A PROOF OF CARLESON'S THEOREM BASED ON A NEW CHARACTERIZATION OF THE LORENTZ SPACES $L(p,1)$ FOR $1 < p < \infty$ AND OTHER APPLICATIONS

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Abstract. In his 1950 Annals of Mathematics paper entitled "Some New Functional Spaces", G. G. Lorentz [1] introduced the function spaces denoted by $\Lambda(\alpha), 0 < \alpha < 1$, defined as the set of real measurable functions for $0 < x < 1$ for which

$$\|f\|_{\Lambda(\alpha)} = \alpha \int_0^1 x^{\alpha-1} f^\star(x)dx < \infty ,$$

where $f^\star$ is the decreasing rearrangement of f. In this paper we give two simple characterizations for $\Lambda(1/p)$ for $1 < p < \infty$ based on a generalization of the special atom space introduced by G. De Souza in earlier works [3], [6], [7], and [11]. The space $\Lambda(1/p)$ is nowadays denoted by $L(p,1)$. As an application, we give a proof of Carleson's Theorem on the convergence of Fourier series on $L(p,1)$ and, more generally, on $L(p,r)$ for $p, r > 1$. Also we have a simple proof of a theorem of Stein and Weiss on operators in $L(p,1)$.

1. Preliminaries

In this section, we state several definitions that will be used throughout this paper with references to the original source.

Definition 1.1. A real-valued function f defined on $[-\pi, \pi]$ belongs to the space $L(p,1)$ for $1 < p < \infty$ if

$$\|f\|_{L(p,1)} = \int_0^{2\pi} f^\star(t) t^{\frac{1}{p}-1} dt < \infty ,$$

where $f^\star$ is the decreasing rearrangement of f. This space was originally introduced by G. G. Lorentz [1] in 1950 where it was denoted by $\Lambda(1/p)$.

Definition 1.2. A generalized special atom is a function $b : [-\pi, \pi] \to \mathbb{R}$, $b(t) = \dfrac{1}{2\pi}$ or for any $\alpha \in (0,1]$ and μ-measurable subsets X, A, B of $[-\pi, \pi]$,

$$b(t) = \frac{1}{\mu(X)^\alpha} \Big[\chi_A(t) - \chi_B(t) \Big]$$

where $X = A \cup B, A \cap B = \emptyset, \mu(A) = \mu(B)$, μ is a measure on subsets of $[-\pi, \pi]$, and χ_E denotes the characteristic function of the set E.

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Definition 1.3. For $0 < \alpha \leq 1$, let $(b_n)_{n\geq 1}$ be a sequence of generalized special atoms, $(C_n)_{n\geq 1}$ a sequence of real numbers, and μ a measure on subsets of $[-\pi, \pi]$. We define the generalized special atom spaces by

$$A(\mu,\alpha) = \left\{ f : [-\pi,\pi] \to \mathbb{R}; f(t) = \sum_{n=1}^{\infty} C_n b_n(t); \sum_{n=1}^{\infty} |C_n| < \infty \right\}.$$

We endow $A(\mu,\alpha)$ with the norm

$$\|f\|_{A(\mu,\alpha)} = \inf \sum_{n=1}^{\infty} |C_n| \ ,$$

where the infimum is taken over all possible representations of f.

The notion of special atoms and the spaces formed by special atoms as well as certain generalized spaces were introduced originally by G. De Souza, see [3], [6],[7], [19]. In those works, intervals and lengths were used.

Definition 1.4. For $0 < \alpha \leq 1$ and μ a measure on sets of $[-\pi, \pi]$, we define the space $B(\mu, \alpha)$ as

$$B(\mu,\alpha) = \left\{ f : [-\pi,\pi] \to \mathbb{R}; f(t) = \sum_{n=1}^{\infty} a_n d_n(t); \sum_{n=1}^{\infty} |a_n| < \infty \right\},$$

where $d_n(t) = \dfrac{1}{\mu^{\alpha}(A_n)} \chi_{A_n}(t)$, A_n are μ-measurable sets in $[-\pi, \pi]$, and a_n's are real numbers. We endow $B(\mu, \alpha)$ with the norm

$$\|f\|_{B(\mu,\alpha)} = \inf \sum_{n=1}^{\infty} |a_n| \ ,$$

where the infimum is taken over all possible representations of f.

This space also was introduced by G. De Souza in his early work, see [3], [4], [6], [7].

Definition 1.5. For $0 < \alpha \leq 1$ and μ a measure on sets of $[-\pi, \pi]$, we define the space $\Lambda(\mu, \alpha)$ as

$$\Lambda(\mu,\alpha) = \left\{ f : [-\pi,\pi] \to \mathbb{R}; \frac{1}{\mu^{\alpha}(X)} \left| \int_A f(x)d\mu(x) - \int_B f(x)d\mu(x) \right| < M \right\}$$

for μ-measurable subsets X, A, B of $[-\pi, \pi]$ such that $X = A \cup B, A \cap B = \emptyset$. We endow $\Lambda(\mu, \alpha)$ with the norm

$$\|f\|_{\Lambda(\mu,\alpha)} = \sup_{X=A\cup B, A\cap B=\emptyset} \left[\frac{1}{\mu^{\alpha}(X)} \left| \int_A f(x)d\mu(x) - \int_B f(x)d\mu(x) \right| \right]$$

Note that this space is a natural generalization of the Lipschitz spaces. In fact if we take μ as the Lebesgue measure, $X = [x-h, x+h]$, $A = [x-h, x)$, $B = [x, x+h]$, and $\mu^{\alpha}(X) = (2h)^{\alpha}$, then for a differentiable f, we get

$$\frac{1}{\mu^{\alpha}(X)} \left| \int_A f'(x)d\mu(x) - \int_B f'(x)d\mu(x) \right| = \frac{|f(x+h) + f(x-h) - 2f(x)|}{(2h)^{\alpha}} \ .$$

The space $\Lambda(\mu, \alpha)$ in this form has been introduced by G. De Souza in his ealier work, see [3], [6], [7], [23].

Definition 1.6. For $0<\alpha\le 1$ and μ a measure on sets of $[-\pi,\pi]$, we define the space $Lip(\mu,\alpha)$ as

$$Lip(\mu,\alpha)=\left\{f:[-\pi,\pi]\to\mathbb{R};\frac{1}{\mu^{\alpha}(X)}\left|\int_A f(x)d\mu(x)\right|<M\right\},$$

where A is a μ-measurable set in $[-\pi,\pi]$. A norm is defined on $Lip(\mu,\alpha)$ as

$$\|f\|_{Lip(\mu,\alpha)}=\sup_A\frac{1}{\mu^{\alpha}(A)}\left|\int_A f(x)d\mu(x)\right|.$$

This space was originally introduced by G. G. Lorentz in 1950, see [1], [2] .

2. Known Results

In this section, we state some known results and sketch very briefly the proof of some of them or make some comments. For more details, see [1], [2], [23], [24].

Theorem 2.1. *The spaces* $(A(\mu,\alpha),\|\cdot\|_{A(\mu,\alpha)})$, $(B(\mu,\alpha),\|\cdot\|_{B(\mu,\alpha)})$, $(\Lambda(\mu,\alpha),\|\cdot\|_{\Lambda(\mu,\alpha)})$, *and* $(Lip(\mu,\alpha),\|\cdot\|_{Lip(\mu,\alpha)})$ *for* $0<\alpha\le 1$ *are Banach spaces.*

The proof follows using direct application of standard techniques for Banach spaces.

Theorem 2.2. *The spaces* $A(\mu,\alpha)$ *and* $B(\mu,\alpha)$ *are the same as Banach spaces and the norms are equivalent, that is* $A(\mu,\alpha)\cong B(\mu,\alpha)$ *with* $M\|f\|_{B(\mu,\alpha)}\le\|f\|_{A(\mu,\alpha)}\le N\|f\|_{B(\mu,\alpha)}$, *where* M *and* N *are absolute constants.*

Clearly $A(\mu,\alpha)$ is continuously contained in $B(\mu,\alpha)$. In fact if $f\in A(\mu,\alpha)$, then

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty}\frac{C_n}{\mu^{\alpha}(X_n)}\Big[\chi_{A_n}(t)-\chi_{B_n}(t)\Big]\\ &= \sum_{n=1}^{\infty}\frac{C_n}{\mu^{\alpha}(X_n)}\chi_{A_n}(t)-\sum_{n=1}^{\infty}\frac{C_n}{\mu^{\alpha}(X_n)}\chi_{A_n}(t)\\ &= \sum_{n=1}^{\infty}C_n\left(\frac{\mu(A_n)}{\mu(X_n)}\right)^{\alpha}\frac{1}{\mu^{\alpha}(A_n)}\chi_{A_n}(t)-\sum_{n=1}^{\infty}C_n\left(\frac{\mu(B_n)}{\mu(X_n)}\right)^{\alpha}\frac{1}{\mu^{\alpha}(B_n)}\chi_{B_n}(t)\end{aligned}$$

Since $X_n=A_n\cup B_n$, $\dfrac{\mu(A_n)}{\mu(X_n)}\le 1$, $\dfrac{\mu(B_n)}{\mu(X_n)}\le 1$, we have $\|f\|_{B(\mu,\alpha)}\le 2\sum_{n=1}^{\infty}|C_n|$. Therefore, $\|f\|_{B(\mu,\alpha)}\le\|f\|_{A(\mu,\alpha)}$.
For the other inequality, please refer to De Souza and Pozo [24].

Theorem 2.3. *The spaces* $\Lambda(\mu,\alpha)$ *and* $Lip(\mu,\alpha)$ *for* $0<\alpha<1$ *are equivalent as Banach spaces that is* $\Lambda(\mu,\alpha)\cong Lip(\mu,\alpha)$ *with*
$M\|f\|_{B(\mu,\alpha)}\le\|f\|_{\Lambda(\mu,\alpha)}\le N\|f\|_{B(\mu,\alpha)}$, *where* M *and* N *are absolute constants.*

Again, one of the inequalities is easily seen, that is $Lip(\mu,\alpha)\subseteq\Lambda(\mu,\alpha)$ and $\|f\|_{\Lambda(\mu,\alpha)}\le 2\|f\|_{Lip(\mu,\alpha)}$. For the other inequality, just note that

$$\frac{1}{\mu^{1/p}(A)}\int_A|f(t)|d\mu(t)\le\sup_{\mu(A\Delta B)\neq 0}\frac{1}{\mu^{1/p}(A\Delta B)}\left|\int_A f(t)d\mu(t)-\int_B f(t)d\mu(t)\right|,$$

where $A\Delta B=(A-B)\cup(B-A)$.

Theorem 2.4 (Duality). *ϕ is a bounded linear functional on $A(\mu,\alpha)$, $0<\alpha<1$ if and only if there is a unique $g \in \Lambda(\mu,\alpha)$ so that $\phi(f) = \int_{-\pi}^{\pi} f(x)g(x)d\mu(x)$ with $\|\phi\| = \|g\|_{\Lambda(\mu,\alpha)}$. That is, $A^\star(\mu,\alpha) \cong \Lambda(\mu,\alpha)$, where $A^\star(\mu,\alpha)$ is the dual space of $A(\mu,\alpha)$.*

Theorem 2.5 (Duality). *ϕ is a bounded linear functional on $B(\mu,\alpha), 0<\alpha<1$ if and only if there is a unique $g \in Lip(\mu,\alpha)$ so that $\phi(f) = \int_{-\pi}^{\pi} f(x)g(x)d\mu(x)$ with $\|\phi\| = \|g\|_{Lip(\mu,\alpha)}$. That is, $B^\star(\mu,\alpha) \cong Lip(\mu,\alpha)$, where $B^\star(\mu,\alpha)$ is the dual space of $B(\mu,\alpha)$.*

The proofs of these two duality Theorems follow easily after a pair of Holder type inequalities. That is

$$\left|\int_{-\pi}^{\pi} f(x)g(x)d\mu(x)\right| \le \|f\|_{A(\mu,\alpha)} \cdot \|g\|_{\Lambda(\mu,\alpha)}, \quad f \in A(\mu,\alpha), g \in \Lambda(\mu,\alpha)$$

and

$$\left|\int_{-\pi}^{\pi} f(x)g(x)d\mu(x)\right| \le \|f\|_{B(\mu,\alpha)} \cdot \|g\|_{Lip(\mu,\alpha)}, \quad f \in B(\mu,\alpha), g \in Lip(\mu,\alpha)\ .$$

For a complete proof see De Souza and Pozo [24].

Theorem 2.6 (Duality-G.G. Lorentz). *ϕ is a bounded linear functional on $L(\frac{1}{\alpha},1), 0<\alpha<1$, if and only if there is a unique $g \in Lip(\mu,\alpha)$ so that $\phi(f) = \int_{-\pi}^{\pi} f(x)g(x)d\mu(x)$ with $\|\phi\| = \|g\|_{Lip(\mu,\alpha)}$. That is, $L^\star(\frac{1}{\alpha},1) \cong Lip(\mu,\alpha)$.*

Again this duality Theorem is due to G.G. Lorentz [1]. It also follows from the Holder type inequality

$$\left|\int_{-\pi}^{\pi} f(x)g(x)d\mu(x)\right| \le \|f\|_{L(\frac{1}{\alpha},1)} \cdot \|g\|_{Lip(\mu,\alpha)}, \quad f \in L(\frac{1}{\alpha},1), g \in Lip(\mu,\alpha)\ .$$

3. Main result

In this section, we state and prove the main result which is the characterization of $L(p,1), 1<p<\infty$ as $B(\mu,1/p)$ and $A(\mu,1/p)$.

Theorem 3.1. *$f \in L(p,1)$ if and only if $f \in B(\mu,1/p)$ for $1<p<\infty$. Moreover $N\|f\|_{B(\mu,1/p)} \le \|f\|_{L(p,1)} \le M\|f\|_{B(\mu,1/p)}$, where N and M are absolute constants.*

Proof. Let us show that $B(\mu,1/p) \subset L(p,1), 1<p<\infty$. To that end, all we need is to estimate $\|f\|_{L(p,1)}$ where $f(t)=\chi_A(t), A$ is a μ-measurable set in $[-\pi,\pi]$. In fact

$$\begin{aligned}
\|\chi_A(t)\|_{L(p,1)} &= \int_0^\infty \chi_A^\star(t) t^{\frac{1}{p}-1}dt \\
&= \int_0^\infty \chi_{[0,\mu(A)]}(t) t^{\frac{1}{p}-1}dt \\
&= \int_0^{\mu(A)} t^{\frac{1}{p}-1}dt \\
&= p(\mu(A))^{\frac{1}{p}}
\end{aligned}$$

That is

$$\left\| \frac{1}{(\mu(A))^{\frac{1}{p}}} \chi_A \right\| \leq p \ .$$

Now if $f \in B(\mu, \alpha)$, then $f(t) = \sum_{n=1}^{\infty} C_n d_n(t)$ with $\sum_{n=1}^{\infty} |C_n| < \infty$, where $d_n(t) = \frac{1}{(\mu(A_n))^{\frac{1}{p}}} \chi_{A_n}(t), A_n$ a μ-measurable set in $[-\pi, \pi]$.

Then $\|f\|_{L(p,1)} \leq \sum_{n=1}^{\infty} |C_n| \|d_n\|_{L(p,1)} \leq p \sum_{n=1}^{\infty} |C_n|$ so that taking the infimum, we get

$$\|f\|_{L(p,1)} \leq p \|f\|_{B(\mu, 1/p)}, \quad 1 < p < \infty \ .$$

□

We have the following situations:

(1) $B(\mu, 1/p) \subseteq L(p,1)$ for $1 < p < \infty$ and $\|f\|_{L(p,1)} \leq p \|f\|_{B(\mu, 1/P)}$
(2) $B^\star(\mu, 1/p) \cong Lip(\mu, 1/p)$ by Theorem 2.5
(3) $L^\star(p,1) \cong Lip(\mu, 1/p)$ by Theorem 2.6
(4) $B(\mu, 1/p)$ is dense in $L(p,1)$. Easily shown with standard technique.

As a consequence of these facts, the embedding operator $I : B(\mu, 1/p) \to L(p,1)$ defined by $I(f) = f$ is a Banach space isomorphism. That is $B(\mu, 1/p) \cong L(p,1)$ with equivalent norms.

Note that $A(\mu, 1/p) \cong B(\mu, 1/p), 1 < p < \infty$ by Theorem 2.2. Therefore we have the following result.

Theorem 3.2. *The spaces $A(\mu, 1/p), B(\mu, 1/p)$ and $L(p,1)$ for $1 < p < \infty$ are equivalent as Banach spaces and the norms are equivalent.*

4. Application

In this section, we give a simple proof of a well-known theorem due to Guido Weiss and Elias Stein given in [25] and [26] concerning linear operators acting on the Lorentz space $L(p,1)$.

Theorem 4.1 (Stein and Weiss)**.** *If T is a linear operator on the space of measurable functions and $\|T\chi_A\|_X \leq M(\mu(A))^{\frac{1}{p}}, 1 < p < \infty$ where X is a Banach space, then T can be extended to all $L(p,1)$; that is $T : L(p,1) \to X$ and $\|Tf\|_X \leq M\|f\|_{L(p,1)}$.*

Proof. After this new characterization of $L(p,1)$ as the space $B(\mu, 1/p), 1 < p < \infty$, given in Theorem 3.1, this result is an immediate consequence of the representation of f as $f \in L(p,1) \Rightarrow f \in B(\mu, 1/p) \Rightarrow f(t) = \sum_{n=1}^{\infty} C_n d_n(t)$ with $\sum_{n=1}^{\infty} |C_n| < \infty$ and $d_n(t) = \frac{1}{(\mu(A_n))^{\frac{1}{p}}} \chi_{A_n}(t), A_n$'s μ-measurable sets in $[-\pi, \pi]$ so that

$$Tf(t) = \sum_{n=1}^{\infty} C_n T(d_n((t)) \ .$$

Consequently, $\|Tf\|_X \leq \sum_{n=1}^{\infty} |C_n| \|Td_n\|_X$ and, by hypothesis, $\|Td_n\|_X \leq M(\mu(A_n))^{\frac{1}{p}}$.

Therefore $\|Tf\|_X \leq \sum_{n=1}^{\infty} |C_n|$ and so $\|Tf\|_X \leq M\| |f|_{L(p,1)}$ □

5. Comments

1. Prof. Richard O'Neil from SUNY at Albany: If $f(t) = \sum_{j=1}^{n} C_j \chi_{A_j}(t)$ where $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_n$ and $C_j > 0$, then $\sum_{j=1}^{n} C_j (\mu(A_j))^{\frac{1}{p}} = \frac{1}{p} \|f\|_{L(p,1)}$. Also he added that if

$$f(t) = \sum_{j=1}^{n} C_j q_j(t)$$

where $q_j(t) = \chi_{A_j}(t) - \chi_{B_j}(t), A_j \cap B_j = \emptyset, C_j > 0$, then

$$\sum_{j=1}^{n} C_j (\mu(A_j \cup B_j))^{\frac{1}{p}} = \frac{1}{p} \|f\|_{L(p,1)} \ .$$

Note $L(p,1) \subseteq L_p$ for $1 < p < \infty$ and because of this new characterization, $L(p,1)$ is a much easier space to work with than L_p. The space which we denoted in this paper by $Lip(\mu, \alpha)$ is denoted in the literature by $L(\frac{1}{\alpha}, \infty)$ for $0 < \alpha < 1$ and is called the weak-L_p space with "norm" given by $\|f\| = \operatorname{Sup}_{t>0}\{t^{\alpha} f^{\star}(t)\}$, where $f^{\star}$ is the decreasing rearrangement of f.
One can show that the "norms" $\|f\|$ and $\|f\|_{Lip(\mu,\alpha)}$ are equivalent.

Finally, if we define $m(f, y) = \mu(\{x : |f(x) > y|\})$, then by a change of variable $y = f^{\star}(t)$, $t = m(f, y)$ and integration by parts, we get

$$\|f\|_{L(p,1)} = \int_0^{\infty} f^{\star}(t) t^{\frac{1}{p}-1} dt = p \int_0^{\infty} (m(f, y))^{\frac{1}{p}} dy \ .$$

Prof. Richard O'Neil:"This last integral is sort of infinitesimal version of your atomic decomposition. Indeed it was this formula that led to the remark that an operator of restricted weak type (p, q) was the same as a strong operator from $L(p, 1)$ to $L(p, \infty)$."

2. One of the most interesting observations that we made in the process to obtain the new characterization of $L(p, 1)$ for $1 < p < \infty$ is that $Tf(x) = \sup_{n\geq 1} |S_n(f, x)|$, where $S_n(f, x)$ is the n^{th} partial sum of the Fourier Series of f is

Theorem 5.1. *If $Tf(x) = \sup_{n\geq 1} |S_n(f, x)|$, then $\|T\chi_A\|_{L(p,1)} \leq M\mu(A)^{\frac{1}{p}}$ for $p > 1$ and so $\|Tf\|_{L(p,1)} \leq M\|f\|_{L(p,1)}$. A is a μ-measurable set.*

Proof. If we take the definition of the norm $\|g\|_{L(p,1)}$ as

$$\|g\|_{L(p,1)} = p \int_0^{\infty} \mu\{x : |g(x)| > \lambda\}^{1/p} d\lambda$$

which is equivalent to the one in Definition 1.1, and use Hunt's inequality given in [27], which is

$$\begin{aligned} \|T\chi_A\|_{L(p,1)} &= p\int_0^\infty \mu\{x : |T\chi_A(x)| \geq \lambda\}^{1/p} d\lambda \\ &\leq \int_0^1 \left[\mu(A)\frac{1}{\lambda}(1+\log\frac{1}{\lambda})\right]^{1/p} d\lambda + C\int_1^\infty \left(\mu(A)e^{-C\lambda}\right)^{1/p} d\lambda \end{aligned}$$

where C is a positive constant, we get $\|T\chi_A\|_{L(p,1)} \leq C\mu(A)^{1/p}$. Consequently, by using the new characterization of $L(p,1)$, we get

$$\|Tf\|_{L(p,1)} \leq C\|f\|_{L(p,1)} \ .$$

Note: This direct proof using Hunt's inequality was mentioned to the author by Loukas Grafakos during the 23^{th} Mini-Conference on Harmonic Analysis and Related Areas held at Auburn University on December 4-5, 2009 after the talk given by the author on the subject. □

As a Corollary of Theorem 5.1, we have that

Corollary 5.2. *If $f \in L(p,1), p > 1$ and $S_n(f,x)$ is the n-th partial sum of the Fourier series of f, then $S_n(f,x) \to f(x)$ almost everywhere.*

Also we note that $L(p,1) \subseteq L(p,\infty)$ with $\|f\|_{L(p,\infty)} \leq C\|f\|_{L(p,1)}$. It follows by Theorem 5.1 that for $p_0 \neq p_1, p_0, p_1 > 1$

a) $\|T\chi_A\|_{L(p_0,\infty)} \leq M_0(\mu(A))^{1/p_0}$

b) $\|T\chi_A\|_{L(p_1,\infty)} \leq M_1(\mu(A))^{1/p_1}$

Therefore using the interpolation Theorem 1.4.19 in [25], we get

$$\|Tf\|_{L(p,r)} \leq M\|f\|_{L(p,r)}, \quad \text{for} \quad \frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, 0 < \theta < 1, \forall r > 1. \tag{5.1}$$

The inequality (5.1) leads to the following corollary.

Corollary 5.3. *If $f \in L(p,r), p, r > 1$, then $S_n(f,x) \to f(x)$ almost everywhere.*

Corollary 5.4 (Carleson's Theorem on Convergence of Fourier Series)**.** *If $f \in L_p$, then $S_n(f,x) \to f(x)$ almost everywhere.*

Proof. Set $p = r$ in Corollary 5.3 since $L(p,p) = L_p$. □

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