Chapter 1

Units and Vectors: Tools for Physics

1.1 The Important Stuff

1.1.1 The SI System

Physics is based on measurement. Measurements are made by comparisons to well–defined **standards** which define the **units** for our measurements.

The **SI system** (popularly known as the *metric system*) is the one used in physics. Its unit of length is the meter, its unit of time is the second and its unit of mass is the kilogram. Other quantities in physics are derived from these. For example the unit of energy is the joule, defined by $1\,\mathrm{J} = 1\,\frac{\mathrm{kg\cdot m^2}}{\mathrm{s^2}}$.

As a convenience in using the SI system we can associate prefixes with the basic units to represent powers of 10. The most commonly used prefixes are given here:

Factor	**Prefix**	**Symbol**
10^{-12}	pico-	p
10^{-9}	nano-	n
10^{-6}	micro-	μ
10^{-3}	milli-	m
10^{-2}	centi-	c
10^{3}	kilo-	k
10^{6}	mega-	M
10^{9}	giga-	G

Other basic units commonly used in physics are:

$$\textbf{Time}: \qquad 1\,\mathrm{minute} = 60\,\mathrm{s} \qquad 1\,\mathrm{hour} = 60\,\mathrm{min} \qquad \mathrm{etc.}$$

$$\textbf{Mass}: \qquad 1\ \text{atomic mass unit} = 1\,\mathrm{u} = 1.6605 \times 10^{-27}\,\mathrm{kg}$$

1.1.2 Changing Units

In all of our mathematical operations we must *always* write down the units and we *always* treat the unit symbols as multiplicative factors. For example, if me multiply 3.0 kg by 2.0 $\frac{\mathrm{m}}{\mathrm{s}}$ we get

$$(3.0\,\mathrm{kg}) \cdot (2.0\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 6.0\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}$$

We use the same idea in changing the units in which some physical quantity is expressed. We can multiply the original quantity by a **conversion factor**, i.e. a ratio of values for which the numerator is the same thing as the denominator. The conversion factor is then *equal to 1*, and so we *do not change* the original quantity when we multiply by the conversion factor.

Examples of conversion factors are:

$$\left(\frac{1\,\mathrm{min}}{60\,\mathrm{s}}\right) \qquad \left(\frac{100\,\mathrm{cm}}{1\,\mathrm{m}}\right) \qquad \left(\frac{1\,\mathrm{yr}}{365.25\,\mathrm{day}}\right) \qquad \left(\frac{1\,\mathrm{m}}{3.28\,\mathrm{ft}}\right)$$

1.1.3 Density

A quantity which will be encountered in your study of liquids and solids is the **density** of a sample. It is usually denoted by ρ and is defined as the ratio of mass to volume:

$$\rho = \frac{m}{V} \tag{1.1}$$

The SI units of density are $\frac{\mathrm{kg}}{\mathrm{m}^3}$ but you often see it expressed in $\frac{\mathrm{g}}{\mathrm{cm}^3}$.

1.1.4 Dimensional Analysis

Every equation that we use in physics must have *the same type of units* on both sides of the equals sign. Our basic unit types (**dimensions**) are length (L), time (T) and mass (M). When we do **dimensional analysis** we focus on the units of a physics equation without worrying about the numerical values.

1.1.5 Vectors; Vector Addition

Many of the quantities we encounter in physics have both *magnitude* ("how much") and *direction*. These are **vector** quantities.

We can represent vectors graphically as arrows and then the sum of two vectors is found (graphically) by joining the head of one to the tail of the other and then connecting head to tail for the combination, as shown in Fig. 1.1 . The sum of two (or more) vectors is often called the **resultant**.

We can add vectors in any order we want: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. We say that vector addition is "commutative".

We express vectors in **component form** using the **unit vectors i**, **j** and **k**, which each have magnitude 1 and point along the x, y and z axes of the coordinate system, respectively.

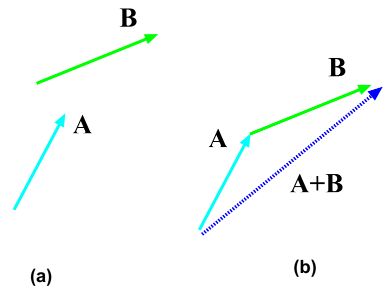

Figure 1.1: Vector addition. (a) shows the vectors **A** and **B** to be summed. (b) shows how to perform the sum graphically.

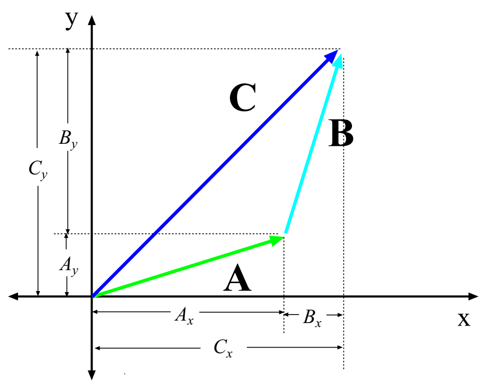

Figure 1.2: Addition of vectors by components (in two dimensions).

Any vector can be expressed as a sum of multiples of these basic vectors; for example, for the vector **A** we would write:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \ .$$

Here we would say that A_x is the x component of the vector **A**; likewise for y and z.

In Fig. 1.2 we illustrate how we get the components for a vector which is the *sum* of two other vectors. If

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \qquad \text{and} \qquad \mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

then

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k} \tag{1.2}$$

Once we have found the (Cartesian) component of two vectors, addition is simple; just add the *corresponding components* of the two vectors to get the components of the resultant vector.

When we multiply a vector by a scalar, the scalar multiplies each component; If **A** is a vector and n is a scalar, then

$$c\mathbf{A} = cA_x\mathbf{i} + cA_y\mathbf{j} + cA_z\mathbf{k} \tag{1.3}$$

In terms of its components, the magnitude ("length") of a vector **A** (which we write as A) is given by:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{1.4}$$

Many of our physics problems will be in two dimensions (x and y) and then we can also represent it in **polar** form. If **A** is a two–dimensional vector and θ as the angle that **A** makes with the $+x$ axis *measured counter-clockwise* then we can express this vector in terms of components A_x and A_y or in terms of its magnitude A and the angle θ. These descriptions are related by:

$$A_x = A\cos\theta \qquad\qquad A_y = A\sin\theta \tag{1.5}$$

$$A = \sqrt{A_x^2 + A_y^2} \qquad\qquad \tan\theta = \frac{A_y}{A_x} \tag{1.6}$$

When we use Eq. 1.6 to find θ from A_x and A_y we need to be careful because the inverse tangent operation (as done on a calculator) might give an angle in the wrong quadrant; one must think about the signs of A_x and A_y.

1.1.6 Multiplying Vectors

There are two ways to "multiply" two vectors together.

The **scalar product** (or **dot product**) of the vectors **a** and **b** is given by

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\phi \tag{1.7}$$

where a is the magnitude of **a**, b is the magnitude of **b** and ϕ is the angle between **a** and **b**.

The scalar product is commutative: $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$. One can show that $\mathbf{a}\cdot\mathbf{b}$ is related to the components of **a** and **b** by:

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z \tag{1.8}$$

If two vectors are perpendicular then their scalar product is *zero*.

The **vector product** (or **cross product**) of vectors **a** and **b** is a vector **c** whose magnitude is given by

$$c = ab\sin\phi \tag{1.9}$$

where ϕ is the *smallest* angle between **a** and **b**. The direction of **c** is perpendicular to the plane containing **a** and **b** with its orientation given by the **right–hand rule**. One way of using the right–hand rule is to let the fingers of the right hand bend (in their natural direction!) from **a** to **b**; the direction of the thumb is the direction of $\mathbf{c} = \mathbf{a}\times\mathbf{b}$. This is illustrated in Fig. 1.3.

The vector product is *anti*–commutative: $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$.

Relations among the unit vectors for vector products are:

$$\mathbf{i}\times\mathbf{j} = \mathbf{k} \qquad \mathbf{j}\times\mathbf{k} = \mathbf{i} \qquad \mathbf{k}\times\mathbf{i} = \mathbf{j} \tag{1.10}$$

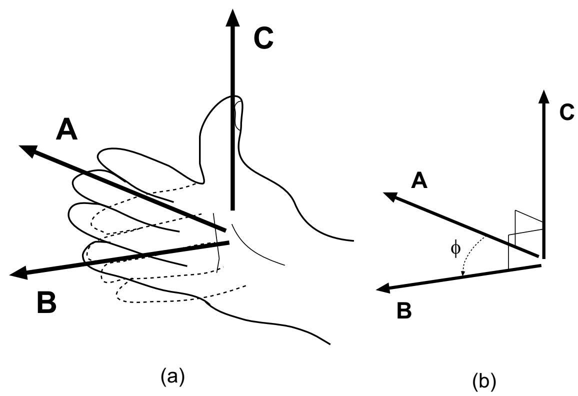

Figure 1.3: (a) Finding the direction of $\mathbf{A} \times \mathbf{B}$. Fingers of the right hand sweep from **A** to **B** in the shortest and least painful way. The extended thumb points in the direction of **C**. (b) Vectors **A**, **B** and **C**. The magnitude of **C** is $C = AB \sin\phi$.

The vector product of **a** and **b** can be computed from the components of these vectors by:

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k} \tag{1.11}$$

which can be abbreviated by the notation of the determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1.12}$$

1.2 Worked Examples

1.2.1 Changing Units

1. The Empire State Building is 1472 ft high. Express this height in both meters and centimeters. [FGT 1-4]

To do the first unit conversion (feet to meters), we can use the relation (see the Conversion Factors in the back of this book):

$$1\,\mathrm{m} = 3.281\,\mathrm{ft}$$

We set up the conversion factor so that "ft" cancels and leaves meters:

$$1472\,\mathrm{ft} = (1472\,\mathrm{ft})\left(\frac{1\,\mathrm{m}}{3.281\,\mathrm{ft}}\right) = 448.6\,\mathrm{m}\ .$$

So the height can be expressed as 448.6 m. To convert this to centimeters, use:

$$1\,\mathrm{m} = 100\,\mathrm{cm}$$

and get:

$$448.6\,\mathrm{m} = (448.6\,\mathrm{m})\left(\frac{100\,\mathrm{cm}}{1\,\mathrm{m}}\right) = 4.486\times 10^4\,\mathrm{cm}$$

The Empire State Building is 4.486×10^4 cm high!

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2. A rectangular building lot is $100.0\,\mathrm{ft}$ by $150.0\,\mathrm{ft}$. Determine the area of this lot in m^2. [Ser4 1-19]

The area of a rectangle is just the product of its length and width so the area of the lot is

$$A = (100.0\,\mathrm{ft})(150.0\,\mathrm{ft}) = 1.500\times 10^4\,\mathrm{ft}^2$$

To convert this to units of m^2 we can use the relation

$$1\,\mathrm{m} = 3.281\,\mathrm{ft}$$

but the conversion factor needs to be applied *twice* so as to cancel "ft^2" and get "m^2". We write:

$$1.500\times 10^4\,\mathrm{ft}^2 = (1.500\times 10^4\,\mathrm{ft}^2)\cdot\left(\frac{1\,\mathrm{m}}{3.281\,\mathrm{ft}}\right)^2 = 1.393\times 10^3\,\mathrm{m}^2$$

The area of the lot is $1.393\times 10^3\,\mathrm{m}^2$.

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3. The Earth is approximately a sphere of radius $6.37\times 10^6\,\mathrm{m}$. (a) What is its circumference in kilometers? (b) What is its surface area in square kilometers? (c) What is its volume in cubic kilometers? [HRW5 1-6]

(a) The circumference of the sphere of radius R, i.e. the distance around any "great circle" is $C = 2\pi R$. Using the given value of R we find:

$$C = 2\pi R = 2\pi(6.37\times 10^6\,\mathrm{m}) = 4.00\times 10^7\,\mathrm{m}\ .$$

To convert this to kilometers, use the relation $1\,\mathrm{km} = 10^3\,\mathrm{m}$ in a conversion factor:

$$C = 4.00\times 10^7\,\mathrm{m} = (4.00\times 10^7\,\mathrm{m})\cdot\left(\frac{1\,\mathrm{km}}{10^3\,\mathrm{m}}\right) = 4.00\times 10^4\,\mathrm{km}$$

The circumference of the Earth is 4.00×10^4 km.

(b) The surface area of a sphere of radius R is $A = 4\pi R^2$. So we get

$$A = 4\pi R^2 = 4\pi(6.37\times 10^6\,\mathrm{m})^2 = 5.10\times 10^{14}\,\mathrm{m}^2$$

Again, use $1\,\mathrm{km} = 10^3\,\mathrm{m}$ but to cancel out the units "m^2" and replace them with "km^2" it must be applied *twice*:

$$A = 5.10\times 10^{14}\,\mathrm{m}^2 = (5.10\times 10^{14}\,\mathrm{m}^2)\cdot\left(\frac{1\,\mathrm{km}}{10^3\,\mathrm{m}}\right)^2 = 5.10\times 10^8\,\mathrm{km}^2$$

The surface area of the Earth is $5.10 \times 10^8\,\mathrm{km}^2$.

(c) The volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$. So we get

$$V = \tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi(6.37 \times 10^6\,\mathrm{m})^3 = 1.08 \times 10^{21}\,\mathrm{m}^3$$

Again, use $1\,\mathrm{km} = 10^3\,\mathrm{m}$ but to cancel out the units "m^3" and replace them with "km^3" it must be applied *three times*:

$$V = 1.08 \times 10^{21}\,\mathrm{m}^3 = (1.08 \times 10^{21}\,\mathrm{m}^3) \cdot \left(\frac{1\,\mathrm{km}}{10^3\,\mathrm{m}}\right)^3 = 1.08 \times 10^{12}\,\mathrm{km}^3$$

The volume of the Earth is $1.08 \times 10^{12}\,\mathrm{km}^3$.

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4. Calculate the number of kilometers in $20.0\,\mathrm{mi}$ using only the following conversion factors: $1\,\mathrm{mi} = 5280\,\mathrm{ft}$, $1\,\mathrm{ft} = 12\,\mathrm{in}$, $1\,\mathrm{in} = 2.54\,\mathrm{cm}$, $1\,\mathrm{m} = 100\,\mathrm{cm}$, $1\,\mathrm{km} = 1000\,\mathrm{m}$. [HRW5 1-7]

Set up the "factors of 1" as follows:

$$\begin{aligned} 20.0\,\mathrm{mi} &= (20.0\,\mathrm{mi}) \cdot \left(\frac{5280\,\mathrm{ft}}{1\,\mathrm{mi}}\right) \cdot \left(\frac{12\,\mathrm{in}}{1\,\mathrm{ft}}\right) \cdot \left(\frac{2.54\,\mathrm{cm}}{1\,\mathrm{in}}\right) \cdot \left(\frac{1\,\mathrm{m}}{100\,\mathrm{cm}}\right) \cdot \left(\frac{1\,\mathrm{km}}{1000\,\mathrm{m}}\right) \\ &= 32.2\,\mathrm{km} \end{aligned}$$

Setting up the "factors of 1" in this way, all of the unit symbols cancel except for km (kilometers) which we keep as the units of the answer.

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5. One gallon of paint (volume $= 3.78 \times 10^{-3}\,\mathrm{m}^3$) covers an area of $25.0\,\mathrm{m}^3$. What is the thickness of the paint on the wall? [Ser4 1-31]

We will assume that the volume which the paint occupies while it's covering the wall is the *same* as it has when it is in the can. (There are reasons why this may not be true, but let's just do this and proceed.)

The paint on the wall covers an area A and has a thickness τ; the volume occupied is the area time the thickness:

$$V = A\tau\ .$$

We have V and A; we just need to solve for τ:

$$\tau = \frac{V}{A} = \frac{3.78 \times 10^{-3}\,\mathrm{m}^3}{25.0\,\mathrm{m}^2} = 1.51 \times 10^{-4}\,\mathrm{m}\ .$$

The thickness is $1.51 \times 10^{-4}\,\mathrm{m}$. This quantity can also be expressed as $0.151\,\mathrm{mm}$.

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6. A certain brand of house paint claims a coverage of $460\,\frac{\mathrm{ft}^2}{\mathrm{gal}}$. (a) Express this quantity in square meters per liter. (b) Express this quantity in SI base units. (c)

What is the inverse of the original quantity, and what is its physical significance? [HRW5 1-15]

(a) Use the following relations in forming the conversion factors: $1\,\mathrm{m} = 3.28\,\mathrm{ft}$ and $1000\,\mathrm{liter} = 264\,\mathrm{gal}$. To get proper cancellation of the units we set it up as:

$$460\,\frac{\mathrm{ft}^2}{\mathrm{gal}} = (460\,\frac{\mathrm{ft}^2}{\mathrm{gal}}) \cdot \left(\frac{1\,\mathrm{m}}{3.28\,\mathrm{ft}}\right)^2 \cdot \left(\frac{264\,\mathrm{gal}}{1000\,\mathrm{L}}\right) = 11.3\,\frac{\mathrm{m}^2}{\mathrm{L}}$$

(b) Even though the units of the answer to part (a) are based on the metric system, they are not made from the *base* units of the SI system, which are m, s, and kg. To make the complete conversion to SI units we need to use the relation $1\,\mathrm{m}^3 = 1000\,\mathrm{L}$. Then we get:

$$11.3\,\frac{\mathrm{m}^2}{\mathrm{L}} = (11.3\,\frac{\mathrm{m}^2}{\mathrm{L}}) \cdot \left(\frac{1000\,\mathrm{L}}{1\,\mathrm{m}^3}\right) = 1.13 \times 10^4\,\mathrm{m}^{-1}$$

So the coverage can also be expressed (not so meaningfully, perhaps) as $1.13 \times 10^4\,\mathrm{m}^{-1}$.

(c) The inverse (reciprocal) of the quantity as it was *originally* expressed is

$$\left(460\,\frac{\mathrm{ft}^2}{\mathrm{gal}}\right)^{-1} = 2.17 \times 10^{-3}\,\frac{\mathrm{gal}}{\mathrm{ft}^2}\ .$$

Of course when we take the reciprocal the *units* in the numerator and denominator also switch places!

Now, the first expression of the quantity tells us that $460\,\mathrm{ft}^2$ are associated with every gallon, that is, each gallon will provide $460\,\mathrm{ft}^2$ of coverage. The new expression tells us that 2.17×10^{-3} gal are associated with every ft^2, that is, to cover one square foot of surface with paint, one needs 2.17×10^{-3} gallons of it.

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7. Express the speed of light, $3.0 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$ **in (a) feet per nanosecond and (b) millimeters per picosecond.** [HRW5 1-19]

(a) For this conversion we can use the following facts:

$$1\,\mathrm{m} = 3.28\,\mathrm{ft} \qquad \text{and} \qquad 1\,\mathrm{ns} = 10^{-9}\,\mathrm{s}$$

to get:

$$\begin{aligned} 3.0 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}} &= (3.0 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}}) \cdot \left(\frac{3.28\,\mathrm{ft}}{1\,\mathrm{m}}\right) \cdot \left(\frac{10^{-9}\,\mathrm{s}}{1\,\mathrm{ns}}\right) \\ &= 0.98\,\frac{\mathrm{ft}}{\mathrm{ns}} \end{aligned}$$

In these new units, the speed of light is $0.98\,\frac{\mathrm{ft}}{\mathrm{ns}}$.

(b) For this conversion we can use:

$$1\,\mathrm{mm} = 10^{-3}\,\mathrm{m} \qquad \text{and} \qquad 1\,\mathrm{ps} = 10^{-12}\,\mathrm{s}$$

and set up the factors as follows:

$$\begin{aligned} 3.0 \times 10^{8}\,\tfrac{\mathrm{m}}{\mathrm{s}} &= (3.0 \times 10^{8}\,\tfrac{\mathrm{m}}{\mathrm{s}}) \cdot \left(\frac{1\,\mathrm{mm}}{10^{-3}\,\mathrm{m}}\right) \cdot \left(\frac{10^{-12}\,\mathrm{s}}{1\,\mathrm{ps}}\right) \\ &= 3.0 \times 10^{-1}\,\tfrac{\mathrm{mm}}{\mathrm{ps}} \end{aligned}$$

In these new units, the speed of light is $3.0 \times 10^{-1}\,\frac{\mathrm{mm}}{\mathrm{ps}}$.

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8. One molecule of water (H_2O) contains two atoms of hydrogen and one atom of oxygen. A hydrogen atom has a mass of $1.0\,\mathrm{u}$ and an atom of oxygen has a mass of $16\,\mathrm{u}$, approximately. (a) What is the mass in kilograms of one molecule of water? (b) How many molecules of water are in the world's oceans, which have an estimated total mass of $1.4 \times 10^{21}\,\mathrm{kg}$? [HRW5 1-33]

(a) We are given the masses of the atoms of H and O in atomic mass units; using these values, one molecule of H_2O has a mass of

$$m_{\mathrm{H_2O}} = 2(1.0\,\mathrm{u}) + 16\,\mathrm{u} = 18\,\mathrm{u}$$

Use the relation between u (atomic mass units) and kilograms to convert this to kg:

$$m_{\mathrm{H_2O}} = (18\,\mathrm{u}) \left(\frac{1.6605 \times 10^{-27}\,\mathrm{kg}}{1\,\mathrm{u}}\right) = 3.0 \times 10^{-26}\,\mathrm{kg}$$

One water molecule has a mass of $3.0 \times 10^{-26}\,\mathrm{kg}$.

(b) To get the number of molecules in all the oceans, divide the mass of *all* the oceans' water by the mass of *one* molecule:

$$N = \frac{1.4 \times 10^{21}\,\mathrm{kg}}{3.0 \times 10^{-26}\,\mathrm{kg}} = 4.7 \times 10^{46} \; .$$

. . . a large number of molecules!

1.2.2 Density

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9. Calculate the density of a solid cube that measures $5.00\,\mathrm{cm}$ on each side and has a mass of $350\,\mathrm{g}$. [Ser4 1-1]

The volume of this cube is

$$V = (5.00\,\mathrm{cm}) \cdot (5.00\,\mathrm{cm}) \cdot (5.00\,\mathrm{cm}) = 125\,\mathrm{cm}^3$$

So from Eq. 1.1 the density of the cube is

$$\rho = \frac{m}{V} = \frac{350\,\mathrm{g}}{125\,\mathrm{cm}^3} = 2.80 \frac{\mathrm{g}}{\mathrm{cm}^3}$$

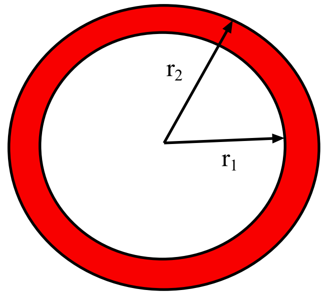

Figure 1.4: Cross–section of copper shell in Example 11.

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10. The mass of the planet Saturn is $5.64 \times 10^{26}\,\mathrm{kg}$ and its radius is $6.00 \times 10^{7}\,\mathrm{m}$. Calculate its density. [Ser4 1-2]

The planet Saturn is roughly a sphere. (But only roughly! Actually its shape is rather distorted.) Using the formula for the volume of a sphere, we find the volume of Saturn:

$$V = \tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi(6.00 \times 10^{7}\,\mathrm{m})^3 = 9.05 \times 10^{23}\,\mathrm{m}^3$$

Now using the definition of density we find:

$$\rho = \frac{m}{V} = \frac{5.64 \times 10^{26}\,\mathrm{kg}}{9.05 \times 10^{23}\,\mathrm{m}^3} = 6.23 \times 10^{2}\,\tfrac{\mathrm{kg}}{\mathrm{m}^3}$$

While this answer is correct, it is useful to express the result in units of $\frac{\mathrm{g}}{\mathrm{cm}^3}$. Using our conversion factors in the usual way, we get:

$$6.23 \times 10^{2}\,\tfrac{\mathrm{kg}}{\mathrm{m}^3} = (6.23 \times 10^{2}\,\tfrac{\mathrm{kg}}{\mathrm{m}^3}) \cdot \left(\frac{10^3\,\mathrm{g}}{1\,\mathrm{kg}}\right) \cdot \left(\frac{1\,\mathrm{m}}{100\,\mathrm{cm}}\right)^3 = 0.623\,\tfrac{\mathrm{g}}{\mathrm{cm}^3}$$

The average density of Saturn is $0.623\,\frac{\mathrm{g}}{\mathrm{cm}^3}$. Interestingly, this is less than the density of water.

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11. How many grams of copper are required to make a hollow spherical shell with an inner radius of $5.70\,\mathrm{cm}$ and an outer radius of $5.75\,\mathrm{cm}$? The density of copper is $8.93\,\mathrm{g}/\,\mathrm{cm}^3$. [Ser4 1-3]

A cross–section of the copper sphere is shown in Fig. 1.4. The outer and inner radii are noted as r_2 and r_1, respectively. We must find the volume of space *occupied by the copper metal*; this volume is the difference in the volumes of the two spherical surfaces:

$$V_{\mathrm{copper}} = V_2 - V_1 = \tfrac{4}{3}\pi r_2^3 - \tfrac{4}{3}\pi r_1^3 = \tfrac{4}{3}\pi(r_2^3 - r_1^3)$$

With the given values of the radii, we find:

$$V_{\mathrm{copper}} = \tfrac{4}{3}\pi((5.75\,\mathrm{cm})^3 - (5.70\,\mathrm{cm})^3) = 20.6\,\mathrm{cm}^3$$

Now use the definition of density to find the mass of the copper contained in the shell:

$$\rho = \frac{m_{\rm copper}}{V_{\rm copper}} \qquad \Longrightarrow \qquad m_{\rm copper} = \rho V_{\rm copper} = \left(8.93\,\tfrac{\rm g}{\rm cm^3}\right)(20.6\,{\rm cm}^3) = 184\,{\rm g}$$

184 grams of copper are required to make the spherical shell of the given dimensions.

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12. One cubic meter ($1.00\,{\rm m}^3$) of aluminum has a mass of $2.70 \times 10^3\,{\rm kg}$, and $1.00\,{\rm m}^3$ of iron has a mass of $7.86 \times 10^3\,{\rm kg}$. Find the radius of a solid aluminum sphere that will balance a solid iron sphere of radius $2.00\,{\rm cm}$ on an equal–arm balance.
[Ser4 1-39]

In the statement of the problem, we are given the densities of aluminum and iron:

$$\rho_{\rm Al} = 2.70 \times 10^3\,\tfrac{\rm kg}{\rm m^3} \qquad \text{and} \qquad \rho_{\rm Fe} = 7.86 \times 10^3\,\tfrac{\rm kg}{\rm m^3}\ .$$

A solid iron sphere of radius $R = 2.00\,{\rm cm} = 2.00 \times 10^{-2}\,{\rm m}$ has a volume

$$V_{\rm Fe} = \tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi(2.00 \times 10^{-2}\,{\rm m})^3 = 3.35 \times 10^{-5}\,{\rm m}^3$$

so that from $M_{\rm Fe} = \rho_{\rm Fe} V_{\rm Fe}$ we find the mass of the iron sphere:

$$M_{\rm Fe} = \rho_{\rm Fe} V_{\rm Fe} = \left(7.86 \times 10^3\,\tfrac{\rm kg}{\rm m^3}\right)(3.35 \times 10^{-5}\,{\rm m}^3) = 2.63 \times 10^{-1}\,{\rm kg}$$

If this sphere balances one made from aluminum in an "equal–arm balance", then they have the same mass. So $M_{\rm Al} = 2.63 \times 10^{-1}\,{\rm kg}$ is the mass of the aluminum sphere. From $M_{\rm Al} = \rho_{\rm Al} V_{\rm Al}$ we can find its volume:

$$V_{\rm Al} = \frac{M_{\rm Al}}{\rho_{\rm Al}} = \frac{2.63 \times 10^{-1}\,{\rm kg}}{2.70 \times 10^3\,\frac{\rm kg}{\rm m^3}} = 9.76 \times 10^{-5}\,{\rm m}^3$$

Having the volume of the sphere, we can find its radius:

$$V_{\rm Al} = \tfrac{4}{3}\pi R^3 \qquad \Longrightarrow \qquad R = \left(\frac{3V_{\rm Al}}{4\pi}\right)^{\frac{1}{3}}$$

This gives:

$$R = \left(\frac{3(9.76 \times 10^{-5}\,{\rm m}^3)}{4\pi}\right)^{\frac{1}{3}} = 2.86 \times 10^{-2}\,{\rm m} = 2.86\,{\rm cm}$$

The aluminum sphere must have a radius of $2.86\,{\rm cm}$ to balance the iron sphere.

1.2.3 Dimensional Analysis

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13. The period T of a simple pendulum is measured in time units and is

$$T = 2\pi\sqrt{\frac{\ell}{g}} \ .$$

where ℓ is the length of the pendulum and g is the free–fall acceleration in units of length divided by the square of time. Show that this equation is dimensionally correct. [Ser4 1-14]

The period (T) of a pendulum is the amount of time it takes to makes one full swing back and forth. It is measured in units of *time* so its dimensions are represented by T.

On the right side of the equation we have the length ℓ, whose dimensions are represented by L. We are told that g is a length divided by the square of a time so its dimensions must be L/T^2. There is a factor of 2π on the right side, but this is a pure number and has no units. So the dimensions of the right side are:

$$\sqrt{\frac{L}{\left(\frac{L}{T^2}\right)}} = \sqrt{T^2} = T$$

so that the right hand side must also have units of time. Both sides of the equation agree in their units, which must be true for it to be a valid equation!

---

14. The volume of an object as a function of time is calculated by $V = At^3 + B/t$, where t is time measured in seconds and V is in cubic meters. Determine the dimension of the constants A and B. [Ser4 1-15]

Both sides of the equation for volume must have the same dimensions, and those must be the dimensions of volume where are L^3 (SI units of m^3). Since we can only add terms with the same dimensions, each of the terms on right side of the equation (At^3 and B/t) must have the same dimensions, namely L^3.

Suppose we denote the units of A by $[A]$. Then our comment about the dimensions of the first term gives us:

$$[A]T^3 = L^3 \qquad \Longrightarrow \qquad [A] = \frac{L^3}{T^3}$$

so A has dimensions L^3/T^3. In the SI system, it would have units of $\mathrm{m}^3/\mathrm{s}^3$.

Suppose we denote the units of B by $[B]$. Then our comment about the dimensions of the second term gives us:

$$\frac{[B]}{T} = L^3 \qquad \Longrightarrow \qquad [B] = L^3T$$

so B has dimensions L^3T. In the SI system, it would have units of $\mathrm{m}^3\,\mathrm{s}$.

---

15. Newton's law of universal gravitation is

$$F = G\frac{Mm}{r^2}$$

Here F is the force of gravity, M and m are masses, and r is a length. Force has the SI units of $\mathrm{kg \cdot m/s^2}$. What are the SI units of the constant G? [Ser4 1-17]

If we denote the *dimensions* of F by $[F]$ (and the same for the other quantities) then then dimensions of the quantities in Newton's Law are:

$$[M] = M \text{ (mass)} \quad [m] = M \quad [r] = L \qquad [F] : \frac{ML}{T^2}$$

What we don't know (yet) is $[G]$, the dimensions of G. Putting the known dimensions into Newton's Law, we must have:

$$\frac{ML}{T^2} = [G]\frac{M \cdot M}{L^2}$$

since the dimensions must be the same on both sides. Doing some algebra with the dimensions, this gives:

$$[G] = \left(\frac{ML}{T^2}\right)\frac{L^2}{M^2} = \frac{L^3}{MT^2}$$

so the dimensions of G are $L^3/(MT^2)$. In the SI system, G has *units* of

$$\frac{\mathrm{m}^3}{\mathrm{kg \cdot s^3}}$$

---

16. In quantum mechanics, the fundamental constant called Planck's constant, h, has dimensions of $[ML^2T^{-1}]$. Construct a quantity with the dimensions of length from h, a mass m, and c, the speed of light. [FGT 1-54]

The problem suggests that there is some product of powers of h, m and c which has dimensions of length. If these powers are r, s and t, respectively, then we are looking for values of r, s and t such that

$$h^r m^s c^t$$

has dimensions of length.

What are the dimensions of this product, as written? We were given the dimensions of h, namely $[ML^2T^{-1}]$; the dimensions of m are M, and the dimensions of c are $\frac{L}{T} = LT^{-1}$ (it is a speed). So the dimensions of $h^r m^s c^t$ are:

$$[ML^2T^{-1}]^r[M]^s[LT^{-1}]^t = M^{r+s}L^{2r+t}T^{-r-t}$$

where we have used the laws of combining exponents which we all remember from algebra.

Now, since this is supposed to have dimensions of length, the power of L must be 1 but the other powers are zero. This gives the equations:

$$\begin{aligned} r + s &= 0 \\ 2r + t &= 1 \\ -r - t &= 0 \end{aligned}$$

which is a set of three equations for three unknowns. Easy to solve!

The last of them gives $r = -t$. Substituting this into the second equation gives

$$2r + t = 2(-t) + t = -t = 1 \qquad \Longrightarrow \qquad t = -1$$

Then $r = +1$ and the first equation gives us $s = -1$. With these values, we can confidently say that

$$h^r m^s c^t = h^1 m^{-1} c^{-1} = \frac{h}{mc}$$

has units of length.

1.2.4 Vectors; Vector Addition

---

17. (a) What is the sum in unit–vector notation of the two vectors $\mathbf{a} = 4.0\mathbf{i} + 3.0\mathbf{j}$ and $\mathbf{b} = -13.0\mathbf{i} + 7.0\mathbf{j}$? (b) What are the magnitude and direction of $\mathbf{a} + \mathbf{b}$? [HRW5 3-20]

(a) Summing the corresponding components of vectors **a** and **b** we find:

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= (4.0 - 13.0)\mathbf{i} + (3.0 + 7.0)\mathbf{j} \\ &= -9.0\mathbf{i} + 10.0\mathbf{j} \end{aligned}$$

This is the sum of the two vectors is unit–vector form.

(b) Using our results from (a), the magnitude of $\mathbf{a} + \mathbf{b}$ is

$$|\mathbf{a} + \mathbf{b}| = \sqrt{(-9.0)^2 + (10.0)^2} = 13.4$$

and if $\mathbf{c} = \mathbf{a} + \mathbf{b}$ points in a direction θ as measured from the positive x axis, then the tangent of θ is found from

$$\tan\theta = \left(\frac{c_y}{c_x}\right) = -1.11$$

If we naively take the arctangent using a calculator, we are told:

$$\theta = \tan^{-1}(-1.11) = -48.0^\circ$$

which is not correct because (as shown in Fig. 1.5), with c_x negative, and c_y positive, the

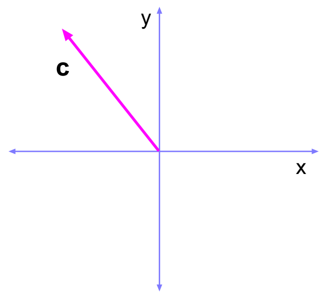

Figure 1.5: Vector **c**, found in Example 17. With $c_x = -9.0$ and $c_y = +10.0$, the direction of **c** is in the second quadrant.

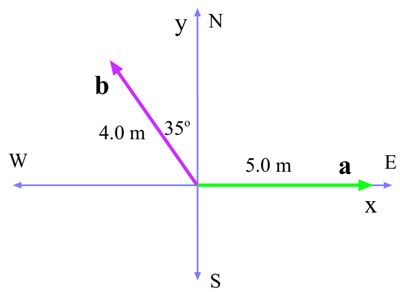

Figure 1.6: Vectors **a** and **b** as given in Example 18.

correct angle must be in the *second* quadrant. The calculator was fooled because angles which differ by multiples of 180° have the same tangent. The direction we really want is

$$\theta = -48.0^\circ + 180.0^\circ = 132.0^\circ$$

---

18. Vector a has magnitude $5.0\,\mathrm{m}$ **and is directed east. Vector b has magnitude** $4.0\,\mathrm{m}$ **and is directed** 35° **west of north. What are (a) the magnitude and (b) the direction of a + b? What are (c) the magnitude and (d) the direction of b – a? Draw a vector diagram for each combination.** [HRW6 3-15]

(a) The vectors are shown in Fig. 1.6. (On the axes are shown the common directions N, S, E, W and also the x and y axes; "North" is the positive y direction, "East" is the positive x direction, etc.) Expressing the vectors in **i**, **j** notation, we have:

$$\mathbf{a} = (5.00\,\mathrm{m})\mathbf{i}$$

and

$$\begin{aligned}\mathbf{b} &= -(4.00\,\mathrm{m})\sin 35^\circ + (4.00\,\mathrm{m})\cos 35^circ \\ &= (-2.29\,\mathrm{m})\mathbf{i} + (3.28\,\mathrm{m})\mathbf{j}\end{aligned}$$

So if vector **c** is the sum of vectors **a** and **b** then:

$$\begin{aligned}c_x = a_x + b_x &= (5.00\,\mathrm{m}) + (-2.29\,\mathrm{m}) = 2.71\,\mathrm{m} \\ c_y = a_y + b_y &= (0.00\,\mathrm{m}) + (3.28\,\mathrm{m}) = 3.28\,\mathrm{m}\end{aligned}$$

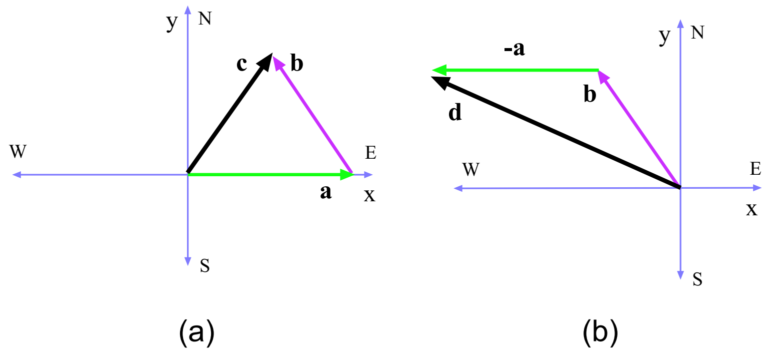

Figure 1.7: (a) Vector diagram showing the addition $\mathbf{a}+\mathbf{b}$. (b) Vector diagram showing $\mathbf{b}-\mathbf{a}$.

The magnitude of $\mathbf{c}$ is

$$c = \sqrt{c_x^2 + c_y^2} = \sqrt{(2.71\,\mathrm{m})^2 + (3.28\,\mathrm{m})^2} = 4.25\,\mathrm{m}$$

(b) If the direction of $\mathbf{c}$, as measured counterclockwise from the $+x$ axis is θ then

$$\tan\theta = \frac{c_y}{c_x} = \frac{3.28\,\mathrm{m}}{2.71\,\mathrm{m}} = 1.211$$

then the $\mathtt{tan}^{-1}$ operation on a calculator gives

$$\theta = \tan^{-1}(1.211) = 50.4^\circ$$

and since vector $\mathbf{c}$ must lie in the first quadrant this angle is correct. We note that this angle is

$$90.0^\circ - 50.4^\circ = 39.6^\circ$$

just shy of the $+y$ axis (the "North" direction). So we can also express the direction by saying it is "39.6° East of North".

A vector diagram showing $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is given in Fig. 1.7(a).

(c) If the vector $\mathbf{d}$ is given by $\mathbf{d} = \mathbf{b} - \mathbf{a}$ then the components of $\mathbf{d}$ are given by

$$d_x = b_x - a_x = (-2.29\,\mathrm{m}) - (5.00\,\mathrm{m}) = -7.29\,\mathrm{m}$$
$$c_y = a_y + b_y = (3.28\,\mathrm{m}) - (0.00\,\mathrm{m}) + (3.28\,\mathrm{m}) = 3.28\,\mathrm{m}$$

The magnitude of $\mathbf{c}$ is

$$d = \sqrt{d_x^2 + d_y^2} = \sqrt{(-7.29\,\mathrm{m})^2 + (3.28\,\mathrm{m})^2} = 8.00\,\mathrm{m}$$

(d) If the direction of $\mathbf{d}$, as measured counterclockwise from the $+x$ axis is θ then

$$\tan\theta = \frac{d_y}{d_x} = \frac{3.28\,\mathrm{m}}{-7.29\,\mathrm{m}} = -0.450$$

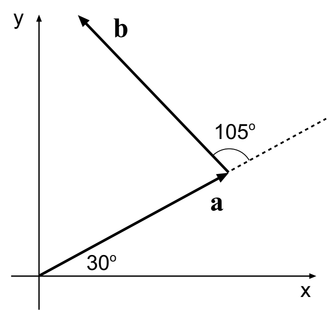

Figure 1.8: Vectors for Example 19.

Naively pushing buttons on the calculator gives

$$\theta = \tan^{-1}(-0.450) = -24.2^\circ$$

which can't be right because from the signs of its components we know that **d** must lie in the second quadrant. We need to add 180° to get the correct answer for the `tan`$^{-1}$ operation:

$$\theta = -24.2^\circ + 180.0^\circ = 156^\circ$$

But we note that this angle is

$$180^\circ - 156^\circ = 24^\circ$$

shy of the $-y$ axis, so the direction can also be expressed as "24° North of West".

A vector diagram showing **a**, **b** and **d** is given in Fig. 1.7(b).

---

19. The two vectors a and b in Fig. 1.8 have equal magnitudes of $10.0\,\mathrm{m}$**. Find (a) the x component and (b) the y component of their vector sum r, (c) the magnitude of r and (d) the angle r makes with the positive direction of the x axis.** [HRW6 3-21]

(a) First, find the x and y components of the vectors **a** and **b**. The vector **a** makes an angle of 30° with the $+x$ axis, so its components are

$$\begin{aligned} a_x &= a\cos 30^\circ = (10.0\,\mathrm{m})\cos 30^\circ = 8.66\,\mathrm{m} \\ a_y &= a\sin 30^\circ = (10.0\,\mathrm{m})\sin 30^\circ = 5.00\,\mathrm{m} \end{aligned}$$

The vector **b** makes an angle of 135° with the $+x$ axis (30° plus 105° more) so its components are

$$\begin{aligned} b_x &= b\cos 135^\circ = (10.0\,\mathrm{m})\cos 135^\circ = -7.07\,\mathrm{m} \\ b_y &= b\sin 135^\circ = (10.0\,\mathrm{m})\sin 135^\circ = 7.07\,\mathrm{m} \end{aligned}$$

Then if $\mathbf{r} = \mathbf{a} + \mathbf{b}$, the x and y components of the vector **r** are:

$$\begin{aligned} r_x &= a_x + b_x = 8.66\,\mathrm{m} - 7.07\,\mathrm{m} = 1.59\,\mathrm{m} \\ r_y &= a_y + b_y = 5.00\,\mathrm{m} + 7.07\,\mathrm{m} = 12.07\,\mathrm{m} \end{aligned}$$

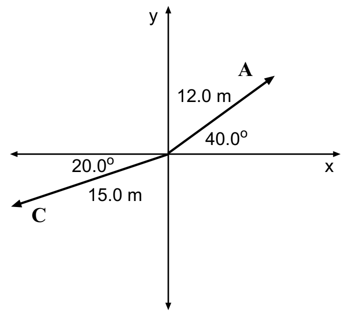

Figure 1.9: Vectors **A** and **C** as described in Example 20.

So the x component of the sum is $r_x = 1.59\,\mathrm{m}$, and...

(b) ...the y component of the sum is $r_y = 12.07\,\mathrm{m}$.

(c) The magnitude of the vector **r** is

$$r = \sqrt{r_x^2 + r_y^2} = \sqrt{(1.59\,\mathrm{m})^2 + (12.07\,\mathrm{m})^2} = 12.18\,\mathrm{m}$$

(d) To get the direction of the vector **r** expressed as an angle θ measured from the $+x$ axis, we note:

$$\tan\theta = \frac{r_y}{r_x} = 7.59$$

and then take the inverse tangent of 7.59:

$$\theta = \tan^{-1}(7.59) = 82.5^\circ$$

Since the components of **r** are both positive, the vector *does* lie in the first quadrant so that the inverse tangent operation has (this time) given the correct answer. So the direction of **r** is given by $\theta = 82.5^\circ$.

---

20. In the sum $\mathbf{A}+\mathbf{B}=\mathbf{C}$, vector A has a magnitude of $12.0\,\mathrm{m}$ and is angled 40.0° counterclockwise from the $+x$ direction, and vector C has magnitude of $15.0\,\mathrm{m}$ and is angled 20.0° counterclockwise from the $-x$ direction. What are (a) the magnitude and (b) the angle (relative to $+x$) of B? [HRW6 3-22]

(a) Vectors **A** and **C** are diagrammed in Fig. 1.9. From these we can get the components of **A** and **C** (watch the signs on vector **C** from the odd way that its angle is given!):

$$A_x = (12.0\,\mathrm{m})\cos(40.0^\circ) = 9.19\,\mathrm{m} \qquad A_y = (12.0\,\mathrm{m})\sin(40.0^\circ) = 7.71\,\mathrm{m}$$

$$C_x = -(15.0\,\mathrm{m})\cos(20.0^\circ) = -14.1\,\mathrm{m} \qquad C_y = -(15.0\,\mathrm{m})\sin(20.0^\circ) = -5.13\,\mathrm{m}$$

(Note, the vectors in this problem have *units* to go along with their magnitudes, namely m (meters).) Then from the relation $\mathbf{A}+\mathbf{B}=\mathbf{C}$ it follows that $\mathbf{B}=\mathbf{C}-\mathbf{A}$, and from this we find the components of **B**:

$$B_x = C_x - A_x = -14.1\,\mathrm{m} - 9.19\,\mathrm{m} = -23.3\,\mathrm{m}$$

$$B_y = C_y - A_y = -5.13\,\mathrm{m} - 7.71\,\mathrm{m} = -12.8\,\mathrm{m}$$

Then we find the magnitude of vector **B**:

$$B = \sqrt{B_x^2 + B_y^2} = \sqrt{(-23.3)^2 + (-12.8)^2}\,\mathrm{m} = 26.6\,\mathrm{m}$$

(b) We find the direction of **B** from:

$$\tan\theta = \left(\frac{B_y}{B_x}\right) = 0.551$$

If we *naively* press the "`atan`" button on our calculators to get θ, we are told:

$$\theta = \tan^{-1}(0.551) = 28.9^\circ \qquad (?)$$

which cannot be correct because from the components of **B** (both negative) we know that vector **B** lies in the third quadrant. So we need to ad 180° to the naive result to get the *correct* answer:

$$\theta = 28.9^\circ + 180.0^\circ = 208.9^\circ \ .$$

This is the angle of **B**, measured counterclockwise from the $+x$ axis.

---

21. If $\mathbf{a} - \mathbf{b} = 2\mathbf{c}$, $\mathbf{a} + \mathbf{b} = 4\mathbf{c}$ and $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j}$, then what are a and b? [HRW5 3-24]

We notice that if we add the first two relations together, the vector **b** will cancel:

$$(\mathbf{a} - \mathbf{b}) + (\mathbf{a} + \mathbf{b}) = (2\mathbf{c}) + (4\mathbf{c})$$

which gives:

$$2\mathbf{a} = 6\mathbf{c} \qquad \Longrightarrow \qquad \mathbf{a} = 3\mathbf{c}$$

and we can use the last of the given equations to substitute for **c**; we get

$$\mathbf{a} = 3\mathbf{c} = 3(3\mathbf{i} + 4\mathbf{j}) = 9\mathbf{i} + 12\mathbf{j}$$

Then we can rearrange the first of the equations to solve for **b**:

$$\begin{aligned} \mathbf{b} &= \mathbf{a} - 2\mathbf{c} = (9\mathbf{i} + 12\mathbf{j}) - 2(3\mathbf{i} + 4\mathbf{j}) \\ &= (9-6)\mathbf{i} + (12-8)\mathbf{j} \\ &= 3\mathbf{i} + 4\mathbf{j} \end{aligned}$$

So we have found:

$$\mathbf{a} = 9\mathbf{i} + 12\mathbf{j} \qquad \text{and} \qquad \mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$$

---

22. If $\mathbf{A} = (6.0\mathbf{i} - 8.0\mathbf{j})$ units, $\mathbf{B} = (-8.0\mathbf{i} + 3.0\mathbf{j})$ units, and $\mathbf{C} = (26.0\mathbf{i} + 19.0\mathbf{j})$ units, determine a and b so that $a\mathbf{A} + b\mathbf{B} + \mathbf{C} = 0$. [Ser4 3-46]

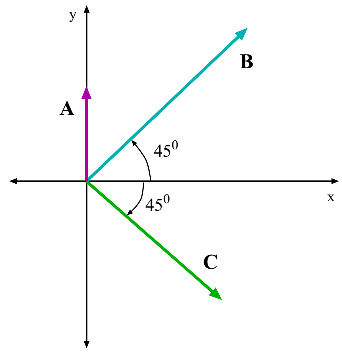

Figure 1.10: Vectors for Example 23

The condition on the vectors given in the problem:

$$a\mathbf{A} + b\mathbf{B} + \mathbf{C} = 0$$

is a condition on the *individual components* of the vectors. It implies:

$$aA_x + bB_x + C_x = 0 \qquad \text{and} \qquad aA_y + bB_y + C_y = 0 \ .$$

So that we have the equations:

$$\begin{aligned} 6.0a - 8.0b + 26.0 &= 0 \\ -8.0a + 3.0b + 19.0 &= = 0 \end{aligned}$$

We have two equations for two unknowns so we *can* find a and b. The are lots of ways to do this; one could multiply the first equation by 4 and the second equation by 3 to get:

$$\begin{aligned} 24.0a - 32.0b + 104.0 &= 0 \\ -24.0a + 9.0b + 57.0 &= = 0 \end{aligned}$$

Adding these gives

$$-23.0b + 161 = 0 \qquad \Longrightarrow \qquad b = \frac{-161.0}{-23.0} = 7.0$$

and then the first of the original equations gives us a:

$$6.0a = 8.0b - 26.0 = 8.0(7.0) - 26.0 = 30.0 \qquad \Longrightarrow \qquad a = \frac{30.0}{6.0} = 5.0$$

and our solution is

$$a = 7.0 \qquad\qquad b = 5.0$$

---

23. Three vectors are oriented as shown in Fig. 1.10, where $|\mathbf{A}| = 20.0\,\text{units}$, $|\mathbf{B}| = 40.0\,\text{units}$, and $|\mathbf{C}| = 30.0\,\text{units}$. Find (a) the x and y components of the

resultant vector and (b) the magnitude and direction of the resultant vector. [Ser4 3-47]

(a) Let's first put these vectors into "unit–vector notation":

$$\begin{aligned} \mathbf{A} &= 20.0\mathbf{j} \\ \mathbf{B} &= (40.0\cos 45^\circ)\mathbf{i} + (40.0\sin 45^\circ)\mathbf{j} = 28.3\mathbf{i} + 28.3\mathbf{j} \\ \mathbf{C} &= (30.0\cos(-45^\circ))\mathbf{i} + (30.0\sin(-45^\circ))\mathbf{j} = 21.2\mathbf{i} - 21.2\mathbf{j} \end{aligned}$$

Adding the components together, the resultant (total) vector is:

$$\begin{aligned} \text{Resultant} &= \mathbf{A} + \mathbf{B} + \mathbf{C} \\ &= (28.3 + 21.2)\mathbf{i} + (20.0 + 28.3 - 21.2)\mathbf{j} \\ &= 49.5\mathbf{i} + 27.1\mathbf{j} \end{aligned}$$

So the x component of the resultant vector is 49.5 and the y component of the resultant is 27.1.

(b) If we call the resultant vector $\mathbf{R}$, then the magnitude of $\mathbf{R}$ is given by

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(49.5)^2 + (27.1)^2} = 56.4$$

To find its direction (given by θ, measured counterclockwise from the x axis), we find:

$$\tan\theta = \frac{R_y}{R_x} = \frac{27.1}{49.5} = 0.547$$

and then taking the inverse tangent gives a *possible* answer for θ:

$$\theta = \tan^{-1}(0.547) = 28.7^\circ \ .$$

Is this the right answer for θ? Since *both* components of $\mathbf{R}$ are *positive*, it must lie in the first quadrant and so θ must be between 0° and 90°. So the direction of $\mathbf{R}$ is given by 28.7°.

---

24. A vector B, when added to the vector $\mathbf{C} = 3.0\mathbf{i} + 4.0\mathbf{j}$, yields a resultant vector that is in the positive y direction and has a magnitude equal to that of C. What is the magnitude of B? [HRW5 3-26]

If the vector $\mathbf{B}$ is denoted by $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j}$ then the resultant of $\mathbf{B}$ and $\mathbf{C}$ is

$$\mathbf{B} + \mathbf{C} = (B_x + 3.0)\mathbf{i} + (B_y + 4.0)\mathbf{j} \ .$$

We are told that the resultant points in the positive y direction, so its x component must be *zero*. Then:

$$B_x + 3.0 = 0 \qquad \Longrightarrow \qquad B_x = -3.0 \ .$$

Now, the magnitude of **C** is

$$C = \sqrt{C_x^2 + C_y^2} = \sqrt{(3.0)^2 + (4.0)^2} = 5.0$$

so that if the magnitude of **B** + **C** is also 5.0 then we get

$$|\mathbf{B} + \mathbf{C}| = \sqrt{(0)^2 + (B_y + 4.0)^2} = 5.0 \qquad \Longrightarrow \qquad (B_y + 4.0)^2 = 25.0 \ .$$

The last equation gives $(B_y + 4.0) = \pm 5.0$ and apparently there are *two* possible answers

$$B_y = +1.0 \qquad \text{and} \qquad B_y = -9.0$$

but the second case gives a resultant vector **B** + **C** which points in the *negative* y direction so we omit it. Then with $B_y = 1.0$ we find the magnitude of **B**:

$$B = \sqrt{(B_x)^2 + (B_y)^2} = \sqrt{(-3.0)^2 + (1.0)^2} = 3.2$$

The magnitude of vector **B** is 3.2.

1.2.5 Multiplying Vectors

---

25. Vector A extends from the origin to a point having polar coordinates $(7, 70^\circ)$ and vector B extends from the origin to a point having polar coordinates $(4, 130^\circ)$. Find $\mathbf{A} \cdot \mathbf{B}$. [Ser4 7-13]

We can use Eq. 1.7 to find $\mathbf{A} \cdot \mathbf{B}$. We have the magnitudes of the two vectors (namely $A = 7$ and $B = 4$) and the angle ϕ between the two is

$$\phi = 130^\circ - 70^\circ = 60^\circ \ .$$

Then we get:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\phi = (7)(4) \cos 60^\circ = 14$$

---

26. Find the angle between $\mathbf{A} = -5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = -2\mathbf{j} - 2\mathbf{k}$. [Ser4 7-20]

Eq. 1.7 allows us to find the cosine of the angle between two vectors as long as we know their magnitudes and their dot product. The magnitudes of the vectors **A** and **B** are:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{(-5)^2 + (-3)^2 + (2)^2} = 6.164$$

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(0)^2 + (-2)^2 + (-2)^2} = 2.828$$

and their dot product is:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z = (-5)(0) + (-3)(-2) + (2)(-2) = 2$$

Then from Eq. 1.7, if ϕ is the angle between **A** and **B**, we have

$$\cos\phi = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{2}{(6.164)(2.828)} = 0.114$$

which then gives

$$\phi = 83.4^\circ \ .$$

---

27. Two vectors a and b have the components, in arbitrary units, $a_x = 3.2$, $a_y = 1.6$, $b_x = 0.50$, $b_y = 4.5$. (a) Find the angle between the directions of a and b. (b) Find the components of a vector c that is perpendicular to a, is in the xy plane and has a magnitude of 5.0 units. [HRW5 3-51]

(a) The scalar product has something to do with the angle between two vectors... if the angle between **a** and **b** is ϕ then from Eq. 1.7 we have:

$$\cos\phi = \frac{\mathbf{a}\cdot\mathbf{b}}{ab} \ .$$

We can compute the right–hand–side of this equation since we know the components of **a** and **b**. First, find $\mathbf{a}\cdot\mathbf{b}$. Using Eq. 1.8 we find:

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= a_x b_x + a_y b_y \\ &= (3.2)(0.50) + (1.6)(4.5) \\ &= 8.8 \end{aligned}$$

Now find the magnitudes of **a** and **b**:

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(3.2)^2 + (1.6)^2} = 3.6$$

$$b = \sqrt{b_x^2 + b_y^2} = \sqrt{(0.50)^2 + (4.5)^2} = 4.5$$

This gives us:

$$\cos\phi = \frac{\mathbf{a}\cdot\mathbf{b}}{ab} = \frac{8.8}{(3.6)(4.5)} = 0.54$$

From which we get ϕ by:

$$\phi = \cos^{-1}(0.54) = 57^\circ$$

(b) Let the components of the vector **c** be c_x and c_y (we are told that it lies in the xy plane). If **c** is perpendicular to **a** then the dot product of the two vectors must give zero. This tells us:

$$\mathbf{a}\cdot\mathbf{c} = a_x c_x + a_y c_y = (3.2)c_x + (1.6)c_y = 0$$

This equation doesn't allow us to solve for the components of **c** but it does give us:

$$c_x = -\frac{1.6}{3.2}c_y = -0.50c_y$$

Since the vector **c** has magnitude 5.0, we know that

$$c = \sqrt{c_x^2 + c_y^2} = 5.0$$

Using the previous equation to substitute for c_x gives:

$$\begin{aligned} c &= \sqrt{c_x^2 + c_y^2} \\ &= \sqrt{(-0.50\,c_y)^2 + c_y^2} \\ &= \sqrt{1.25\,c_y^2} = 5.0 \end{aligned}$$

Squaring the last line gives

$$1.25c_y^2 = 25 \qquad \Longrightarrow \qquad c_y^2 = 20. \qquad \Longrightarrow \qquad c_y = \pm 4.5$$

One must be careful... there are *two* possible solutions for c_y here. If $c_y = 4.5$ then we have

$$c_x = -0.50\,c_y = (-0.50)(4.5) = -2.2$$

But if $c_y = -4.5$ then we have

$$c_x = -0.50\,c_y = (-0.50)(-4.5) = 2.2$$

So the two possibilities for the vector **c** are

$$c_x = -2.2 \qquad c_y = 4.5$$

and

$$c_x = 2.2 \qquad c_y = -4.5$$

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28. Two vectors are given by $\mathbf{A} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j}$. Find (a) $\mathbf{A} \times \mathbf{B}$ and (b) the angle between A and B. [Ser4 11-7]

(a) Setting up the determinant in Eq. 1.12 (or just using Eq. 1.11 for the cross product) we find:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 0 \\ 2 & 3 & 0 \end{vmatrix} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + ((-9)-(8))\mathbf{k} = -17\mathbf{k}$$

(b) To get the angle between **A** and **B** it is easiest to use the dot product and Eq. 1.7. The magnitudes of **A** and **B** are:

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{(-3)^2 + (4)^2} = 5 \qquad B = \sqrt{B_x^2 + B_y^2} = \sqrt{(2)^2 + (3)^2} = 3.61$$

and the dot product of the two vectors is

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z = (-3)(2) + (4)(3) = 6$$

so then if ϕ is the angle between $\mathbf{A}$ and $\mathbf{B}$ we get:

$$\cos\phi = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{6}{(5)(3.61)} = 0.333$$

which gives

$$\phi = 70.6^\circ \ .$$

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29. Prove that two vectors must have equal magnitudes if their sum is perpendicular to their difference. [HRW6 3-23]

Suppose the condition stated in this problem holds for the two vectors $\mathbf{a}$ and $\mathbf{b}$. If the sum $\mathbf{a}+\mathbf{b}$ is perpendicular to the difference $\mathbf{a}-\mathbf{b}$ then the dot product of these two vectors is zero:

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = 0$$

Use the distributive property of the dot product to expand the left side of this equation. We get:

$$\mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b}$$

But the dot product of a vector with itself gives the magnitude squared:

$$\mathbf{a}\cdot\mathbf{a} = a_x^2 + a_y^2 + a_z^2 = a^2$$

(likewise $\mathbf{b}\cdot\mathbf{b} = b^2$) and the dot product is commutative: $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$. Using these facts, we then have

$$a^2 - \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b} + b^2 = 0 \ ,$$

which gives:

$$a^2 - b^2 = 0 \qquad \Longrightarrow \qquad a^2 = b^2$$

Since the magnitude of a vector must be a positive number, this implies $a = b$ and so vectors $\mathbf{a}$ and $\mathbf{b}$ have the same magnitude.

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30. For the following three vectors, what is $3\mathbf{C}\cdot(2\mathbf{A}\times\mathbf{B})$?

$$\mathbf{A} = 2.00\mathbf{i} + 3.00\mathbf{j} - 4.00\mathbf{k}$$

$$\mathbf{B} = -3.00\mathbf{i} + 4.00\mathbf{j} + 2.00\mathbf{k} \qquad\qquad \mathbf{C} = 7.00\mathbf{i} - 8.00\mathbf{j}$$

[HRW6 3-36]

Actually, from the properties of scalar multiplication we can combine the factors in the desired vector product to give:

$$3\mathbf{C} \cdot (2\mathbf{A} \times \mathbf{B}) = 6\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \ .$$

Evaluate $\mathbf{A} \times \mathbf{B}$ first:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2.0 & 3.0 & -4.0 \\ -3.0 & 4.0 & 2.0 \end{vmatrix} = (6.0 + 16.0)\mathbf{i} + (12.0 - 4.0)\mathbf{j} + (8.0 + 9.0)\mathbf{k}$$

$$= 22.0\mathbf{i} + 8.0\mathbf{j} + 17.0\mathbf{k}$$

Then:

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = (7.0)(22.0) - (8.0)(8.0) + (0.0)(17.0) = 90$$

So the answer we want is:

$$6\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = (6)(90.0) = 540$$

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31. A student claims to have found a vector A such that

$$(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times \mathbf{A} = (4\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \ .$$

Do you believe this claim? Explain. [Ser4 11-8]

Frankly, I've been in this teaching business so long and I've grown so cynical that I don't believe *anything* any student claims anymore, and this case is no exception. But enough about *me*; let's see if we can provide a mathematical answer.

We might try to work out a solution for **A**, but let's think about some of the basic properties of the cross product. We know that the cross product of two vectors must be perpendicular to *each* of the "multiplied" vectors. So if the student is telling the truth, it must be true that $(4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ is perpendicular to $(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$. Is it?

We can test this by taking the dot product of the two vectors:

$$(4\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = (4)(2) + (3)(-3) + (-1)(4) = -5 \ .$$

The dot product does *not* give zero as it must if the two vectors are perpendicular. So we have a contradiction. There can't be *any* vector **A** for which the relation is true.