
Craps	Pass/Come w/10X Odds	0.18%
Craps	Pass/Come w/3/4/5X Odds	0.37%
Craps	Don’t Pass/Don’t Come	1.36%
Craps	Don’t Pass/Don’t Come w/1X Odds	0.68%
Craps	Don’t Pass/Don’t Come w/2X Odds	0.45%
Craps	Don’t Pass/Don’t Come w/3X Odds	0.34%
Craps	Don’t Pass/Don’t Come w/5X Odds	0.23%
Craps	Don’t Pass/Don’t Come w/10X Odds	0.12%
Craps	Don’t Pass/Don’t Come w/3/4/5X Odds	0.27%
Craps	Place 4 or 10	6.67%
Craps	Place 5 or 9	4.00%
Craps	Place 6 or 8	1.52%
Craps	Don’t Place 4 or 10	3.03%
Craps	Don’t Place 5 or 9	2.50%
Craps	Don’t Place 6 or 8	1.82%
Craps	Buy (Any)	4.76%
Craps	Lay 4 or 10	2.44%
Craps	Lay 5 or 9	3.23%
Craps	Lay 6 or 8	4.00%
Craps	Any Craps	11.11%
Craps	Any Seven	16.67%
Craps	Big 6, Big 8	9.09%
Craps	C&E	11.11%
Craps	Field (2 And 12 Pay Double)	5.56%
Craps	Field (2 Or 12 Pays Triple)	2.78%
Craps	Hard 4, Hard 10	11.11%

Craps	Hard 6, Hard 8	9.09%
Craps	Three, Eleven (14-1)	16.67%
Craps	Three, Eleven (15-1)	11.11%
Craps	Two, Twelve (29-1)	16.67%
Craps	Two, Twelve (30-1)	13.89%
Craps	Hop Bet – Easy (14-1)	16.67%
Craps	Hop Bet – Easy (15-1)	11.11%
Craps	Hop Bet – Hard (29-1)	16.67%
Craps	Hop Bet – Hard (30-1)	13.89%
Craps	Horn Bet (30-1 & 15-1)	12.50%
Craps	Horn High – Any (29-1 & 14-1)	16.67%
Craps	Horn High 2, Horn High 12 (30-1 & 15-1)	12.78%
Craps	Horn High 3, Horn High 11 (30-1 & 15-1)	12.22%
Keno	Typical	27.00%
Let It Ride	Base Bet	3.51%
Pai Gow Poker	Skilled Player (Non-Banker)	2.54%
Pai Gow Poker	Average Player (Non-Banker)	2.84%
Red Dog	Basic Bet (Six Decks)	2.80%
Roulette	Single-Zero	2.70%
Roulette	Double-Zero (Except Five-Number)	5.26%
Roulette	Double-Zero, Five-Number Bet	7.89%
Sic Bo	Big/Small	2.78%
Sic Bo	One of a Kind	7.87%
Sic Bo	7, 14	9.72%
Sic Bo	8, 13	12.50%
Sic Bo	10, 11	12.50%
Sic Bo	Any Three Of A Kind	13.89%
Sic Bo	5, 16	13.89%
Sic Bo	4, 17	15.28%
Sic Bo	Three of a Kind	16.20%
Sic Bo	Two-Dice Combination	16.67%
Sic Bo	6, 15	16.67%
Sic Bo	Two of a Kind	18.52%
Sic Bo	9, 12	18.98%
Slots	Dollar Slots (Good)	4.00%
Slots	Quarter Slots (Good)	5.00%
Slots	Dollar Slots (Average)	6.00%
Slots	Quarter Slots (Average)	8.00%
Sports Betting	Bet $11 To Win $10	4.55%
Three Card Poker	Pair Plus	2.32%
Three Card Poker	Ante	3.37%
Video Poker	Selected Machines	–0.50%

**House Advantages under typical conditions, expressed "per hand" and including ties, where appropriate. Optimal strategy assumed unless otherwise noted.*

TABLE 4. UNIT NORMAL LINEAR LOSS INTEGRAL (UNLLI) TABLE

Unit Normal Linear Loss Integral (UNLLI)					
Z	**.00**	**.02**	**.04**	**.06**	**.08**
0.0	.3989	.3890	.3793	.3697	.3602
0.1	.3509	.3418	.3329	.3240	.3154
0.2	.3069	.2986	.2904	.2824	.2745
0.3	.2668	.2592	.2518	.2445	.2374
0.4	.2304	.2236	.2169	.2104	.2040
0.5	.1978	.1917	.1857	.1799	.1742
0.6	.1687	.1633	.1580	.1528	.1478
0.7	.1429	.1381	.1334	.1289	.1245
0.8	.1202	.1160	.1120	.1080	.1042
0.9	.1004	.0968	.0933	.0899	.0865
1.0	.0833	.0802	.0772	.0742	.0714
1.1	.0686	.0659	.0634	.0609	.0584
1.2	.0561	.0538	.0517	.0495	.0475
1.3	.0455	.0436	.0418	.0400	.0383
1.4	.0367	.0351	.0336	.0321	.0307
1.5	.0293	.0280	.0267	.0255	.0244
1.6	.0232	.0222	.0211	.0201	.0192
1.7	.0183	.0174	.0166	.0158	.0150
1.8	.0143	.0136	.0129	.0123	.0116
1.9	.0111	.0105	.0010	.0094	.0090
2.0	.0085	.0080	.0076	.0072	.0068
2.1	.0065	.0061	.0058	.0055	.0052
2.2	.0049	.0046	.0044	.0041	.0039
2.3	.0037	.0035	.0033	.0031	.0029
2.4	.0027	.0026	.0024	.0023	.0021
2.5	.0020	.0019	.0018	.0017	.0016

Table entries are values of *UNLLI(a)* = $\int_a^\infty (z-a)\cdot f(z)dz$, where $f(z)$ is the standard normal probability density function. For example, *UNLLI*(1.46) = .0321.

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