CHAPTER 3

Exponential, Logistic, and Logarithmic Functions

The loudness of a sound we hear is based on the intensity of the associated sound wave. This sound intensity is the energy per unit time of the wave over a given area, measured in watts per square meter (W/m^2). The intensity is greatest near the source and decreases as you move away, whether the sound is rustling leaves or rock music. Because of the wide range of audible sound intensities, they are generally converted into *decibels*, which are based on logarithms. See page 307.

BIBLIOGRAPHY

For students: *Beyond Numeracy*, John Allen Paulos. Alfred A. Knopf, 1991.

For teachers: *e: The Story of a Number,* Eli Maor. Princeton University Press, 1993.

Learning Mathematics for a New Century, 2000 Yearbook, Maurice J. Burke and Frances R. Curcio (Eds.), National Council of Teachers of Mathematics, 2000.

Chapter 3 Overview

In this chapter, we study three interrelated families of functions: exponential, logistic, and logarithmic functions. Polynomial functions, rational functions, and power functions with rational exponents are **algebraic functions**— functions obtained by adding, subtracting, multiplying, and dividing constants and an independent variable, and raising expressions to integer powers and extracting roots. In this chapter and the next one, we explore **transcendental functions**, which go beyond, or transcend, these algebraic operations.

Just like their algebraic cousins, exponential, logistic, and logarithmic functions have wide application. Exponential functions model growth and decay over time, such as *unrestricted* population growth and the decay of radioactive substances. Logistic functions model *restricted* population growth, certain chemical reactions, and the spread of rumors and diseases. Logarithmic functions are the basis of the Richter scale of earthquake intensity, the pH acidity scale, and the decibel measurement of sound.

The chapter closes with a study of the mathematics of finance, an application of exponential and logarithmic functions often used when making investments.

3.1 Exponential and Logistic Functions

What you'll learn about

- Exponential Functions and Their Graphs
- The Natural Base e
- Logistic Functions and Their Graphs
- Population Models

. . . and why

Exponential and logistic functions model many growth patterns, including the growth of human and animal populations.

Exponential Functions and Their Graphs

The functions $f(x) = x^2$ and $g(x) = 2^x$ each involve a base raised to a power, but the roles are reversed:

- For $f(x) = x^2$, the base is the variable x, and the exponent is the constant 2; f is a familiar monomial and power function.
- For $g(x) = 2^x$, the base is the constant 2, and the exponent is the variable x; g is an *exponential function*. See Figure 3.1.

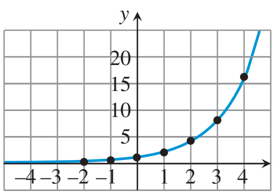


FIGURE 3.1 Sketch of $g(x) = 2^x$.

DEFINITION Exponential Functions

Let a and b be real number constants. An **exponential function** in x is a function that can be written in the form

$$f(x) = a \cdot b^x,$$

where a is nonzero, b is positive, and $b \neq 1$. The constant a is the *initial value* of f (the value at $x = 0$), and b is the **base**.

Exponential functions are defined and continuous for all real numbers. It is important to recognize whether a function is an exponential function.

OBJECTIVE

Students will be able to evaluate exponential expressions and identify and graph exponential and logistic functions.

MOTIVATE

Ask . . .
If the population of a town increases by 10% every year, what will a graph of the population function look like?

LESSON GUIDE

Day 1: Exponential Functions and Their Graphs; The Natural Base e
Day 2: Logistic Functions and Their Graphs; Population Models

EXAMPLE 1 Identifying Exponential Functions

(a) $f(x) = 3^x$ is an exponential function, with an initial value of 1 and base of 3.

(b) $g(x) = 6x^{-4}$ is *not* an exponential function because the base x is a variable and the exponent is a constant; g is a power function.

(c) $h(x) = -2 \cdot 1.5^x$ is an exponential function, with an initial value of -2 and base of 1.5.

(d) $k(x) = 7 \cdot 2^{-x}$ is an exponential function, with an initial value of 7 and base of 1/2 because $2^{-x} = (2^{-1})^x = (1/2)^x$.

(e) $q(x) = 5 \cdot 6^{\pi}$ is *not* an exponential function because the exponent π is a constant; q is a constant function. *Now try Exercise 1.*

One way to evaluate an exponential function, when the inputs are rational numbers, is to use the properties of exponents.

EXAMPLE 2 Computing Exponential Function Values for Rational Number Inputs

For $f(x) = 2^x$,

(a) $f(4) = 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

(b) $f(0) = 2^0 = 1$

(c) $f(-3) = 2^{-3} = \frac{1}{2^3} = \frac{1}{8} = 0.125$

(d) $f\left(\frac{1}{2}\right) = 2^{1/2} = \sqrt{2} = 1.4142\ldots$

(e) $f\left(-\frac{3}{2}\right) = 2^{-3/2} = \frac{1}{2^{3/2}} = \frac{1}{\sqrt{2^3}} = \frac{1}{\sqrt{8}} = 0.35355\ldots$ *Now try Exercise 7.*

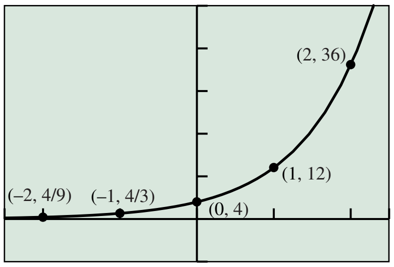
There is no way to use properties of exponents to express an exponential function's value for *irrational* inputs. For example, if $f(x) = 2^x$, $f(\pi) = 2^{\pi}$, but what does 2^{π} mean? Using properties of exponents, $2^3 = 2 \cdot 2 \cdot 2$, $2^{3.1} = 2^{31/10} = \sqrt[10]{2^{31}}$. So we can find meaning for 2^{π} by using successively closer *rational* approximations to π as shown in Table 3.1.

Table 3.1 Values of $f(x) = 2^x$ for Rational Numbers x Approaching $\pi = 3.14159265\ldots$

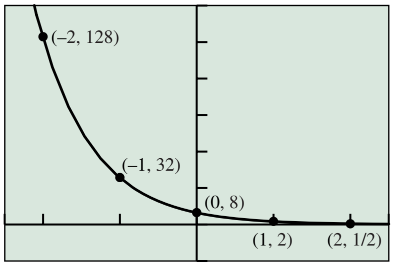
x	3	3.1	3.14	3.141	3.1415	3.14159
2^x	8	8.5. . .	8.81. . .	8.821. . .	8.8244. . .	8.82496. . .

We can conclude that $f(\pi) = 2^{\pi} \approx 8.82$, which could be found directly using a grapher. The methods of calculus permit a more rigorous definition of exponential functions than we give here, a definition that allows for both rational and irrational inputs.

The way exponential functions change makes them useful in applications. This pattern of change can best be observed in tabular form.



[–2.5, 2.5] by [–10, 50]
(a)



[–2.5, 2.5] by [–25, 150]
(b)

FIGURE 3.2 Graphs of (a) $g(x) = 4 \cdot 3^x$ and (b) $h(x) = 8 \cdot (1/4)^x$. (Example 3)

EXAMPLE 3 Finding an Exponential Function from its Table of Values

Determine formulas for the exponential functions g and h whose values are given in Table 3.2.

Table 3.2 Values for Two Exponential Functions

x	$g(x)$		$h(x)$	
-2	4/9	$\times 3$	128	$\times \frac{1}{4}$
-1	4/3	$\times 3$	32	$\times \frac{1}{4}$
0	4	$\times 3$	8	$\times \frac{1}{4}$
1	12	$\times 3$	2	$\times \frac{1}{4}$
2	36		1/2	

SOLUTION Because g is exponential, $g(x) = a \cdot b^x$. Because $g(0) = 4$, the initial value a is 4. Because $g(1) = 4 \cdot b^1 = 12$, the base b is 3. So,

$$g(x) = 4 \cdot 3^x.$$

Because h is exponential, $h(x) = a \cdot b^x$. Because $h(0) = 8$, the initial value a is 8. Because $h(1) = 8 \cdot b^1 = 2$, the base b is 1/4. So,

$$h(x) = 8 \cdot \left(\frac{1}{4}\right)^x.$$

Figure 3.2 shows the graphs of these functions pass through the points whose coordinates are given in Table 3.2. *Now try Exercise 11.*

Observe the patterns in the $g(x)$ and $h(x)$ columns of Table 3.2. The $g(x)$ values increase by a factor of 3 and the $h(x)$ values decrease by a factor of 1/4, as we add 1 to x moving from one row of the table to the next. In each case, the change factor is the base of the exponential function. This pattern generalizes to all exponential functions as illustrated in Table 3.3.

Table 3.3 Values for a General Exponential Function $f(x) = a \cdot b^x$

x	$a \cdot b^x$	
-2	ab^{-2}	$\times b$
-1	ab^{-1}	$\times b$
0	a	$\times b$
1	ab	$\times b$
2	ab^2	

In Table 3.3, as x increases by 1, the function value is multiplied by the base b. This relationship leads to the following *recursive formula*.

TEACHING NOTE

Recursive formulas tell us how to obtain a new function value from a known function value. The recursive formula for an exponential function shows its close relationship to a geometric sequence, as discussed in Chapter 9.

Exponential Growth and Decay

For any exponential function $f(x) = a \cdot b^x$ and any real number x,

$$f(x + 1) = b \cdot f(x).$$

If $a > 0$ and $b > 1$, the function f is increasing and is an **exponential growth function**. The base b is its **growth factor**.

If $a > 0$ and $b < 1$, f is decreasing and is an **exponential decay function**. The base b is its **decay factor**.

In Example 3, g is an exponential growth function, and h is an exponential decay function. As x increases by 1, $g(x) = 4 \cdot 3^x$ grows by a factor of 3, and $h(x) = 8 \cdot (1/4)^x$ decays by a factor of 1/4. The base of an exponential function, like the slope of a linear function, tells us whether the function is increasing or decreasing and by how much.

So far, we have focused most of our attention on the algebraic and numerical aspects of exponential functions. We now turn our attention to the graphs of these functions.

EXPLORATION EXTENSIONS

Graph (a) $y_1 = 2^x$, (b) $y_2 = 2^{-x}$, (c) $y_3 = -2^x$. Describe how y_2 compares to y_1 and how y_3 compares to y_1.

EXPLORATION 1 Graphs of Exponential Functions

1. Graph each function in the viewing window $[-2, 2]$ by $[-1, 6]$.

   (a) $y_1 = 2^x$ **(b)** $y_2 = 3^x$ **(c)** $y_3 = 4^x$ **(d)** $y_4 = 5^x$

   - Which point is common to all four graphs?
   - Analyze the functions for domain, range, continuity, increasing or decreasing behavior, symmetry, boundedness, extrema, asymptotes, and end behavior.

2. Graph each function in the viewing window $[-2, 2]$ by $[-1, 6]$.

   (a) $y_1 = \left(\frac{1}{2}\right)^x$ **(b)** $y_2 = \left(\frac{1}{3}\right)^x$

   (c) $y_3 = \left(\frac{1}{4}\right)^x$ **(d)** $y_4 = \left(\frac{1}{5}\right)^x$

   - Which point is common to all four graphs?
   - Analyze the functions for domain, range, continuity, increasing or decreasing behavior, symmetry, boundedness, extrema, asymptotes, and end behavior.

We summarize what we have learned about exponential functions with an initial value of 1.

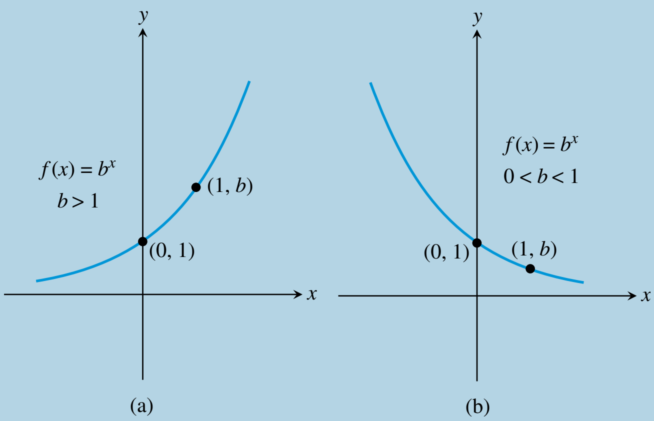


FIGURE 3.3 Graphs of $f(x) = b^x$ for (a) $b > 1$ and (b) $0 < b < 1$.

Exponential Functions $f(x) = b^x$

Domain: All reals
Range: $(0, \infty)$
Continuous
No symmetry: neither even nor odd
Bounded below, but not above
No local extrema
Horizontal asymptote: $y = 0$
No vertical asymptotes

If $b > 1$ (see Figure 3.3a), then

- f is an increasing function,
- $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = \infty$.

If $0 < b < 1$ (see Figure 3.3b), then

- f is a decreasing function,
- $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = 0$.

The translations, reflections, stretches, and shrinks studied in Section 1.5 together with our knowledge of the graphs of basic exponential functions allow us to predict the graphs of the functions in Example 4.

EXAMPLE 4 Transforming Exponential Functions

Describe how to transform the graph of $f(x) = 2^x$ into the graph of the given function. Sketch the graphs by hand and support your answer with a grapher.

(a) $g(x) = 2^{x-1}$ **(b)** $h(x) = 2^{-x}$ **(c)** $k(x) = 3 \cdot 2^x$

SOLUTION

(a) The graph of $g(x) = 2^{x-1}$ is obtained by translating the graph of $f(x) = 2^x$ by 1 unit to the right (Figure 3.4a).

(b) We can obtain the graph of $h(x) = 2^{-x}$ by reflecting the graph of $f(x) = 2^x$ across the y-axis (Figure 3.4b). Because $2^{-x} = (2^{-1})^x = (1/2)^x$, we can also think of h as an exponential function with an initial value of 1 and a base of 1/2.

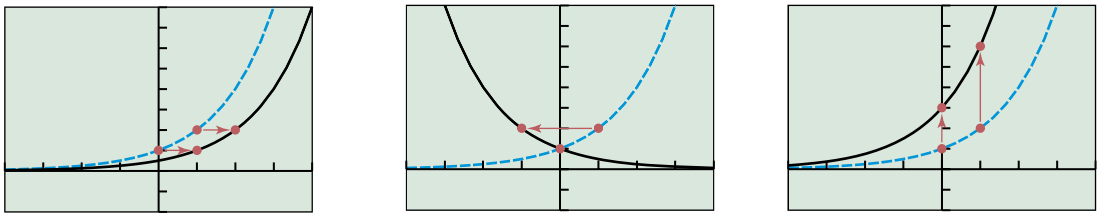
(c) We can obtain the graph of $k(x) = 3 \cdot 2^x$ by vertically stretching the graph of $f(x) = 2^x$ by a factor of 3 (Figure 3.4c). *Now try Exercise 15.*

TEACHING NOTE

Exponential and logistic functions often require a large range of y-values in order to show a "global" view of their graphs.

ALERT

Some students may have trouble entering exponents on a grapher. Many of the exponents are more complicated than the students have encountered in previous mathematics courses. Careful attention must be given to the syntax and placement of parentheses.



[–4, 4] by [–2, 8]
(a)

[–4, 4] by [–2, 8]
(b)

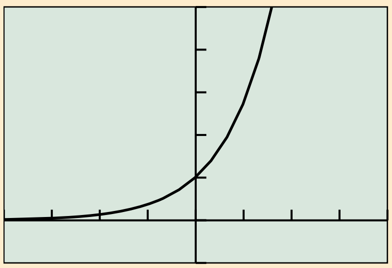
[–4, 4] by [–2, 8]
(c)

FIGURE 3.4 The graph of $f(x) = 2^x$ shown with (a) $g(x) = 2^{x-1}$, (b) $h(x) = 2^{-x}$, and (c) $k(x) = 3 \cdot 2^x$. (Example 4)

The Natural Base *e*

The function $f(x) = e^x$ is one of the basic functions introduced in Section 1.3, and is an exponential growth function.

BASIC FUNCTION The Exponential Function



[–4, 4] by [–1, 5]

FIGURE 3.5 The graph of $f(x) = e^x$.

$f(x) = e^x$
Domain: All reals
Range: $(0, \infty)$
Continuous
Increasing for all x
No symmetry
Bounded below, but not above
No local extrema
Horizontal asymptote: $y = 0$
No vertical asymptotes
End behavior: $\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to \infty} e^x = \infty$

Because $f(x) = e^x$ is increasing, it is an exponential growth function, so $e > 1$. But what is e, and what makes this exponential function *the* exponential function?

The letter e is the initial of the last name of Leonhard Euler (1707–1783), who introduced the notation. Because $f(x) = e^x$ has special calculus properties that simplify many calculations, e is the *natural base* of exponential functions for calculus purposes, and $f(x) = e^x$ is considered the *natural exponential function*.

DEFINITION The Natural Base *e*

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

We cannot compute the irrational number e directly, but using this definition we can obtain successively closer approximations to e, as shown in Table 3.4. Continuing the process in Table 3.4 with a sufficiently accurate computer, it can be shown that $e \approx 2.718281828459$.

Table 3.4 Approximations Approaching the Natural Base *e*

x	1	10	100	1000	10,000	100,000
$(1 + 1/x)^x$	2	2.5. . .	2.70. . .	2.716. . .	2.7181. . .	2.71826. . .

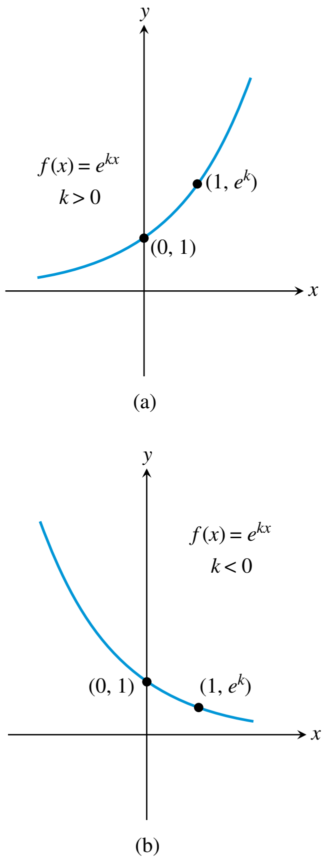


FIGURE 3.6 Graphs of $f(x) = e^{kx}$ for (a) $k > 0$ and (b) $k < 0$.

We are usually more interested in the exponential function $f(x) = e^x$ and variations of this function than in the irrational number e. In fact, *any* exponential function can be expressed in terms of the natural base e.

THEOREM Exponential Functions and the Base *e*

Any exponential function $f(x) = a \cdot b^x$ can be rewritten as

$$f(x) = a \cdot e^{kx},$$

for an appropriately chosen real number constant k.

If $a > 0$ and $k > 0$, $f(x) = a \cdot e^{kx}$ is an exponential growth function. (See Figure 3.6a.)

If $a > 0$ and $k < 0$, $f(x) = a \cdot e^{kx}$ is an exponential decay function. (See Figure 3.6b.)

In Section 3.3 we will develop some mathematics so that, for any positive number $b \neq 1$, we can easily find the value of k such that $e^{kx} = b^x$. In the meantime, we can use graphical and numerical methods to approximate k, as you will discover in Exploration 2.

EXPLORATION 2 Choosing *k* so that $e^{kx} = 2^x$

1. Graph $f(x) = 2^x$ in the viewing window $[-4, 4]$ by $[-2, 8]$.
2. One at a time, overlay the graphs of $g(x) = e^{kx}$ for $k = 0.4, 0.5, 0.6, 0.7$, and 0.8. For which of these values of k does the graph of g most closely match the graph of f? $k = 0.7$
3. Using tables, find the 3-decimal-place value of k for which the values of g most closely approximate the values of f. $k \approx 0.693$

EXPLORATION EXTENSION

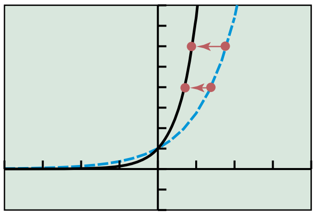
Calculate the values of $e^{0.4}$, $e^{0.5}$, $e^{0.6}$, $e^{0.7}$, and $e^{0.8}$ and discuss how these values relate to the results from Steps 2 and 3.

EXAMPLE 5 Transforming Exponential Functions

Describe how to transform the graph of $f(x) = e^x$ into the graph of the given function. Sketch the graphs by hand and support your answer with a grapher.

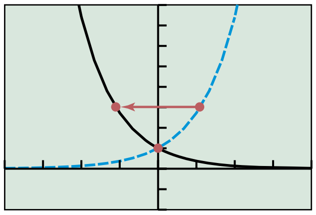
(a) $g(x) = e^{2x}$ **(b)** $h(x) = e^{-x}$ **(c)** $k(x) = 3e^x$

continued



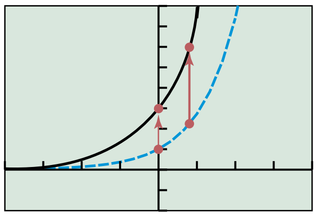
[–4, 4] by [–2, 8]

(a)



[–4, 4] by [–2, 8]

(b)



[–4, 4] by [–2, 8]

(c)

FIGURE 3.7 The graph of $f(x) = e^x$ shown with (a) $g(x) = e^{2x}$, (b) $h(x) = e^{-x}$, and (c) $k(x) = 3e^x$. (Example 5)

ALIASES FOR LOGISTIC GROWTH

Logistic growth is also known as *restricted, inhibited,* or *constrained exponential growth.*

SOLUTION

(a) The graph of $g(x) = e^{2x}$ is obtained by horizontally shrinking the graph of $f(x) = e^x$ by a factor of 2 (Figure 3.7a).

(b) We can obtain the graph of $h(x) = e^{-x}$ by reflecting the graph of $f(x) = e^x$ across the y-axis (Figure 3.7b).

(c) We can obtain the graph of $k(x) = 3e^x$ by vertically stretching the graph of $f(x) = e^x$ by a factor of 3 (Figure 3.7c). *Now try Exercise 21.*

Logistic Functions and Their Graphs

Exponential growth is *unrestricted*. An exponential growth function increases at an ever increasing rate and is not bounded above. In many growth situations, however, there is a limit to the possible growth. A plant can only grow so tall. The number of goldfish in an aquarium is limited by the size of the aquarium. In such situations the growth often begins in an exponential manner, but the growth eventually slows and the graph levels out. The associated growth function is bounded both below and above by horizontal asymptotes.

DEFINITION Logistic Growth Functions

Let a, b, c, and k be positive constants, with $b < 1$. A **logistic growth function** in x is a function that can be written in the form

$$f(x) = \frac{c}{1 + a \cdot b^x} \text{ or } f(x) = \frac{c}{1 + a \cdot e^{-kx}}$$

where the constant c is the **limit to growth**.

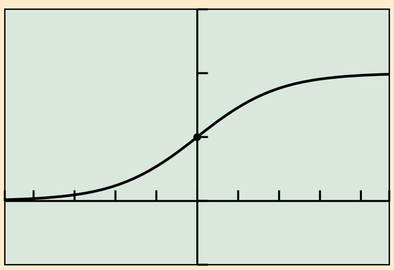
If $b > 1$ or $k < 0$, these formulas yield **logistic decay functions**. Unless otherwise stated, all *logistic functions* in this book will be logistic growth functions.

By setting $a = c = k = 1$, we obtain the **logistic function**

$$f(x) = \frac{1}{1 + e^{-x}}.$$

This function, though related to the exponential function e^x, *cannot* be obtained from e^x by translations, reflections, and horizontal and vertical stretches and shrinks. So we give the logistic function a formal introduction:

BASIC FUNCTION The Logistic Function



[–4.7, 4.7] by [–0.5, 1.5]

FIGURE 3.8 The graph of $f(x) = 1/(1 + e^{-x})$.

$$f(x) = \frac{1}{1 + e^{-x}}$$

Domain: All reals

Range: (0, 1)

Continuous

Increasing for all x

Symmetric about (0, 1/2), but neither even nor odd

Bounded below and above

No local extrema

Horizontal asymptotes: $y = 0$ and $y = 1$

No vertical asymptotes

End behavior: $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 1$

All logistic growth functions have graphs much like the basic logistic function. Their end behavior is always described by the equations

$$\lim_{x\to-\infty} f(x) = 0 \text{ and } \lim_{x\to\infty} f(x) = c,$$

where c is the limit to growth (see Exercise 73). All logistic functions are bounded by their horizontal asymptotes, $y = 0$ and $y = c$, and have a range of $(0, c)$. Although every logistic function is symmetric about the point of its graph with y-coordinate $c/2$, this point of symmetry is usually not the y-intercept, as we can see in Example 6.

EXAMPLE 6 Graphing Logistic Growth Functions

Graph the function. Find the y-intercept and the horizontal asymptotes.

(a) $f(x) = \dfrac{8}{1 + 3 \cdot 0.7^x}$ **(b)** $g(x) = \dfrac{20}{1 + 2e^{-3x}}$

SOLUTION

(a) The graph of $f(x) = 8/(1 + 3 \cdot 0.7^x)$ is shown in Figure 3.9a. The y-intercept is

$$f(0) = \frac{8}{1 + 3 \cdot 0.7^0} = \frac{8}{1 + 3} = 2.$$

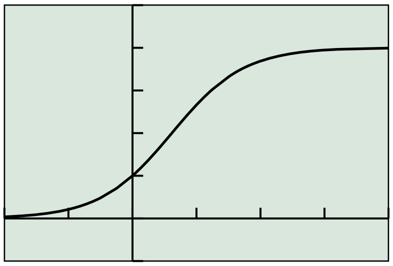
Because the limit to growth is 8, the horizontal asymptotes are $y = 0$ and $y = 8$.

(b) The graph of $g(x) = 20/(1 + 2e^{-3x})$ is shown in Figure 3.9b. The y-intercept is

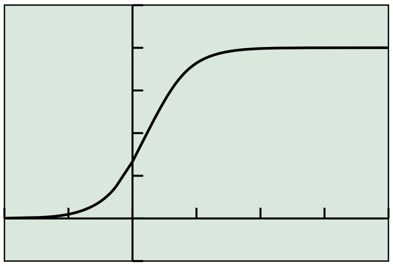
$$g(0) = \frac{20}{1 + 2e^{-3 \cdot 0}} = \frac{20}{1 + 2} = 20/3 \approx 6.67.$$

Because the limit to growth is 20, the horizontal asymptotes are $y = 0$ and $y = 20$.

Now try Exercise 41.



[–10, 20] by [–2, 10]
(a)



[–2, 4] by [–5, 25]
(b)

FIGURE 3.9 The graphs of (a) $f(x) = 8/(1 + 3 \cdot 0.7^x)$ and (b) $g(x) = 20/(1 + 2e^{-3x})$. (Example 6)

Population Models

Exponential and logistic functions have many applications. One area where both types of functions are used is in modeling population. Between 1990 and 2000, both Phoenix and San Antonio passed the 1 million mark. With its Silicon Valley industries, San Jose,

FOLLOW-UP

Ask . . .
If $a > 0$, how can you tell whether $y = a \cdot b^x$ represents an increasing or decreasing function? **(The function is increasing if $b > 1$ and decreasing if $0 < b < 1$.)**

ASSIGNMENT GUIDE

Day 1: Ex. 1–13 odd, 15–39, multiples of 3, 45, 48
Day 2: Ex. 41, 44, 49, 52, 53, 55, 65, 67, 68, 70, 71

COOPERATIVE LEARNING

Group Activity: Ex. 39, 40

NOTES ON EXERCISES

Ex. 15–30 encourage students to think about the appearance of functions without using a grapher.
Ex. 59–64 provide practice for standardized tests.
Ex. 69–72 require students to think about the meaning of different kinds of functions.

ONGOING ASSESSMENT

Self-Assessment: Ex. 1, 7, 11, 15, 21, 41, 51, 55
Embedded Assessment: Ex. 53, 68

NOTES ON EXAMPLES

The base year chosen in Example 7 (1990) and Example 8 (1900) are arbitrary and chosen for convenience. Examples 7 and 8 set the stage for the regression modeling in Section 3.2.

We have chosen to use graphical solutions here because the algebraic methods needed to solve the exponential equations are presented in Sections 3.5 and 3.6.

California appears to be the next U.S. city destined to surpass 1 million residents. When a city's population is growing rapidly, as in the case of San Jose, exponential growth is a reasonable model.

EXAMPLE 7 Modeling San Jose's Population

Using the data in Table 3.5 and assuming the growth is exponential, when will the population of San Jose surpass 1 million persons?

SOLUTION

Model Let $P(t)$ be the population of San Jose t years after 1990. Because P is exponential, $P(t) = P_0 \cdot b^t$, where P_0 is the initial (1990) population of 782,248. From Table 3.5 we see that $P(10) = 782{,}248b^{10} = 895{,}193$. So,

$$b = \sqrt[10]{\frac{895{,}193}{782{,}248}} \approx 1.0136$$

and $P(t) = 782{,}248 \cdot 1.0136^t$.

Solve Graphically Figure 3.10 shows that this population model intersects $y = 1{,}000{,}000$ when the independent variable is about 18.18.

Interpret Because $1990 + 18 = 2008$, if the growth of its population is exponential, San Jose would surpass the 1 million mark in 2008. *Now try Exercise 51.*

While San Jose's population is soaring, in other major cities, such as Dallas, the population growth is slowing. The once sprawling Dallas is now *constrained* by its neighboring cities. *A logistic function is often an appropriate model for restricted growth*, such as the growth that Dallas is experiencing.

EXAMPLE 8 Modeling Dallas's Population

Based on recent census data, a logistic model for the population of Dallas, t years after 1900, is as follows:

$$P(t) = \frac{1{,}301{,}642}{1 + 21.602e^{-0.05054t}}$$

According to this model, when was the population 1 million?

SOLUTION

Figure 3.11 shows that the population model intersects $y = 1{,}000{,}000$ when the independent variable is about 84.51. Because $1900 + 85 = 1985$, if Dallas's population has followed this logistic model, its population was 1 million at the beginning of 1985. *Now try Exercise 55.*

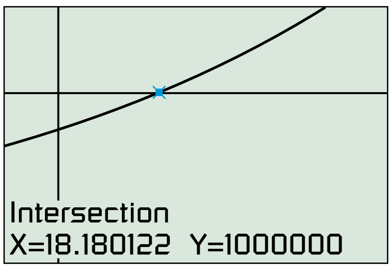
A NOTE ON POPULATION DATA

When the U.S. Census Bureau reports a population for a given year, it generally represents the population at the middle of the year, or July 1. We will assume this to be the case when interpreting our results to population problems.

Table 3.5 The Population of San Jose, California

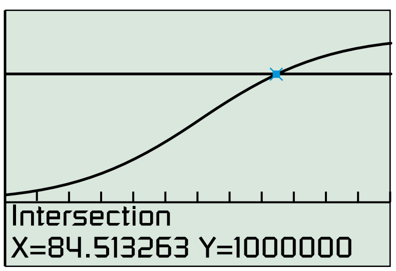
Year	Population
1990	782,248
2000	895,193

Source: World Almanac and Book of Facts 2005.



[–10, 60] by [0, 1 500 000]

FIGURE 3.10 A population model for San Jose, California. (Example 7)



[0, 120] by [–500 000, 1 500 000]

FIGURE 3.11 A population model for Dallas, Texas. (Example 8)

QUICK REVIEW 3.1 *(For help, go to Sections A.1 and P.1.)*

In Exercises 1–4, evaluate the expression without using a calculator.

1. $\sqrt[3]{-216}$ -6
2. $\sqrt[3]{\dfrac{125}{8}}$ $\dfrac{5}{2} = 2.5$
3. $27^{2/3}$ 9
4. $4^{5/2}$ 32

In Exercises 5–8, rewrite the expression using a single positive exponent.

5. $(2^{-3})^4$ $1/2^{12}$
6. $(3^4)^{-2}$ $1/3^8$
7. $(a^{-2})^3$ $1/a^6$
8. $(b^{-3})^{-5}$ b^{15}

In Exercises 9–10, use a calculator to evaluate the expression.

9. $\sqrt[5]{-5.37824}$ -1.4
10. $\sqrt[4]{92.3521}$ 3.1

SECTION 3.1 EXERCISES

In Exercises 1–6, which of the following are exponential functions? For those that are exponential functions, state the initial value and the base. For those that are not, explain why not.

1. $y = x^8$ Not exponential, a monomial function
2. $y = 3^x$ Exponential function, initial value of 1 and base of 3
3. $y = 5^x$ Exponential function initial value of 1 and base of 5
4. $y = 4^2$ Not exponential, a constant function
5. $y = x^{\sqrt{x}}$ Not exponential, variable base
6. $y = x^{1.3}$ Not exponential, a power function

In Exercises 7–10, compute the exact value of the function for the given x-value without using a calculator.

7. $f(x) = 3 \cdot 5^x$ for $x = 0$ 3
8. $f(x) = 6 \cdot 3^x$ for $x = -2$ $2/3$
9. $f(x) = -2 \cdot 3^x$ for $x = 1/3$ $-2\sqrt[3]{3}$
10. $f(x) = 8 \cdot 4^x$ for $x = -3/2$ 1

In Exercises 11 and 12, determine a formula for the exponential function whose values are given in Table 3.6.

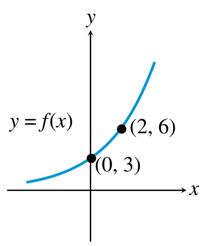
11. $f(x)$ $3/2 \cdot (1/2)^x$
12. $g(x)$ $12(1/3)^x$

Table 3.6 Values for Two Exponential Functions

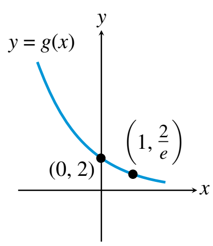
x	$f(x)$	$g(x)$
−2	6	108
−1	3	36
0	3/2	12
1	3/4	4
2	3/8	4/3

In Exercises 13 and 14, determine a formula for the exponential function whose graph is shown in the figure.

13. $f(x)$ $3 \cdot 2^{x/2}$



14. $g(x)$ $2e^{-x}$



In Exercises 15–24, describe how to transform the graph of f into the graph of g. Sketch the graphs by hand and support your answer with a grapher.

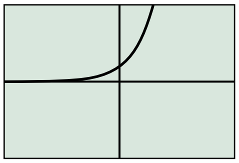
15. $f(x) = 2^x$, $g(x) = 2^{x-3}$ Translate $f(x) = 2^x$ by 3 units to the right.
16. $f(x) = 3^x$, $g(x) = 3^{x+4}$ Translate $f(x) = 3^x$ by 4 units to the left.
17. $f(x) = 4^x$, $g(x) = 4^{-x}$ Reflect $f(x) = 4^x$ over the y-axis.
18. $f(x) = 2^x$, $g(x) = 2^{5-x}$
19. $f(x) = 0.5^x$, $g(x) = 3 \cdot 0.5^x + 4$
20. $f(x) = 0.6^x$, $g(x) = 2 \cdot 0.6^{3x}$
21. $f(x) = e^x$, $g(x) = e^{-2x}$
22. $f(x) = e^x$, $g(x) = -e^{-3x}$
23. $f(x) = e^x$, $g(x) = 2e^{3-3x}$
24. $f(x) = e^x$, $g(x) = 3e^{2x} - 1$

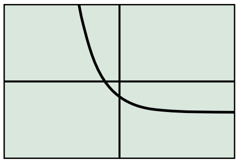
In Exercises 25–30, **(a)** match the given function with its graph. **(b) Writing to Learn** Explain how to make the choice without using a grapher.

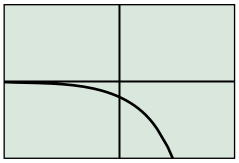
25. $y = 3^x$
26. $y = 2^{-x}$ Graph (d) is the reflection of $y = 2^x$ across the y-axis.
27. $y = -2^x$ Graph (c) is the reflection of $y = 2^x$ across the x-axis.
28. $y = -0.5^x$ Graph (e) is the reflection of $y = 0.5^x$ across the x-axis.

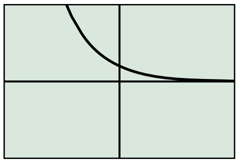
29. $y = 3^{-x} - 2$

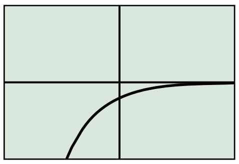
30. $y = 1.5^{x} - 2$

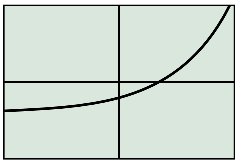

(a)


(b)


(c)


(d)


(e)


(f)

In Exercises 31–34, state whether the function is an exponential growth function or exponential decay function, and describe its end behavior using limits.

31. $f(x) = 3^{-2x}$

32. $f(x) = \left(\frac{1}{e}\right)^{x}$ Exponential decay; $\lim_{x \to \infty} f(x) = 0$, $\lim_{x \to -\infty} f(x) = \infty$

33. $f(x) = 0.5^{x}$ Exponential decay; $\lim_{x \to \infty} f(x) = 0$, $\lim_{x \to -\infty} f(x) = \infty$

34. $f(x) = 0.75^{-x}$ Exponential growth; $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = 0$

In Exercises 35–38, solve the inequality graphically.

35. $9^{x} < 4^{x}$ $x < 0$

36. $6^{-x} > 8^{-x}$ $x > 0$

37. $\left(\frac{1}{4}\right)^{x} > \left(\frac{1}{3}\right)^{x}$ $x < 0$

38. $\left(\frac{1}{3}\right)^{x} < \left(\frac{1}{2}\right)^{x}$ $x > 0$

Group Activity In Exercises 39 and 40, use the properties of exponents to prove that two of the given three exponential functions are identical. Support graphically.

39. (a) $y_1 = 3^{2x+4}$

(b) $y_2 = 3^{2x} + 4$

(c) $y_3 = 9^{x+2}$

$y_1 = y_3$ since $3^{2x+4} = 3^{2(x+2)} = (3^2)^{x+2} = 9^{x+2}$

40. (a) $y_1 = 4^{3x-2}$

(b) $y_2 = 2(2^{3x-2})$

(c) $y_3 = 2^{3x-1}$

In Exercises 41–44, use a grapher to graph the function. Find the y-intercept and the horizontal asymptotes.

41. $f(x) = \dfrac{12}{1 + 2 \cdot 0.8^{x}}$

42. $f(x) = \dfrac{18}{1 + 5 \cdot 0.2^{x}}$

43. $f(x) = \dfrac{16}{1 + 3e^{-2x}}$

44. $g(x) = \dfrac{9}{1 + 2e^{-x}}$

In Exercises 45–50, graph the function and analyze it for domain, range, continuity, increasing or decreasing behavior, symmetry, boundedness, extrema, asymptotes, and end behavior.

45. $f(x) = 3 \cdot 2^{x}$

46. $f(x) = 4 \cdot 0.5^{x}$

47. $f(x) = 4 \cdot e^{3x}$

48. $f(x) = 5 \cdot e^{-x}$

49. $f(x) = \dfrac{5}{1 + 4 \cdot e^{-2x}}$

50. $f(x) = \dfrac{6}{1 + 2 \cdot e^{-x}}$

51. Population Growth Using the data in Table 3.7 and assuming the growth is exponential, when would the population of Austin surpass 800,000 persons? In 2006

52. Population Growth Using the data in Table 3.7 and assuming the growth is exponential, when would the population of Columbus surpass 800,000 persons? In 2010

53. Population Growth Using the data in Table 3.7 and assuming the growth is exponential, when would the populations of Austin and Columbus be equal? Near the end of 2003

54. Population Growth Using the data in Table 3.7 and assuming the growth is exponential, which city—Austin or Columbus—would reach a population of 1 million first, and in what year? Austin, 2012

Table 3.7 Populations of Two Major U.S. Cities

City	1990 Population	2000 Population
Austin, Texas	465,622	656,562
Columbus, Ohio	632,910	711,265

Source: World Almanac and Book of Facts 2005.

55. Population Growth Using 20th-century U.S. census data, the population of Ohio can be modeled by $P(t) = 12.79/(1 + 2.402e^{-0.0309x})$, where P is the population in millions and t is the number of years since 1900. Based on this model, when was the population of Ohio 10 million? In 1970

56. Population Growth Using 20th century U.S. census data, the population of New York state can be modeled by

$$P(t) = \frac{19.875}{1 + 57.993e^{-0.035005t}},$$

where P is the population in millions and t is the number of years since 1800. Based on this model,

(a) What was the population of New York in 1850? 1,794,558

(b) What will New York state's population be in 2010? 19,161,673

(c) What is New York's *maximum sustainable population* (limit to growth)? 19,875,000

57. Bacteria Growth The number B of bacteria in a petri dish culture after t hours is given by

$$B = 100e^{0.693t}.$$

(a) What was the initial number of bacteria present? 100

(b) How many bacteria are present after 6 hours? ≈ 6394

58. Carbon Dating The amount C in grams of carbon-14 present in a certain substance after t years is given by

$$C = 20e^{-0.0001216t}.$$

(a) What was the initial amount of carbon-14 present? 20 g

(b) How much is left after 10,400 years? When will the amount left be 10 g? ≈ 5.647 g; after about 5700.22 yr

Standardized Test Questions

59. True or False Every exponential function is strictly increasing. Justify your answer.

60. True or False Every logistic growth function has two horizontal asymptotes. Justify your answer.

In Exercises 61–64, solve the problem without using a calculator.

61. Multiple Choice Which of the following functions is exponential? E

(A) $f(x) = a^2$

(B) $f(x) = x^3$

(C) $f(x) = x^{2/3}$

(D) $f(x) = \sqrt[3]{x}$

(E) $f(x) = 8^x$

62. Multiple Choice What point do all functions of the form $f(x) = b^x$ $(b > 0)$ have in common? C

(A) (1, 1)

(B) (1, 0)

(C) (0, 1)

(D) (0, 0)

(E) $(-1, -1)$

63. Multiple Choice The growth factor for $f(x) = 4 \cdot 3^x$ is A

(A) 3. **(B)** 4. **(C)** 12.

(D) 64. **(E)** 81.

64. Multiple Choice For $x > 0$, which of the following is **true?** B

(A) $3^x > 4^x$ **(B)** $7^x > 5^x$ **(C)** $(1/6)^x > (1/2)^x$

(D) $9^{-x} > 8^{-x}$ **(E)** $0.17^x > 0.32^x$

Explorations

65. Graph each function and analyze it for domain, range, increasing or decreasing behavior, boundedness, extrema, asymptotes, and end behavior.

(a) $f(x) = x \cdot e^x$ **(b)** $g(x) = \dfrac{e^{-x}}{x}$

66. Use the properties of exponents to solve each equation. Support graphically.

(a) $2^x = 4^2$ $x = 4$ **(b)** $3^x = 27$ $x = 3$

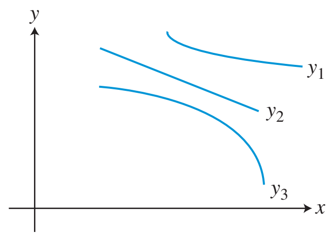
(c) $8^{x/2} = 4^{x+1}$ $x = -4$ **(d)** $9^x = 3^{x+1}$ $x = 1$

Extending the Ideas

67. Writing to Learn Table 3.8 gives function values for $y = f(x)$ and $y = g(x)$. Also, three different graphs are shown.

Table 3.8 Data for Two Functions

x	$f(x)$	$g(x)$
1.0	5.50	7.40
1.5	5.35	6.97
2.0	5.25	6.44
2.5	5.17	5.76
3.0	5.13	4.90
3.5	5.09	3.82
4.0	5.06	2.44
4.5	5.05	0.71



(a) Which curve of those shown in the graph most closely resembles the graph of $y = f(x)$? Explain your choice.

(b) Which curve most closely resembles the graph of $y = g(x)$? Explain your choice.

68. Writing to Learn Let $f(x) = 2^x$. Explain why the graph of $f(ax + b)$ can be obtained by applying one transformation to the graph of $y = c^x$ for an appropriate value of c. What is c?

Exercises 69–72 refer to the expression $f(a, b, c) = a \cdot b^c$. If $a = 2$, $b = 3$, and $c = x$, the expression is $f(2, 3, x) = 2 \cdot 3^x$, an exponential function.

69. If $b = x$, state conditions on a and c under which the expression $f(a, b, c)$ is a quadratic power function. $a \neq 0, c = 2$

70. If $b = x$, state conditions on a and c under which the expression $f(a, b, c)$ is a decreasing linear function. $a < 0, c = 1$

71. If $c = x$, state conditions on a and b under which the expression $f(a, b, c)$ is an increasing exponential function.

72. If $c = x$, state conditions on a and b under which the expression $f(a, b, c)$ is a decreasing exponential function.

73. Prove that $\lim_{x\to-\infty} \frac{c}{1 + a \cdot b^x} = 0$ and $\lim_{x\to\infty} \frac{c}{1 + a \cdot b^x} = c$, for constants a, b, and c, with $a > 0$, $0 < b < 1$, and $c > 0$.

68. $c = 2^a$: to the graph of $(2^a)^x$ apply a vertical stretch by 2^b because $f(ax + b) = 2^{ax+b} = 2^{ax}2^b = (2^b)(2^a)^x$.

71. $a > 0$ and $b > 1$, or $a < 0$ and $0 < b < 1$

72. $a > 0$ and $0 < b < 1$, or $a < 0$ and $b > 1$

73. As $x \to -\infty$, $b^x \to \infty$, so $1 + a \cdot b^x \to \infty$ and $\frac{c}{1 + a \cdot b^x} \to 0$;

As $x \to \infty$, $b^x \to 0$, so $1 + a \cdot b^x \to 1$ and $\frac{c}{1 + a \cdot b^x} \to c$

3.2 Exponential and Logistic Modeling

What you'll learn about

- Constant Percentage Rate and Exponential Functions
- Exponential Growth and Decay Models
- Using Regression to Model Population
- Other Logistic Models

. . . and why

Exponential functions model many types of unrestricted growth; logistic functions model restricted growth, including the spread of disease and the spread of rumors.

OBJECTIVE

Students will be able to use exponential growth, decay, and regression to model real-life problems.

MOTIVATE

Ask. . .
If a culture of 100 bacteria is put into a petri dish and the culture doubles every hour, how long will it take to reach 400,000? Is there a limit to growth?

LESSON GUIDE

Day 1: Constant Percentage Rate and Exponential Functions; Exponential Growth and Decay Models
Day 2: Using Regression to Model Population; Other Logisitic Models

Constant Percentage Rate and Exponential Functions

Suppose that a population is changing at a **constant percentage rate** r, where r is the percent rate of change expressed in decimal form. Then the population follows the pattern shown.

Time in years	Population
0	$P(0) = P_0 =$ initial population
1	$P(1) = P_0 + P_0r = P_0(1 + r)$
2	$P(2) = P(1) \cdot (1 + r) = P_0(1 + r)^2$
3	$P(3) = P(2) \cdot (1 + r) = P_0(1 + r)^3$
⋮	⋮
t	$P(t) = P_0(1 + r)^t$

So, in this case, the population is an exponential function of time.

Exponential Population Model

If a population P is changing at a constant percentage rate r each year, then

$$P(t) = P_0(1 + r)^t,$$

where P_0 is the initial population, r is expressed as a decimal, and t is time in years.

If $r > 0$, then $P(t)$ is an exponential growth function, and its *growth factor* is the base of the exponential function, $1 + r$.

On the other hand, if $r < 0$, the base $1 + r < 1$, $P(t)$ is an exponential decay function, and $1 + r$ is the *decay factor* for the population.

EXAMPLE 1 Finding Growth and Decay Rates

Tell whether the population model is an exponential growth function or exponential decay function, and find the constant percentage rate of growth or decay.

(a) San Jose: $P(t) = 782{,}248 \cdot 1.0136^t$

(b) Detroit: $P(t) = 1{,}203{,}368 \cdot 0.9858^t$

SOLUTION

(a) Because $1 + r = 1.0136$, $r = 0.0136 > 0$. So, P is an exponential growth function with a growth rate of 1.36%.

(b) Because $1 + r = 0.9858$, $r = -0.0142 < 0$. So, P is an exponential decay function with a decay rate of 1.42%.

Now try Exercise 1.

EXAMPLE 2 Finding an Exponential Function

Determine the exponential function with initial value = 12, increasing at a rate of 8% per year.

SOLUTION

Because $P_0 = 12$ and $r = 8\% = 0.08$, the function is $P(t) = 12(1 + 0.08)^t$ or $P(t) = 12 \cdot 1.08^t$. We could write this as $f(x) = 12 \cdot 1.08^x$, where x represents time.

Now try Exercise 7.

Exponential Growth and Decay Models

Exponential growth and decay models are used for populations of animals, bacteria, and even radioactive atoms. Exponential growth and decay apply to any situation where the growth is proportional to the current size of the quantity of interest. Such situations are frequently encountered in biology, chemistry, business, and the social sciences.

Exponential growth models can be developed in terms of the time it takes a quantity to double. On the flip side, exponential decay models can be developed in terms of the time it takes for a quantity to be halved. Examples 3–5 use these strategies.

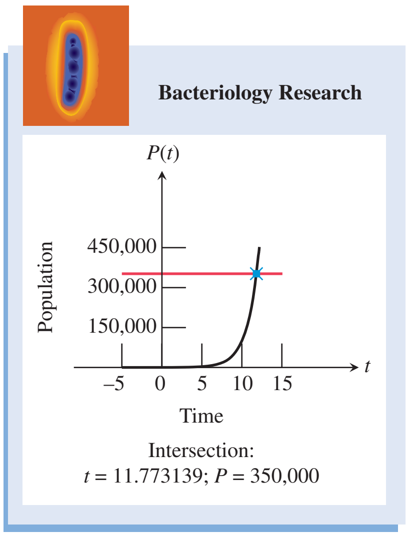


FIGURE 3.12 Rapid growth of a bacteria population. (Example 3)

EXAMPLE 3 Modeling Bacteria Growth

Suppose a culture of 100 bacteria is put into a petri dish and the culture doubles every hour. Predict when the number of bacteria will be 350,000.

SOLUTION

Model

$$200 = 100 \cdot 2 \quad \text{Total bacteria after 1 hr}$$
$$400 = 100 \cdot 2^2 \quad \text{Total bacteria after 2 hr}$$
$$800 = 100 \cdot 2^3 \quad \text{Total bacteria after 3 hr}$$
$$\vdots$$
$$P(t) = 100 \cdot 2^t \quad \text{Total bacteria after } t \text{ hr}$$

So the function $P(t) = 100 \cdot 2^t$ represents the bacteria population t hr after it is placed in the petri dish.

Solve Graphically Figure 3.12 shows that the population function intersects $y = 350{,}000$ when $t \approx 11.77$.

Interpret The population of the bacteria in the petri dish will be 350,000 in about 11 hr and 46 min.

Now try Exercise 15.

Exponential decay functions model the amount of a radioactive substance present in a sample. The number of atoms of a specific element that change from a radioactive state to a nonradioactive state is a fixed fraction per unit time. The process is called **radioactive decay**, and the time it takes for half of a sample to change its state is the **half-life** of the radioactive substance.

NOTES ON EXAMPLES

In Examples 4 and 5, it may be helpful to have students carry the modeling through a few more lines before deciding what the pattern is in the final step.

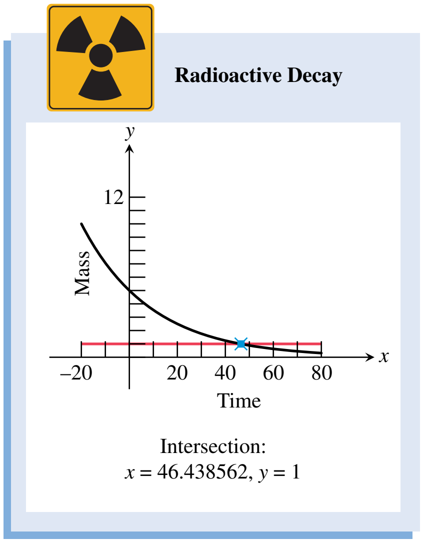


FIGURE 3.13 Radioactive decay. (Example 4)

EXAMPLE 4 Modeling Radioactive Decay

Suppose the half-life of a certain radioactive substance is 20 days and there are 5 g (grams) present initially. Find the time when there will be 1 g of the substance remaining.

SOLUTION

Model If t is the time in days, the number of half-lives will be $t/20$.

$$\frac{5}{2} = 5\left(\frac{1}{2}\right)^{20/20} \qquad \text{Grams after 20 days}$$

$$\frac{5}{4} = 5\left(\frac{1}{2}\right)^{40/20} \qquad \text{Grams after } 2(20) = 40 \text{ days}$$

$$\vdots$$

$$f(t) = 5\left(\frac{1}{2}\right)^{t/20} \qquad \text{Grams after } t \text{ days}$$

Thus the function $f(t) = 5 \cdot 0.5^{t/20}$ models the mass in grams of the radioactive substance at time t.

Solve Graphically Figure 3.13 shows that the graph of $f(t) = 5 \cdot 0.5^{t/20}$ intersects $y = 1$ when $t \approx 46.44$.

Interpret There will be 1 g of the radioactive substance left after approximately 46.44 days, or about 46 days, 11 hr. *Now try Exercise 33.*

Scientists have established that atmospheric pressure at sea level is 14.7 lb/in.^{2}, and the pressure is reduced by half for each 3.6 mi above sea level. For example, the pressure 3.6 mi above sea level is $(1/2)(14.7) = 7.35$ lb/in.^{2}. This rule for atmospheric pressure holds for altitudes up to 50 mi above sea level. Though the context is different, the mathematics of atmospheric pressure closely resembles the mathematics of radioactive decay.

EXAMPLE 5 Determining Altitude from Atmospheric Pressure

Find the altitude above sea level at which the atmospheric pressure is 4 lb/in.^{2}.

SOLUTION

Model

$$7.35 = 14.7 \cdot 0.5^{3.6/3.6} \qquad \text{Pressure at 3.6 mi}$$

$$3.675 = 14.7 \cdot 0.5^{7.2/3.6} \qquad \text{Pressure at } 2(3.6) = 7.2 \text{ mi}$$

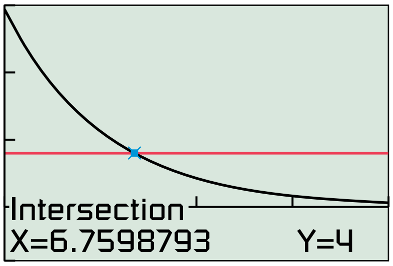
$$\vdots$$

$$P(h) = 14.7 \cdot 0.5^{h/3.6} \qquad \text{Pressure at } h \text{ mi}$$

So $P(h) = 14.7 \cdot 0.5^{h/3.6}$ models the atmospheric pressure P (in pounds per square inch) as a function of the height h (in miles above sea level). We must find the value of h that satisfies the equation

$$14.7 \cdot 0.5^{h/3.6} = 4.$$

continued



[0, 20] by [–4, 15]

FIGURE 3.14 A model for atmospheric pressure. (Example 5)

Solve Graphically Figure 3.14 shows that the graph of $P(h) = 14.7 \cdot 0.5^{h/3.6}$ intersects $y = 4$ when $h \approx 6.76$.

Interpret The atmospheric pressure is 4 lb/in.^{2} at an altitude of approximately 6.76 mi above sea level. *Now try Exercise 41.*

Using Regression to Model Population

So far, our models have been given to us or developed algebraically. We now use exponential and logistic regression to build models from population data.

Due to the post-World War II baby boom and other factors, exponential growth is not a perfect model for the U.S. population. It does, however, provide a means to make approximate predictions, as illustrated in Example 6.

EXAMPLE 6 Modeling U.S. Population Using Exponential Regression

Use the 1900–2000 data in Table 3.9 and exponential regression to predict the U.S. population for 2003. Compare the result with the listed value for 2003.

Table 3.9 U.S. Population (in millions)

Year	Population
1900	76.2
1910	92.2
1920	106.0
1930	123.2
1940	132.2
1950	151.3
1960	179.3
1970	203.3
1980	226.5
1990	248.7
2000	281.4
2003	290.8

Source: World Almanac and Book of Facts 2005.

SOLUTION

Model

Let $P(t)$ be the population (in millions) of the United States t years after 1900. Figure 3.15a shows a scatter plot of the data. Using exponential regression, we find a model for the 1990–2000 data

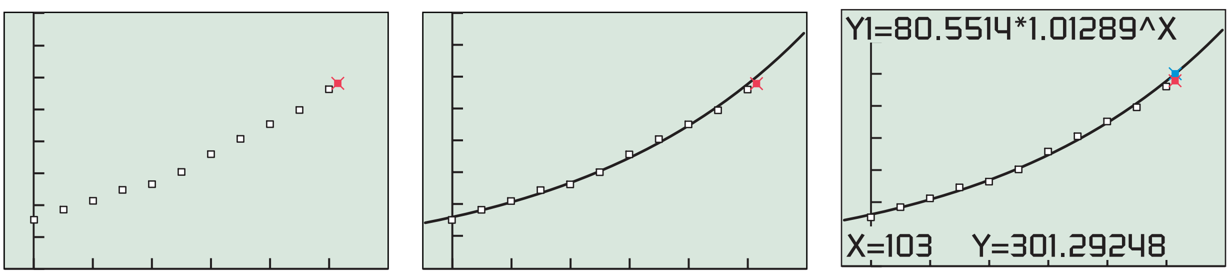
$$P(t) = 80.5514 \cdot 1.01289^t.$$

Figure 3.15b shows the scatter plot of the data with a graph of the population model just found. You can see that the curve fits the data fairly well. The coefficient of determination is $r^2 \approx 0.995$, indicating a close fit and supporting the visual evidence.

Solve Graphically

To predict the 2003 U.S. population we substitute $t = 103$ into the regression model. Figure 3.15c reports that $P(103) = 80.5514 \cdot 1.01289^{103} \approx 301.3$.

continued



[–10, 120] by [0, 400] (a) [–10, 120] by [0, 400] (b) [–10, 120] by [0, 400] (c)

FIGURE 3.15 Scatter plots and graphs for Example 6. The red “x” denotes the data point for 2003. The blue “x” in (c) denotes the model's prediction for 2003.

Interpret

The model predicts the U.S. population was 301.3 million in 2003. The actual population was 290.8 million. We overestimated by 10.5 million, less than a 4% error.

Now try Exercise 43.

Exponential growth is unrestricted, but population growth often is not. For many populations, the growth begins exponentially, but eventually slows and approaches a limit to growth called the **maximum sustainable population**.

In Section 3.1 we modeled Dallas's population with a logistic function. We now use logistic regression to do the same for the populations of Florida and Pennsylvania. As the data in Table 3.10 suggest, Florida had rapid growth in the second half of the 20th century, whereas Pennsylvania appears to be approaching its maximum sustainable population.

Table 3.10 Populations of Two U.S. States (in millions)

Year	Florida	Pennsylvania
1900	0.5	6.3
1910	0.8	7.7
1920	1.0	8.7
1930	1.5	9.6
1940	1.9	9.9
1950	2.8	10.5
1960	5.0	11.3
1970	6.8	11.8
1980	9.7	11.9
1990	12.9	11.9
2000	16.0	12.3

Source: U.S. Census Bureau.

EXAMPLE 7 Modeling Two States' Populations Using Logistic Regression

Use the data in Table 3.10 and logistic regression to predict the maximum sustainable populations for Florida and Pennsylvania. Graph the logistic models and interpret their significance.

SOLUTION Let $F(t)$ and $P(t)$ be the populations (in millions) of Florida and Pennsylvania, respectively, t years *after 1800*. Figure 3.16a shows a scatter plot of the data for both states; the data for Florida is shown in black, and for Pennsylvania, in red. Using logistic regression, we obtain the models for the two states:

$$F(t) = \frac{28.021}{1 + 9018.63e^{-0.047015t}} \quad \text{and} \quad P(t) = \frac{12.579}{1 + 29.0003e^{-0.034315t}}$$

continued

FOLLOW-UP

Ask. . .
If you were to get paid a quarter on the first day of the month, fifty cents on the second day, one dollar on the third day, and this pattern continues throughout the month how much would you get paid on day 23 of the month? on day 30?
(Day 23: $(0.25)2^{22}$ = $1,048,576; Day 30: $(0.25)2^{29}$ = $134,217,728)

ASSIGNMENT GUIDE

Day 1: Ex. 3–21, multiples of 3, 30, 33, 34, 35
Day 2: Ex. 20, 24, 27, 37, 40, 45, 47, 50, 58, 59

NOTES ON EXERCISES

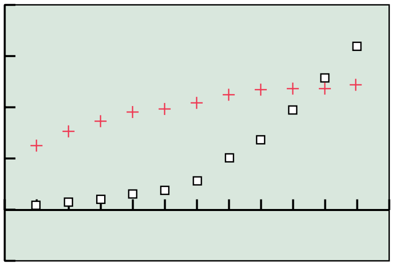
Ex. 7–18 allow students to practice writing exponential functions given initial values and rates of growth or decay.
Ex. 29–34 and 39–50 are real life problems that can be modeled with exponential functions.
Ex. 51–56 provide practice for standardized tests.
Ex. 58 could serve as a project.

ONGOING ASSIGNMENT

Self-Assessment: Ex. 1, 7, 15, 33, 41, 43, 45, 50
Embedded Assessment: Ex. 34, 47

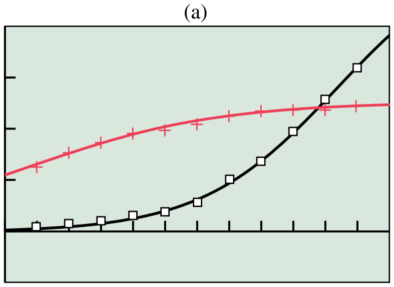
NOTES ON EXAMPLES

A base year of 1800 is used in Example 7 to avoid negative time values.



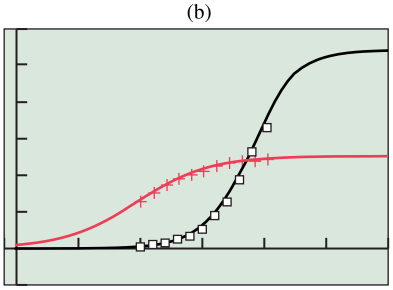
[90, 210] by [–5, 20]

(a)



[90, 210] by [–5, 20]

(b)



[–10, 300] by [–5, 30]

(c)

FIGURE 3.16 Scatter plots and graphs for Example 7.

Figure 3.16b shows the scatter plots of the data with graphs of the two population models. You can see that the curves fit the data fairly well. From the numerators of the models we see that

$$\lim_{t\to\infty} F(t) = 28.021 \quad \text{and} \quad \lim_{t\to\infty} P(t) = 12.579.$$

So the maximum sustainable population for Florida is about 28.0 million, and for Pennsylvania is about 12.6 million.

Figure 3.16c shows a three-century span for the two states. Pennsylvania had rapid growth in the 19th century and first half of the 20th century, and is now approaching its limit to growth. Florida, on the other hand, is currently experiencing extremely rapid growth but should be approaching its maximum sustainable population by the end of the 21st century. *Now try Exercise 50.*

Other Logistic Models

In Example 3, the bacteria cannot continue to grow exponentially forever because they cannot grow beyond the confines of the petri dish. In Example 7, though Florida's population is booming now, it will eventually level off, just as Pennsylvania's has done. Sunflowers and many other plants grow to a natural height following a logistic pattern. Chemical acid-base titration curves are logistic. Yeast cultures grow logistically. Contagious diseases and even rumors spread according to logistic models.

EXAMPLE 8 Modeling a Rumor

Watauga High School has 1200 students. Bob, Carol, Ted, and Alice start a rumor, which spreads logistically so that $S(t) = 1200/(1 + 39 \cdot e^{-0.9t})$ models the number of students who have heard the rumor by the end of t days, where $t = 0$ is the day the rumor begins to spread.

(a) How many students have heard the rumor by the end of Day 0?

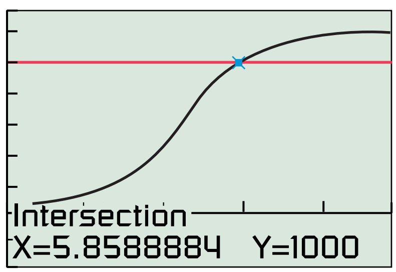
(b) How long does it take for 1000 students to hear the rumor?

SOLUTION

(a) $S(0) = \dfrac{1200}{1 + 39 \cdot e^{-0.9 \cdot 0}} = \dfrac{1200}{1 + 39} = 30$. So, 30 students have heard the rumor by the end of Day 0.

(b) We need to solve $\dfrac{1200}{1 + 39e^{-0.9t}} = 1000$.

Figure 3.17 shows that the graph of $S(t) = 1200/(1 + 39 \cdot e^{-0.9t})$ intersects $y = 1000$ when $t \approx 5.86$. So toward the end of Day 6 the rumor has reached the ears of 1000 students. *Now try Exercise 45.*



[0, 10] by [–400, 1400]

FIGURE 3.17 The spread of a rumor. (Example 8)

QUICK REVIEW 3.2 *(For help, go to Section P.5.)*

In Exercises 1 and 2, convert the percent to decimal form or the decimal into a percent.

1. 15% 0.15

2. 0.04 4%

3. Show how to increase 23 by 7% using a single multiplication. $23 \cdot 1.07$

4. Show how to decrease 52 by 4% using a single multiplication. $52 \cdot 0.96$

In Exercises 5 and 6, solve the equation algebraically.

5. $40 \cdot b^2 = 160$ ± 2

6. $243 \cdot b^3 = 9$ 1/3

In Exercises 7–10, solve the equation numerically.

7. $782b^6 = 838$ 1.01

8. $93b^5 = 521$ 1.41

9. $672b^4 = 91$ 0.61

10. $127b^7 = 56$ 0.89

SECTION 3.2 EXERCISES

In Exercises 1–6, tell whether the function is an exponential growth function or exponential decay function, and find the constant percentage rate of growth or decay.

1. $P(t) = 3.5 \cdot 1.09^t$

2. $P(t) = 4.3 \cdot 1.018^t$

3. $f(x) = 78{,}963 \cdot 0.968^x$

4. $f(x) = 5607 \cdot 0.9968^x$

5. $g(t) = 247 \cdot 2^t$

6. $g(t) = 43 \cdot 0.05^t$

In Exercises 7–18, determine the exponential function that satisfies the given conditions.

7. Initial value = 5, increasing at a rate of 17% per year $5 \cdot 1.17^x$

8. Initial value = 52, increasing at a rate of 2.3% per day

9. Initial value = 16, decreasing at a rate of 50% per month

10. Initial value = 5, decreasing at a rate of 0.59% per week

11. Initial population = 28,900, decreasing at a rate of 2.6% per year $28{,}900 \cdot 0.974^x$

12. Initial population = 502,000, increasing at a rate of 1.7% per year $502{,}000 \cdot 1.017^x$

13. Initial height = 18 cm, growing at a rate of 5.2% per week

14. Initial mass = 15 g, decreasing at a rate of 4.6% per day

15. Initial mass = 0.6 g, doubling every 3 days $0.6 \cdot 2^{x/3}$

16. Initial population = 250, doubling every 7.5 hours $250 \cdot 2^{2x/15}$

17. Initial mass = 592 g, halving once every 6 years $592 \cdot 2^{-x/6}$

18. Initial mass = 17 g, halving once every 32 hours $17 \cdot 2^{-x/32}$

In Exercises 19 and 20, determine a formula for the exponential function whose values are given in Table 3.11.

19. $f(x)$ $2.3 \cdot 1.25^x$

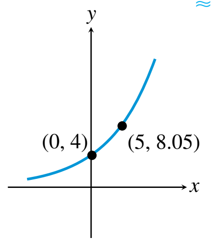
20. $g(x)$ $-5.8 \cdot 0.8^x$

Table 3.11 Values for Two Exponential Functions

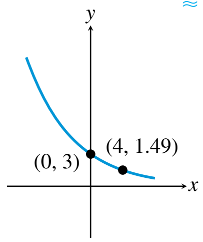
x	$f(x)$	$g(x)$
−2	1.472	−9.0625
−1	1.84	−7.25
0	2.3	−5.8
1	2.875	−4.64
2	3.59375	−3.7123

In Exercises 21 and 22, determine a formula for the exponential function whose graph is shown in the figure.

21. $\approx 4 \cdot 1.15^x$



22. $\approx 3 \cdot 0.84^x$



In Exercises 23–26, find the logistic function that satisfies the given conditions.

23. Initial value = 10, limit to growth = 40, passing through (1, 20). $40/[1 + 3 \cdot (1/3)^x]$

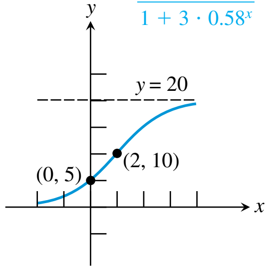
24. Initial value = 12, limit to growth = 60, passing through (1, 24). $60/[1 + 4(3/8)^x]$

25. Initial population = 16, maximum sustainable population = 128, passing through (5, 32). $\approx 128/(1 + 7 \cdot 0.844^x)$

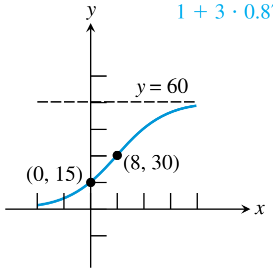
26. Initial height = 5, limit to growth = 30, passing through (3, 15). $\approx 30/(1 + 5 \cdot 0.585^x)$

In Exercises 27 and 28, determine a formula for the logistic function whose graph is shown in the figure.

27. $\frac{20}{1 + 3 \cdot 0.58^x}$



28. $\frac{60}{1 + 3 \cdot 0.87^x}$



29. Exponential Growth The 2000 population of Jacksonville, Florida was 736,000 and was increasing at the rate of 1.49% each year. At that rate, when will the population be 1 million? In 2021

30. Exponential Growth The 2000 population of Las Vegas, Nevada was 478,000 and is increasing at the rate of 6.28% each year. At that rate, when will the population be 1 million? In 2012

31. Exponential Growth The population of Smallville in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.

(a) Estimate the population in 1915 and 1940. 12,315; 24,265

(b) Predict when the population reached 50,000. 1966

32. Exponential Growth The population of River City in the year 1910 was 4200. Assume the population increased at a rate of 2.25% per year.

(a) Estimate the population in 1930 and 1945. 6554; 9151

(b) Predict when the population reached 20,000. 1980

33. Radioactive Decay The half-life of a certain radioactive substance is 14 days. There are 6.6 g present initially.

(a) Express the amount of substance remaining as a function of time *t*.

(b) When will there be less than 1 g remaining? After 38.11 days

34. Radioactive Decay The half-life of a certain radioactive substance is 65 days. There are 3.5 g present initially.

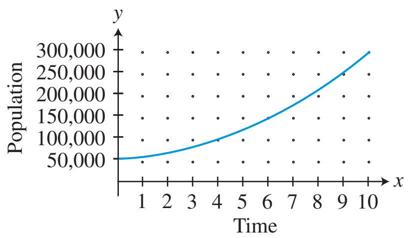
(a) Express the amount of substance remaining as a function of time *t*.

(b) When will there be less than 1 g remaining? After 117.48 days

35. Writing to Learn Without using formulas or graphs, compare and contrast exponential functions and linear functions.

36. Writing to Learn Without using formulas or graphs, compare and contrast exponential functions and logistic functions.

37. Writing to Learn Using the population model that is graphed, explain why the time it takes the population to double (doubling time) is independent of the population size.



38. Writing to Learn Explain why the half-life of a radioactive substance is independent of the initial amount of the substance that is present.

39. Bacteria Growth The number *B* of bacteria in a petri dish culture after *t* hours is given by

$$B = 100e^{0.693t}.$$

When will the number of bacteria be 200? Estimate the doubling time of the bacteria. when $t = 1$; every hour

40. Radiocarbon Dating The amount *C* in grams of carbon-14 present in a certain substance after *t* years is given by

$$C = 20e^{-0.0001216t}.$$

Estimate the half-life of carbon-14. about 5700 years

41. Atmospheric Pressure Determine the atmospheric pressure outside an aircraft flying at 52,800 ft (10 mi above sea level). 2.14 lb/in.^2

42. Atmospheric Pressure Find the altitude above sea level at which the atmospheric pressure is 2.5 lb/in.^2.

43. Population Modeling Use the 1950–2000 data in Table 3.12 and exponential regression to predict Los Angeles's population for 2003. Compare the result with the listed value for 2003.

44. Population Modeling Use the 1950–2000 data in Table 3.12 and exponential regression to predict Phoenix's population for 2003. Compare the result with the listed value for 2003. Repeat these steps using 1960–2000 data to create the model.

Table 3.12 Populations of Two U.S. Cities (in thousands)

Year	Los Angeles	Phoenix
1950	1970	107
1960	2479	439
1970	2812	584
1980	2969	790
1990	3485	983
2000	3695	1321
2003	3820	1388

Source: World Almanac and Book of Facts, 2002, 2005.

45. Spread of Flu The number of students infected with flu at Springfield High School after t days is modeled by the function

$$P(t) = \frac{800}{1 + 49e^{-0.2t}}.$$

(a) What was the initial number of infected students? 16

(b) When will the number of infected students be 200? about 14 days

(c) The school will close when 300 of the 800-student body are infected. When will the school close? in about 17 days

46. Population of Deer The population of deer after t years in Cedar State Park is modeled by the function

$$P(t) = \frac{1001}{1 + 90e^{-0.2t}}.$$

(a) What was the initial population of deer? 11

(b) When will the number of deer be 600? 24 or 25 years

(c) What is the maximum number of deer possible in the park? 1001

47. Population Growth Using all of the data in Table 3.9, compute a logistic regression model, and use it to predict the U.S. population in 2010. ≈ 311,400,000

48. Population Growth Using the data in Table 3.13, confirm the model used in Example 8 of Section 3.1.

Table 3.13 Population of Dallas, Texas

Year	Population
1950	434,462
1960	679,684
1970	844,401
1980	904,599
1990	1,006,877
2000	1,188,589

Source: U.S. Census Bureau.

49. Population Growth Using the data in Table 3.14, confirm the model used in Exercise 56 of Section 3.1.

50. Population Growth Using the data in Table 3.14, compute a logistic regression model for Arizona's population for t years since 1800. Based on your model and the New York population model from Exercise 56 of Section 3.1, will the population of Arizona ever surpass that of New York? If so, when? No

Table 3.14 Populations of Two U.S. States (in millions)

Year	Arizona	New York
1900	0.1	7.3
1910	0.2	9.1
1920	0.3	10.3
1930	0.4	12.6
1940	0.5	13.5
1950	0.7	14.8
1960	1.3	16.8
1970	1.8	18.2
1980	2.7	17.6
1990	3.7	18.0
2000	5.1	19.0

Source: U.S. Census Bureau.

Standardized Test Questions

51. True or False Exponential population growth is constrained with a maximum sustainable population. Justify your answer.

52. True or False If the constant percentage rate of an exponential function is negative, then the base of the function is negative. Justify your answer.

In Exercises 53–56, you may use a graphing calculator to solve the problem.

53. Multiple Choice What is the constant percentage growth rate of $P(t) = 1.23 \cdot 1.049^t$? C

(A) 49% **(B)** 23% **(C)** 4.9% **(D)** 2.3% **(E)** 1.23%

54. Multiple Choice What is the constant percentage decay rate of $P(t) = 22.7 \cdot 0.834^t$? B

(A) 22.7% **(B)** 16.6% **(C)** 8.34%

(D) 2.27% **(E)** 0.834%

55. Multiple Choice A single cell amoeba doubles every 4 days. About how long will it take one amoeba to produce a population of 1000? D

(A) 10 days **(B)** 20 days **(C)** 30 days

(D) 40 days **(E)** 50 days

56. Multiple Choice A rumor spreads logistically so that $S(t) = 789/(1 + 16 \cdot e^{-0.8t})$ models the number of persons who have heard the rumor by the end of t days. Based on this model, which of the following is **true**? E

(A) After 0 days, 16 people have heard the rumor.

(B) After 2 days, 439 people have heard the rumor.

(C) After 4 days, 590 people have heard the rumor.

(D) After 6 days, 612 people have heard the rumor.

(E) After 8 days, 769 people have heard the rumor.

Explorations

57. Population Growth **(a)** Use the 1900–1990 data in Table 3.9 and *logistic* regression to predict the U.S. population for 2000. ≈277,900,000

(b) Writing to Learn Compare the prediction with the value listed in the table for 2000. Underestimates actual population by 3.5 million.

(c) Noting the results of Example 6, which model—exponential or logistic—makes the better prediction in this case? logistic model

58. Population Growth Use the data in Tables 3.9 and 3.15.

(a) Based on exponential growth models, will Mexico's population surpass that of the United States, and if so, when? Yes, in 2249.

(b) Based on logistic growth models, will Mexico's population surpass that of the United States, and if so, when? No, will not exceed

(c) What are the maximum sustainable populations for the two countries? USA: 799 million, Mex: 165 million

(d) Writing to Learn Which model—exponential or logistic—is more valid in this case? Justify your choice. Logistic because there is a limit to growth

Table 3.15 Population of Mexico (in millions)

Year	Population
1900	13.6
1950	25.8
1960	34.9
1970	48.2
1980	66.8
1990	88.1
2001	101.9
2025	130.2
2050	154.0

Sources: 1992 Statesman's Yearbook and World Almanac and Book of Facts 2002.

Extending the Ideas

59. The **hyperbolic sine function** is defined by $\sinh(x) = (e^x - e^{-x})/2$. Prove that sinh is an odd function.

60. The **hyperbolic cosine function** is defined by $\cosh(x) = (e^x + e^{-x})/2$. Prove that cosh is an even function.

61. The **hyperbolic tangent function** is defined by $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$.

(a) Prove that $\tanh(x) = \sinh(x)/\cosh(x)$.

(b) Prove that tanh is an odd function.

(c) Prove that $f(x) = 1 + \tanh(x)$ is a logistic function.

59. $\sinh(-x) = \dfrac{e^{-x} - e^{-(-x)}}{2} = -\dfrac{e^x - e^{-x}}{2} = -\sinh(x)$

60. $\cosh(-x) = \dfrac{e^{-x} + e^{-(-x)}}{2} = \dfrac{e^x + e^{-x}}{2} = \cosh(x)$

61. (a) $\dfrac{\sinh(x)}{\cosh(x)} = \dfrac{(e^x - e^{-x})/2}{(e^x + e^{-x})/2} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh(x)$

(b) $\tanh(-x) = \dfrac{\sinh(-x)}{\cosh(-x)} = \dfrac{-\sinh(x)}{\cosh(x)} = -\tanh(x)$

(c) $f(x) = 1 + \tanh(x) = 1 + \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^x + e^{-x}}{e^x + e^{-x}} + \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{2e^x}{e^x + e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \dfrac{2}{1 + e^{-2x}}$, which is logistic.

3.3 Logarithmic Functions and Their Graphs

What you'll learn about

- Inverses of Exponential Functions
- Common Logarithms—Base 10
- Natural Logarithms—Base *e*
- Graphs of Logarithmic Functions
- Measuring Sound Using Decibels

. . . and why

Logarithmic functions are used in many applications, including the measurement of the relative intensity of sounds.

Inverses of Exponential Functions

In Section 1.4 we learned that, if a function passes the *horizontal line test*, then the inverse of the function is also a function. So an exponential function $f(x) = b^x$, has an inverse that is a function. See Figure 3.18. This inverse is the **logarithmic function with base** b, denoted $\log_b(x)$ or $\log_b x$. That is, if $f(x) = b^x$ with $b > 0$ and $b \neq 1$, then $f^{-1}(x) = \log_b x$. See Figure 3.19.

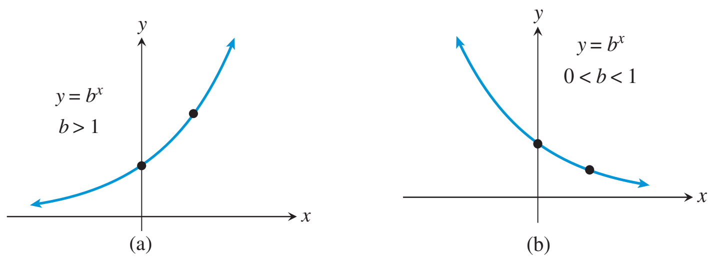


FIGURE 3.18 Exponential functions are either (a) increasing or (b) decreasing.

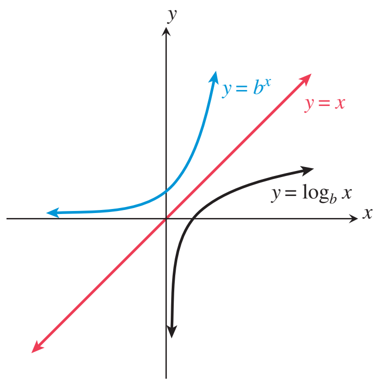


FIGURE 3.19 Because logarithmic functions are inverses of exponential functions, we can obtain the graph of a logarithmic function by the mirror or rotational methods discussed in Section 1.4.

An immediate and useful consequence of this definition is the link between an exponential equation and its logarithmic counterpart.

Changing Between Logarithmic and Exponential Form

If $x > 0$ and $0 < b \neq 1$, then

$$y = \log_b(x) \quad \text{if and only if} \quad b^y = x.$$

This linking statement says that *a logarithm is an exponent*. Because logarithms are exponents, we can evaluate simple logarithmic expressions using our understanding of exponents.

A BIT OF HISTORY

Logarithmic functions were developed around 1594 as computational tools by Scottish mathematician John Napier (1550–1617). He originally called them "artificial numbers," but changed the name to logarithms, which means "reckoning numbers."

EXAMPLE 1 Evaluating Logarithms

(a) $\log_2 8 = 3$ because $2^3 = 8$.

(b) $\log_3 \sqrt{3} = 1/2$ because $3^{1/2} = \sqrt{3}$.

(c) $\log_5 \dfrac{1}{25} = -2$ because $5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$.

(d) $\log_4 1 = 0$ because $4^0 = 1$.

(e) $\log_7 7 = 1$ because $7^1 = 7$.

Now try Exercise 1.

We can generalize the relationships observed in Example 1.

Basic Properties of Logarithms

For $0 < b \neq 1$, $x > 0$, and any real number y,

- $\log_b 1 = 0$ because $b^0 = 1$.
- $\log_b b = 1$ because $b^1 = b$.
- $\log_b b^y = y$ because $b^y = b^y$.
- $b^{\log_b x} = x$ because $\log_b x = \log_b x$.

GENERALLY $b > 1$

In practice, logarithmic bases are almost always greater than 1.

These properties give us efficient ways to evaluate simple logarithms and some exponential expressions. The first two parts of Example 2 are the same as the first two parts of Example 1.

TEACHING NOTE

You may wish to have students review the Inverse Relations and Inverse Function material in Section 1.4.

OBJECTIVE

Students will be able to convert equations between logarithmic form and exponential form, evaluate common and natural logarithms, and graph common and natural logarithmic functions.

MOTIVATE

Graph an exponential function such as $y = 2^x$ and have students discuss the graph of its inverse function.

LESSON GUIDE

Day 1: Inverses of Exponential Functions; Common Logarithms; Natural Logarithms
Day 2: Graphs of Logarithmic Functions; Measuring Sound Using Decibels

EXAMPLE 2 Evaluating Logarithmic and Exponential Expressions

(a) $\log_2 8 = \log_2 2^3 = 3$.

(b) $\log_3 \sqrt{3} = \log_3 3^{1/2} = 1/2$.

(c) $6^{\log_6 11} = 11$.

Now try Exercise 5.

Logarithmic functions are inverses of exponential functions. So the inputs and outputs are switched. Table 3.16 illustrates this relationship for $f(x) = 2^x$ and $f^{-1}(x) = \log_2 x$.

Table 3.16 An Exponential Function and Its Inverse

x	$f(x) = 2^x$	x	$f^{-1}(x) = \log_2 x$
−3	1/8	1/8	−3
−2	1/4	1/4	−2
−1	1/2	1/2	−1
0	1	1	0
1	2	2	1
2	4	4	2
3	8	8	3

This relationship can be used to produce both tables and graphs for logarithmic functions, as you will discover in Exploration 1.

EXPLORATION 1 Comparing Exponential and Logarithmic Functions

1. Set your grapher to Parametric mode and Simultaneous graphing mode.

   Set X1T = T and Y1T = 2^T.

   Set X2T = 2^T and Y2T = T.

   Creating Tables. Set TblStart = −3 and ΔTbl = 1. Use the Table feature of your grapher to obtain the decimal form of both parts of Table 3.16. Be sure to scroll to the right to see X2T and Y2T.

   Drawing Graphs. Set Tmin = −6, Tmax = 6, and Tstep = 0.5. Set the (x, y) window to $[-6, 6]$ by $[-4, 4]$. Use the Graph feature to obtain the simultaneous graphs of $f(x) = 2^x$ and $f^{-1}(x) = \log_2 x$. Use the Trace feature to explore the numerical relationships within the graphs.

2. *Graphing in Function mode.* Graph $y = 2^x$ in the same window. Then use the "draw inverse" command to draw the graph of $y = \log_2 x$.

EXPLORATION EXTENSIONS

Analyze the characteristics of the graphs (domain, range, increasing or decreasing behavior, symmetry, asymptotes, end behavior) and discuss how the two graphs are related to each other visually.

Common Logarithms—Base 10

Logarithms with base 10 are called **common logarithms**. Because of their connection to our base-ten number system, the metric system, and scientific notation, common logarithms are especially useful. We often drop the subscript of 10 for the base when using common logarithms. The common logarithmic function $\log_{10} x = \log x$ is the inverse of the exponential function $f(x) = 10^x$. So

$$y = \log x \quad \text{if and only if} \quad 10^y = x.$$

Applying this relationship, we can obtain other relationships for logarithms with base 10.

Basic Properties of Common Logarithms

Let x and y be real numbers with $x > 0$.

- $\log 1 = 0$ because $10^0 = 1$.
- $\log 10 = 1$ because $10^1 = 10$.
- $\log 10^y = y$ because $10^y = 10^y$.
- $10^{\log x} = x$ because $\log x = \log x$.

Using the definition of common logarithm or these basic properties, we can evaluate expressions involving a base of 10.

SOME WORDS OF WARNING

In Figure 3.20, notice we used "10^Ans" instead of "10^1.537819095" to check log (34.5). This is because graphers generally store more digits than they display and so we can obtain a more accurate check. Even so, because log (34.5) is an irrational number, a grapher cannot produce its exact value, so checks like those shown in Figure 3.20 may not always work out so perfectly.

EXAMPLE 3 Evaluating Logarithmic and Exponential Expressions—Base 10

(a) $\log 100 = \log_{10} 100 = 2$ because $10^2 = 100$.

(b) $\log \sqrt[5]{10} = \log 10^{1/5} = \frac{1}{5}$.

(c) $\log \frac{1}{1000} = \log \frac{1}{10^3} = \log 10^{-3} = -3$.

(d) $10^{\log 6} = 6$.

Now try Exercise 7.

Common logarithms can be evaluated by using the LOG key on a calculator, as illustrated in Example 4.

EXAMPLE 4 Evaluating Common Logarithms with a Calculator

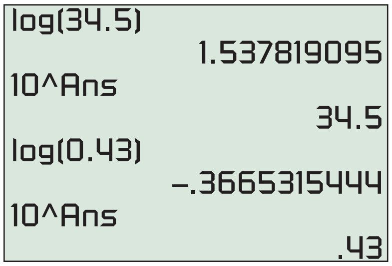


FIGURE 3.20 Doing and checking common logarithmic computations. (Example 4)

Use a calculator to evaluate the logarithmic expression if it is defined, and check your result by evaluating the corresponding exponential expression.

(a) $\log 34.5 = 1.537\ldots$ because $10^{1.537\ldots} = 34.5$.

(b) $\log 0.43 = -0.366\ldots$ because $10^{-0.366\ldots} = 0.43$.

See Figure 3.20.

(c) $\log(-3)$ is undefined because there is no real number y such that $10^y = -3$. A grapher will yield either an error message or a complex number answer for entries such as $\log(-3)$. We shall restrict the domain of logarithmic functions to the set of positive real numbers and ignore such complex number answers.

Now try Exercise 25.

Changing from logarithmic form to exponential form sometimes is enough to solve an equation involving logarithmic functions.

NOTES ON EXAMPLES

Example 5 involves simple equations that can be solved by switching between exponential and logarithmic forms. Tougher equations are addressed in Section 3.5.

EXAMPLE 5 Solving Simple Logarithmic Equations

Solve each equation by changing it to exponential form.

(a) $\log x = 3$ **(b)** $\log_2 x = 5$

SOLUTION

(a) Changing to exponential form, $x = 10^3 = 1000$.

(b) Changing to exponential form, $x = 2^5 = 32$.

Now try Exercise 33.

Natural Logarithms—Base *e*

Because of their special calculus properties, logarithms with the natural base e are used in many situations. Logarithms with base e are **natural logarithms**. We often use the special abbreviation "ln" (without a subscript) to denote a natural logarithm. Thus, the natural

READING A NATURAL LOG

The expression $\ln x$ is pronounced "el en of ex." The "l" is for logarithm, and the "n" is for natural.

logarithmic function $\log_e x = \ln x$. It is the inverse of the exponential function $f(x) = e^x$. So

$$y = \ln x \quad \text{if and only if} \quad e^y = x.$$

Applying this relationship, we can obtain other fundamental relationships for logarithms with the natural base e.

TEACHING NOTE

Don't miss the opportunity to show or explore that e^x and $\ln x$ are inverses. Students will not remember how to graph logarithmic functions without this knowledge. It is not easy to visualize a data table for $\ln x$, but they can visualize the behavior of e^x.

Basic Properties of Natural Logarithms

Let x and y be real numbers with $x > 0$.

- $\ln 1 = 0$ because $e^0 = 1$.
- $\ln e = 1$ because $e^1 = e$.
- $\ln e^y = y$ because $e^y = e^y$.
- $e^{\ln x} = x$ because $\ln x = \ln x$.

Using the definition of natural logarithm or these basic properties, we can evaluate expressions involving the natural base e.

EXAMPLE 6 Evaluating Logarithmic and Exponential Expressions—Base *e*

(a) $\ln \sqrt{e} = \log_e \sqrt{e} = 1/2$ because $e^{1/2} = \sqrt{e}$.

(b) $\ln e^5 = \log_e e^5 = 5$.

(c) $e^{\ln 4} = 4$.

Now try Exercise 13.

Natural logarithms can be evaluated by using the LN key on a calculator, as illustrated in Example 7.

EXAMPLE 7 Evaluating Natural Logarithms with a Calculator

Use a calculator to evaluate the logarithmic expression, if it is defined, and check your result by evaluating the corresponding exponential expression.

(a) $\ln 23.5 = 3.157\ldots$ because $e^{3.157\ldots} = 23.5$.

(b) $\ln 0.48 = -0.733\ldots$ because $e^{-0.733\ldots} = 0.48$.

See Figure 3.21.

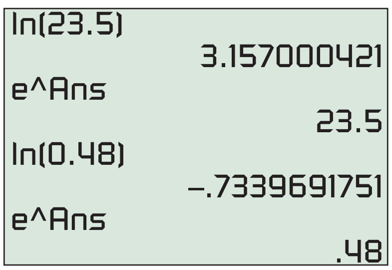


FIGURE 3.21 Doing and checking natural logarithmic computations. (Example 7)

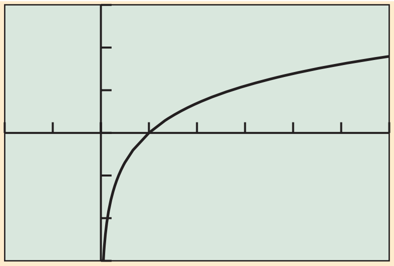
(c) $\ln(-5)$ is undefined because there is no real number y such that $e^y = -5$. A grapher will yield either an error message or a complex number answer for entries such as $\ln(-5)$. We will continue to restrict the domain of logarithmic functions to the set of positive real numbers and ignore such complex number answers.

Now try Exercise 29.

Graphs of Logarithmic Functions

The natural logarithmic function $f(x) = \ln x$ is one of the basic functions introduced in Section 1.3. We now list its properties.

BASIC FUNCTION The Natural Logarithmic Function



[–2, 6] by [–3, 3]

$f(x) = \ln x$
Domain: $(0,\infty)$
Range: All reals
Continuous on $(0,\infty)$
Increasing on $(0,\infty)$
No symmetry
Not bounded above or below
No local extrema
No horizontal asymptotes
Vertical asymptote: $x = 0$
End behavior: $\lim_{x\to\infty} \ln x = \infty$

Any logarithmic function $g(x) = \log_b x$ with $b > 1$ has the same domain, range, continuity, increasing behavior, lack of symmetry, and other general behavior as $f(x) = \ln x$. It is rare that we are interested in logarithmic functions $g(x) = \log_b x$ with $0 < b < 1$. So, the graph and behavior of $f(x) = \ln x$ is typical of logarithmic functions.

We now consider the graphs of the common and natural logarithmic functions and their geometric transformations. To understand the graphs of $y = \log x$ and $y = \ln x$, we can compare each to the graph of its inverse, $y = 10^x$ and $y = e^x$, respectively. Figure 3.23a shows that the graphs of $y = \ln x$ and $y = e^x$ are reflections of each other across the line $y = x$. Similarly, Figure 3.23b shows that the graphs of $y = \log x$ and $y = 10^x$ are reflections of each other across this same line.

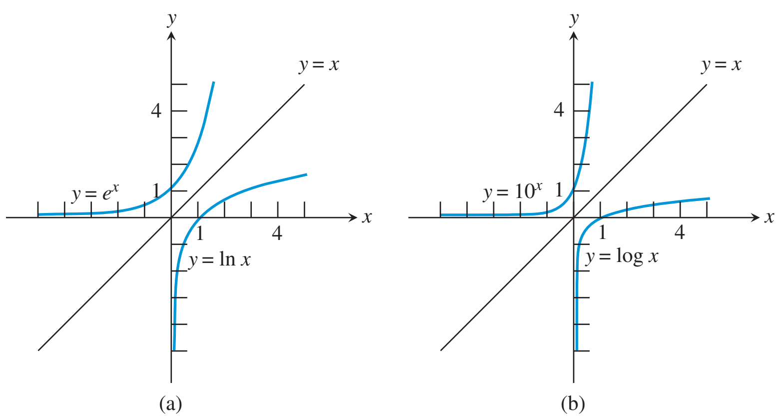
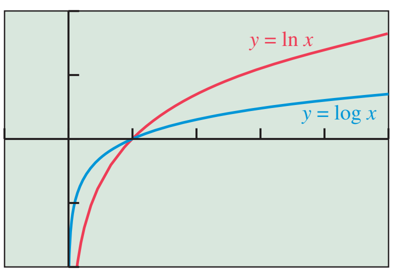


FIGURE 3.23 Two pairs of inverse functions.



[–1, 5] by [–2, 2]

FIGURE 3.24 The graphs of the common and natural logarithmic functions.

From Figure 3.24 we can see that the graphs of $y = \log x$ and $y = \ln x$ have much in common. Figure 3.24 also shows how they differ.

The geometric transformations studied in Section 1.5 together with our knowledge of the graphs of $y = \ln x$ and $y = \log x$ allow us to predict the graphs of the functions in Example 8.

EXAMPLE 8 Transforming Logarithmic Graphs

Describe how to transform the graph of $y = \ln x$ or $y = \log x$ into the graph of the given function.

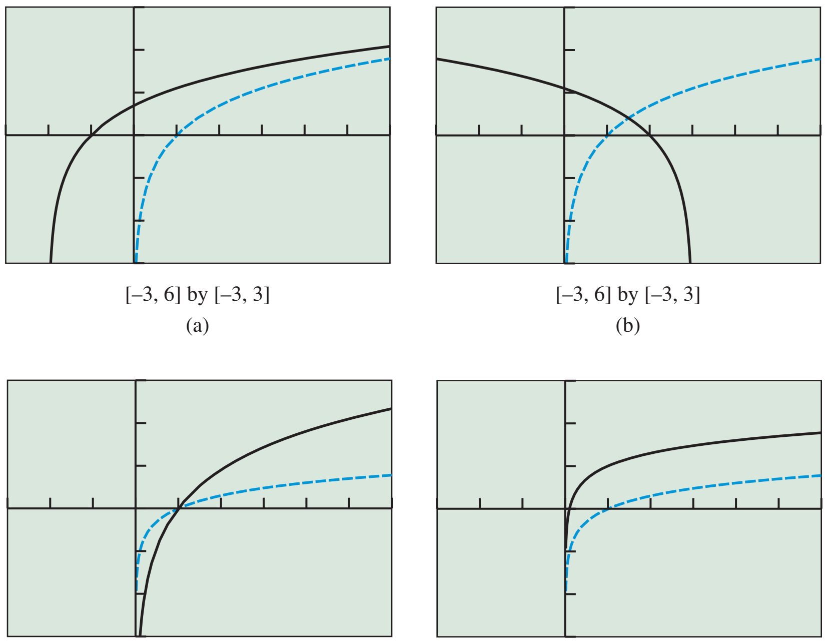
(a) $g(x) = \ln (x + 2)$ **(b)** $h(x) = \ln (3 - x)$

(c) $g(x) = 3 \log x$ **(d)** $h(x) = 1 + \log x$

SOLUTION

(a) The graph of $g(x) = \ln (x + 2)$ is obtained by translating the graph of $y = \ln (x)$ 2 units to the left. See Figure 3.25a.

(b) $h(x) = \ln (3 - x) = \ln [-(x - 3)]$. So we obtain the graph of $h(x) = \ln (3 - x)$ from the graph of $y = \ln x$ by applying, in order, a reflection across the y-axis followed by a translation 3 units to the right. See Figure 3.25b.



[–3, 6] by [–3, 3]
(a)

[–3, 6] by [–3, 3]
(b)

[–3, 6] by [–3, 3]
(c)

[–3, 6] by [–3, 3]
(d)

FIGURE 3.25 Transforming $y = \ln x$ to obtain (a) $g(x) = \ln(x + 2)$ and (b) $h(x) = \ln(3 - x)$; and $y = \log x$ to obtain (c) $g(x) = 3 \log x$ and (d) $h(x) = 1 + \log x$. (Example 8)

(c) The graph of $g(x) = 3 \log x$ is obtained by vertically stretching the graph of $f(x) = \log x$ by a factor of 3. See Figure 3.25c.

(d) We can obtain the graph of $h(x) = 1 + \log x$ from the graph of $f(x) = \log x$ by a translation 1 unit up. See Figure 3.25d. ***Now try Exercise 41.***

FOLLOW-UP

Ask students to determine the domain and range of the function $y = \log_a x$. Do the domain and range depend on the value of a? **(No, assuming $a > 0$, $a \neq 1$)**

ASSIGNMENT GUIDE

Day 1: Ex. 3–36, multiples of 3
Day 2: Ex. 37, 40, 43, 45, 49, 52, 53, 56, 59, 60, 69, 72, 74

COOPERATIVE LEARNING

Group Activity: Ex. 71

NOTES ON EXERCISES

Ex. 1–24 require students to evaluate logarithmic expressions without a calculator.
Ex. 33–36 give students practice in solving equations by converting from logarithmic form to exponential form.
Ex. 41–52 give students practice in graphing logarithmic functions by hand.
Ex. 63–68 provide practice for standardized tests.

ONGOING ASSESSMENT

Self-Assessment: Ex. 1, 5, 7, 13, 25, 29, 33, 41
Embedded Assessment: Ex. 59, 70

Measuring Sound Using Decibels

Table 3.17 lists assorted sounds. Notice that a jet at takeoff is 100 trillion times as loud as a soft whisper. Because the range of audible sound intensities is so great, common logarithms (powers of 10) are used to compare how loud sounds are.

DEFINITION Decibels

The level of sound intensity in **decibels** (dB) is

$$\beta = 10 \log(I/I_0),$$

where β (beta) is the number of decibels, I is the sound intensity in W/m^2, and $I_0 = 10^{-12}$ W/m^2 is the threshold of human hearing (the quietest audible sound intensity).

BEL IS FOR BELL

The original unit for sound intensity level was the *bel* (B), which proved to be inconveniently large. So the decibel, one-tenth of a bel, has replaced it. The bel was named in honor of Scottish-born American Alexander Graham Bell (1847–1922), inventor of the telephone.

Table 3.17 Approximate Intensities of Selected Sounds

Sound	Intensity (W/m^2)
Hearing threshold	10^{-12}
Soft whisper at 5 m	10^{-11}
City traffic	10^{-5}
Subway train	10^{-2}
Pain threshold	10^{0}
Jet at takeoff	10^{3}

Source: Adapted from R. W. Reading, Exploring Physics: Concepts and Applications (Belmont, CA: Wadsworth, 1984).

CHAPTER OPENER PROBLEM (from page 275)

PROBLEM: How loud is a train inside a subway tunnel?

SOLUTION: Based on the data in Table 3.17,

$$\begin{aligned}\beta &= 10 \log(I/I_0)\\ &= 10 \log(10^{-2}/10^{-12})\\ &= 10 \log(10^{10})\\ &= 10 \cdot 10 = 100\end{aligned}$$

So the sound intensity level inside the subway tunnel is 100 dB.

QUICK REVIEW 3.3 (*For help, go to Section A.2.*)

In Exercises 1–6, evaluate the expression without using a calculator.

1. 5^{-2} 1/25 = 0.04

2. 10^{-3} 1/1000 = 0.001

3. $\frac{4^0}{5}$ 1/5 = 0.2

4. $\frac{1^0}{2}$ 1/2 = 0.5

5. $\frac{8^{11}}{2^{28}}$ 32

6. $\frac{9^{13}}{27^8}$ 9

In Exercises 7–10, rewrite as a base raised to a rational number exponent.

7. $\sqrt{5}$ $5^{1/2}$

8. $\sqrt[3]{10}$ $10^{1/3}$

9. $\frac{1}{\sqrt{e}}$ $e^{-1/2}$

10. $\frac{1}{\sqrt[3]{e^2}}$ $e^{-2/3}$

SECTION 3.3 EXERCISES

In Exercises 1–18, evaluate the logarithmic expression without using a calculator.

1. $\log_4 4$ 1

2. $\log_6 1$ 0

3. $\log_2 32$ 5

4. $\log_3 81$ 4

5. $\log_5 \sqrt[3]{25}$ 2/3

6. $\log_6 \frac{1}{\sqrt[5]{36}}$ −2/5

7. $\log 10^3$ 3

8. $\log 10{,}000$ 4

9. $\log 100{,}000$ 5

10. $\log 10^{-4}$ −4

11. $\log \sqrt[3]{10}$ 1/3

12. $\log \frac{1}{\sqrt{1000}}$ −3/2

13. $\ln e^3$ 3

14. $\ln e^{-4}$ −4

15. $\ln \frac{1}{e}$ −1

16. $\ln 1$ 0

17. $\ln \sqrt[4]{e}$ 1/4

18. $\ln \frac{1}{\sqrt{e^7}}$ −7/2

In Exercises 19–24, evaluate the expression without using a calculator.

19. $7^{\log_7 3}$ 3

20. $5^{\log_5 8}$ 8

21. $10^{\log(0.5)}$ 0.5

22. $10^{\log 14}$ 14

23. $e^{\ln 6}$ 6

24. $e^{\ln(1/5)}$ 1/5

In Exercises 25–32, use a calculator to evaluate the logarithmic expression if it is defined, and check your result by evaluating the corresponding exponential expression.

25. $\log 9.43$ ≈ 0.975

26. $\log 0.908$ ≈ −0.042

27. $\log(-14)$ undefined

28. $\log(-5.14)$ undefined

29. $\ln 4.05$ ≈ 1.399

30. $\ln 0.733$ ≈ −0.311

31. $\ln(-0.49)$ undefined

32. $\ln(-3.3)$ undefined

In Exercises 33–36, solve the equation by changing it to exponential form.

33. $\log x = 2$ 100

34. $\log x = 4$ 10,000

35. $\log x = -1$ 0.1

36. $\log x = -3$ 0.001

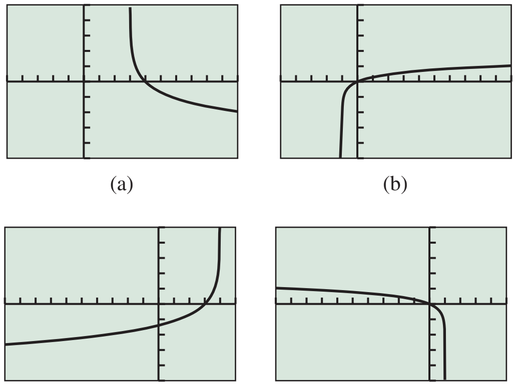
In Exercises 37–40, match the function with its graph.

37. $f(x) = \log(1 - x)$ (d)

38. $f(x) = \log(x + 1)$ (b)

39. $f(x) = -\ln(x - 3)$ (a)

40. $f(x) = -\ln(4 - x)$ (c)



(a) (b) (c) (d)

In Exercises 41–46, describe how to transform the graph of $y = \ln x$ into the graph of the given function. Sketch the graph by hand and support your sketch with a grapher.

41. $f(x) = \ln(x + 3)$

42. $f(x) = \ln(x) + 2$

43. $f(x) = \ln(-x) + 3$

44. $f(x) = \ln(-x) - 2$

45. $f(x) = \ln(2 - x)$

46. $f(x) = \ln(5 - x)$

In Exercises 47–52, describe how to transform the graph of $y = \log x$ into the graph of the given function. Sketch the graph by hand and support with a grapher.

47. $f(x) = -1 + \log(x)$

48. $f(x) = \log(x - 3)$

49. $f(x) = -2\log(-x)$

50. $f(x) = -3\log(-x)$

51. $f(x) = 2\log(3 - x) - 1$

52. $f(x) = -3\log(1 - x) + 1$

In Exercises 53–58, graph the function, and analyze it for domain, range, continuity, increasing or decreasing behavior, boundedness, extrema, symmetry, asymptotes, and end behavior.

53. $f(x) = \log(x - 2)$

54. $f(x) = \ln(x + 1)$

55. $f(x) = -\ln(x - 1)$

56. $f(x) = -\log(x + 2)$

57. $f(x) = 3\log(x) - 1$

58. $f(x) = 5\ln(2 - x) - 3$

59. Sound Intensity Use the data in Table 3.17 to compute the sound intensity in decibels for **(a)** a soft whisper, **(b)** city traffic, and **(c)** a jet at takeoff.

60. Light Absorption The Beer-Lambert law of absorption applied to Lake Erie states that the light intensity I (in lumens), at a depth of x feet, satisfies the equation

$$\log \frac{I}{12} = -0.00235x.$$

Find the intensity of the light at a depth of 30 ft. ≈ 10.2019 lumens

61. Population Growth Using the data in Table 3.18, compute a logarithmic regression model, and use it to predict when the population of San Antonio will be 1,500,000. 2023

Table 3.18 Population of San Antonio

Year	Population
1970	654,153
1980	785,940
1990	935,933
2000	1,151,305

Source: World Alamanac and Book of Facts 2005.

62. Population Decay Using the data in Table 3.19, compute a logarithmic regression model, and use it to predict when the population of Milwaukee will be 500,000. 2024

Table 3.19 Population of Milwaukee

Year	Population
1970	717,372
1980	636,297
1990	628,088
2000	596,974

Source: World Alamanac and Book of Facts 2005.

Standardized Test Questions

63. True or False A logarithmic function is the inverse of an exponential function. Justify your answer.

64. True or False Common logarithms are logarithms with base 10. Justify your answer.

In Exercises 65–68, you may use a graphing calculator to solve the problem.

65. Multiple Choice What is the approximate value of the common log of 2? C

(A) 0.10523 **(B)** 0.20000

(C) 0.30103 **(D)** 0.69315

(E) 3.32193

66. Multiple Choice Which statement is **false**? A

(A) $\log 5 = 2.5 \log 2$ **(B)** $\log 5 = 1 - \log 2$

(C) $\log 5 > \log 2$ **(D)** $\log 5 < \log 10$

(E) $\log 5 = \log 10 - \log 2$

67. Multiple Choice Which statement is **false** about $f(x) = \ln x$? B

(A) It is increasing on its domain.

(B) It is symmetric about the origin.

(C) It is continuous on its domain.

(D) It is unbounded.

(E) It has a vertical asymptote.

68. Multiple Choice Which of the following is the inverse of $f(x) = 2 \cdot 3^x$? A

(A) $f^{-1}(x) = \log_3(x/2)$ **(B)** $f^{-1}(x) = \log_2(x/3)$

(C) $f^{-1}(x) = 2\log_3(x)$ **(D)** $f^{-1}(x) = 3\log_2(x)$

(E) $f^{-1}(x) = 0.5\log_3(x)$

Explorations

69. Writing to Learn Parametric Graphing In the manner of Exploration 1, make tables and graphs for $f(x) = 3^x$ and its inverse $f^{-1}(x) = \log_3 x$. Write a comparative analysis of the two functions regarding domain, range, intercepts, and asymptotes.

70. Writing to Learn Parametric Graphing In the manner of Exploration 1, make tables and graphs for $f(x) = 5^x$ and its inverse $f^{-1}(x) = \log_5 x$. Write a comparative analysis of the two functions regarding domain, range, intercepts, and asymptotes.

71. Group Activity Parametric Graphing In the manner of Exploration 1, find the number $b > 1$ such that the graphs of $f(x) = b^x$ and its inverse $f^{-1}(x) = \log_b x$ have exactly one point of intersection. What is the one point that is in common to the two graphs? $b = \sqrt[e]{e}$; (e, e)

72. Writing to Learn Explain why zero is not in the domain of the logarithmic functions $y = \log_3 x$ and $y = \log_5 x$.

Extending the Ideas

73. Describe how to transform the graph of $f(x) = \ln x$ into the graph of $g(x) = \log_{1/e} x$. reflect across the x-axis

74. Describe how to transform the graph of $f(x) = \log x$ into the graph of $g(x) = \log_{0.1} x$. reflect across the x-axis

3.4 Properties of Logarithmic Functions

What you'll learn about

- Properties of Logarithms
- Change of Base
- Graphs of Logarithmic Functions with Base b
- Re-expressing Data

. . . and why

The applications of logarithms are based on their many special properties, so learn them well.

Properties of Logarithms

Logarithms have special algebraic traits that historically made them indispensable for calculations and that still make them valuable in many areas of applications and modeling. In Section 3.3 we learned about the inverse relationship between exponents and logarithms and how to apply some basic properties of logarithms. We now delve deeper into the nature of logarithms to prepare for equation solving and modeling.

Properties of Logarithms

Let b, R, and S be positive real numbers with $b \neq 1$, and c any real number.

- **Product rule**: $\log_b (RS) = \log_b R + \log_b S$
- **Quotient rule**: $\log_b \frac{R}{S} = \log_b R - \log_b S$
- **Power rule**: $\log_b R^c = c \log_b R$

PROPERTIES OF EXPONENTS

Let b, x, and y be real numbers with $b > 0$.

1. $b^x \cdot b^y = b^{x+y}$
2. $\frac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$

The properties of exponents in the margin are the basis for these three properties of logarithms. For instance, the first exponent property listed in the margin is used to verify the product rule.

EXAMPLE 1 Proving the Product Rule for Logarithms

Prove $\log_b (RS) = \log_b R + \log_b S$.

SOLUTION Let $x = \log_b R$ and $y = \log_b S$. The corresponding exponential statements are $b^x = R$ and $b^y = S$. Therefore,

$$RS = b^x \cdot b^y$$

$$= b^{x+y} \quad \text{First property of exponents}$$

$$\log_b (RS) = x + y \quad \text{Change to logarithmic form.}$$

$$= \log_b R + \log_b S \quad \text{Use the definitions of } x \text{ and } y.$$

Now try Exercise 37.

OBJECTIVE

Students will be able to apply the properties of logarithms to evaluate expressions and graph functions, and be able to re-express data.

MOTIVATE

Have students use a grapher to compare the graphs of $y = \log x$ and $y = \log (10x)$. Discuss the graphs.

LESSON GUIDE

Day 1: Properties of Logarithms; Change of Base; Graphs of Logarithmic Functions with Base b

Day 2: Re-expressing Data

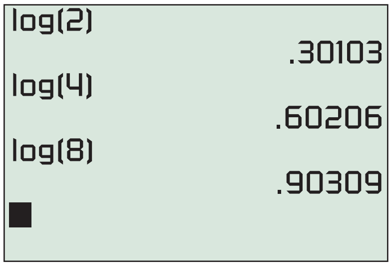


FIGURE 3.26 An arithmetic pattern of logarithms. (Exploration 1)

EXPLORATION 1 Exploring the Arithmetic of Logarithms

Use the 5-decimal place approximations shown in Figure 3.26 to support the properties of logarithms numerically.

1. Product $\log(2 \cdot 4) = \log 2 + \log 4$ $0.90309 = 0.30103 + 0.60206$
2. Quotient $\log\left(\frac{8}{2}\right) = \log 8 - \log 2$ $0.60206 = 0.90309 - 0.30103$
3. Power $\log 2^3 = 3 \log 2$ $0.90309 = 3 \times 0.30103$

Now evaluate the common logs of other positive integers using the information given in Figure 3.26 and without using your calculator.

4. Use the fact that $5 = 10/2$ to evaluate log 5. 0.69897
5. Use the fact that 16, 32, and 64 are powers of 2 to evaluate log 16, log 32, and log 64. 1.20412; 1.50515; 1.80618
6. Evaluate log 25, log 40, and log 50. 1.39794; 1.60206; 1.69897

List all of the positive integers less than 100 whose common logs can be evaluated knowing only log 2 and the properties of logarithms and without using a calculator. 1, 2, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50, 64, 80

EXPLORATION EXTENSION

Using the information given in Figure 3.26 and the value of log 5 found in Step 4, evaluate log (16/5), log (5/16), and log (2/5) without using a calculator.

When we solve equations algebraically that involve logarithms, we often have to rewrite expressions using properties of logarithms. Sometimes we need to expand as far as possible, and other times we condense as much as possible. The next three examples illustrate how properties of logarithms can be used to change the form of expressions involving logarithms.

EXAMPLE 2 Expanding the Logarithm of a Product

Assuming x and y are positive, use properties of logarithms to write $\log(8xy^4)$ as a sum of logarithms or multiples of logarithms.

SOLUTION

$$\begin{aligned}\log(8xy^4) &= \log 8 + \log x + \log y^4 && \text{Product rule}\\ &= \log 2^3 + \log x + \log y^4 && 8 = 2^3\\ &= 3\log 2 + \log x + 4\log y && \text{Power rule}\end{aligned}$$

Now try Exercise 1.

EXAMPLE 3 Expanding the Logarithm of a Quotient

Assuming x is positive, use properties of logarithms to write $\ln(\sqrt{x^2 + 5}/x)$ as a sum or difference of logarithms or multiples of logarithms.

SOLUTION

$$\begin{aligned}\ln\frac{\sqrt{x^2+5}}{x} &= \ln\frac{(x^2+5)^{1/2}}{x}\\ &= \ln(x^2+5)^{1/2} - \ln x && \text{Quotient rule}\\ &= \frac{1}{2}\ln(x^2+5) - \ln x && \text{Power rule}\end{aligned}$$

Now try Exercise 9.

EXAMPLE 4 Condensing a Logarithmic Expression

Assuming x and y are positive, write $\ln x^5 - 2\ln(xy)$ as a single logarithm.

SOLUTION

$$\begin{aligned} \ln x^5 - 2\ln(xy) &= \ln x^5 - \ln(xy)^2 && \text{Power rule} \\ &= \ln x^5 - \ln(x^2y^2) \\ &= \ln \frac{x^5}{x^2y^2} && \text{Quotient rule} \\ &= \ln \frac{x^3}{y^2} \end{aligned}$$

Now try Exercise 13.

As we have seen, logarithms have some surprising properties. It is easy to overgeneralize and fall into misconceptions about logarithms. Exploration 2 should help you discern what is true and false about logarithmic relationships.

EXPLORATION 2 Discovering Relationships and Nonrelationships

Of the eight relationships suggested here, four are *true* and four are *false* (using values of x within the domains of both sides of the equations). Thinking about the properties of logarithms, make a prediction about the truth of each statement. Then test each with some specific numerical values for x. Finally, compare the graphs of the two sides of the equation.

1. $\ln(x+2) = \ln x + \ln 2$ false
2. $\log_3(7x) = 7\log_3 x$ false
3. $\log_2(5x) = \log_2 5 + \log_2 x$ true
4. $\ln \frac{x}{5} = \ln x - \ln 5$ true
5. $\log \frac{x}{4} = \frac{\log x}{\log 4}$ false
6. $\log_4 x^3 = 3\log_4 x$ true
7. $\log_5 x^2 = (\log_5 x)(\log_5 x)$ false
8. $\log|4x| = \log 4 + \log|x|$ true

Which four are true, and which four are false?

EXPLORATION EXTENSIONS

Repeat the Exploration exercise with the following relationships:

9. $\log_3 5 = (\log_3 x)(\log_x 5)$
10. $e^{\ln 6 + \ln x} = 6x$

Change of Base

When working with a logarithmic expression with an undesirable base, it is possible to change the expression into a quotient of logarithms with a different base. For example, it is hard to evaluate $\log_4 7$ because 7 is not a simple power of 4 and there is no $\boxed{\log_4}$ key on a calculator or grapher.

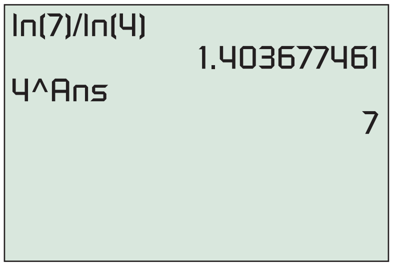


FIGURE 3.27 Evaluating and checking $\log_4 7$.

We can work around this problem with some algebraic trickery. First let $y = \log_4 7$. Then

$$4^y = 7 \quad \text{Switch to exponential form.}$$
$$\ln 4^y = \ln 7 \quad \text{Apply ln.}$$
$$y \ln 4 = \ln 7 \quad \text{Power rule}$$
$$y = \frac{\ln 7}{\ln 4} \quad \text{Divide by ln 4.}$$

So using our grapher (see Figure 3.27), we see that

$$\log_4 7 = \frac{\ln 7}{\ln 4} = 1.4036\ldots$$

We generalize this useful trickery as the change-of-base formula:

TEACHING NOTE

As a mnemonic device to help students remember the change-of-base formula, tell them to place the "log of the base in the basement."

Change-of-Base Formula for Logarithms

For positive real numbers a, b, and x with $a \neq 1$ and $b \neq 1$,

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

Calculators and graphers generally have two logarithm keys—LOG and LN—which correspond to the bases 10 and e, respectively. So we often use the change-of-base formula in one of the following two forms:

$$\log_b x = \frac{\log x}{\log b} \quad \text{or} \quad \log_b x = \frac{\ln x}{\ln b}$$

These two forms are useful in evaluating logarithms and graphing logarithmic functions.

EXAMPLE 5 Evaluating Logarithms by Changing the Base

(a) $\log_3 16 = \dfrac{\ln 16}{\ln 3} = 2.523\ldots \approx 2.52$

(b) $\log_6 10 = \dfrac{\log 10}{\log 6} = \dfrac{1}{\log 6} = 1.285\ldots \approx 1.29$

(c) $\log_{1/2} 2 = \dfrac{\ln 2}{\ln (1/2)} = \dfrac{\ln 2}{\ln 1 - \ln 2} = \dfrac{\ln 2}{-\ln 2} = -1$

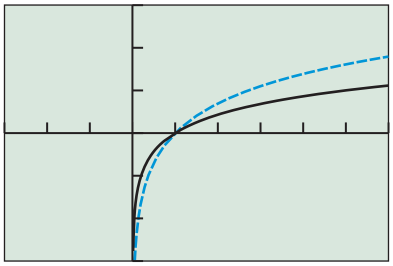
Now try Exercise 23.

Graphs of Logarithmic Functions with Base *b*

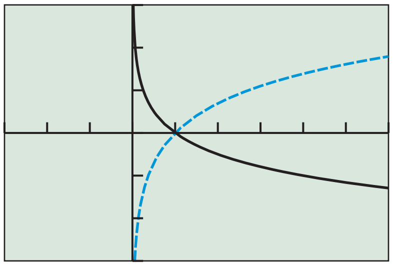
Using the change-of-base formula we can rewrite any logarithmic function $g(x) = \log_b x$ as

$$g(x) = \frac{\ln x}{\ln b} = \frac{1}{\ln b} \ln x.$$

So every logarithmic function is a constant multiple of the natural logarithmic function $f(x) = \ln x$. If the base is $b > 1$, the graph of $g(x) = \log_b x$ is a vertical stretch or shrink of the graph of $f(x) = \ln x$ by the factor $1/\ln b$. If $0 < b < 1$, a reflection across the x-axis is required as well.



[–3, 6] by [–3, 3]
(a)



[–3, 6] by [–3, 3]
(b)

FIGURE 3.28 Transforming $f(x) = \ln x$ to obtain (a) $g(x) = \log_5 x$ and (b) $h(x) = \log_{1/4} x$. (Example 6)

EXAMPLE 6 Graphing logarithmic functions

Describe how to transform the graph of $f(x) = \ln x$ into the graph of the given function. Sketch the graph by hand and support your answer with a grapher.

(a) $g(x) = \log_5 x$

(b) $h(x) = \log_{1/4} x$

NOTE ON EXAMPLE 6

You may wish to have students graph these functions as inverses of (a) $y = 5^x$ and (b) $y = (1/4)^x$.

SOLUTION

(a) Because $g(x) = \log_5 x = \ln x/\ln 5$, its graph is obtained by vertically shrinking the graph of $f(x) = \ln x$ by a factor of $1/\ln 5 \approx 0.62$. See Figure 3.28a.

(b) $h(x) = \log_{1/4} x = \frac{\ln x}{\ln 1/4} = \frac{\ln x}{\ln 1 - \ln 4} = \frac{\ln x}{-\ln 4} = -\frac{1}{\ln 4} \ln x$. So we can obtain the graph of h from the graph of $f(x) = \ln x$ by applying, in either order, a reflection across the x-axis and a vertical shrink by a factor of $1/\ln 4 \approx 0.72$. See Figure 3.28b.

Now try Exercise 39.

We can generalize Example 6b in the following way: If $b > 1$, then $0 < 1/b < 1$ and

$$\log_{1/b} x = -\log_b x.$$

So when given a function like $h(x) = \log_{1/4} x$, with a base between 0 and 1, we can immediately rewrite it as $h(x) = -\log_4 x$. Because we can so readily change the base of logarithms with bases between 0 and 1, such logarithms are rarely encountered or used. Instead, we work with logarithms that have bases $b > 1$, which behave much like natural and common logarithms, as we now summarize.

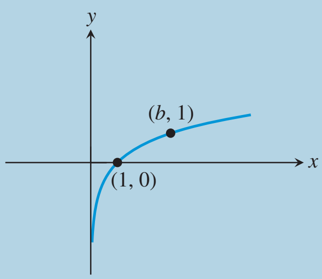


FIGURE 3.29 $f(x) = \log_b x,\ b > 1$

Logarithmic Functions $f(x) = \log_b x$, with $b > 1$

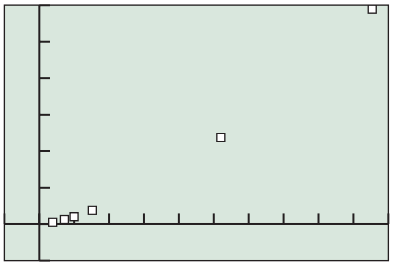
Domain: $(0, \infty)$
Range: All reals
Continuous
Increasing on its domain
No symmetry: neither even nor odd
Not bounded above or below
No local extrema
No horizontal asymptotes
Vertical asymptote: $x = 0$
End behavior: $\lim_{x\to\infty} \log_b x = \infty$

ASTRONOMICALLY SPEAKING

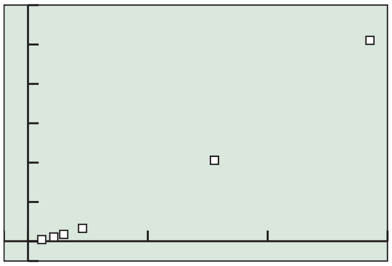
An astronomical unit (AU) is the average distance between the Earth and the Sun, about 149.6 million kilometers (149.6 Gm).

Re-expressing Data

When seeking a model for a set of data it is often helpful to transform the data by applying a function to one or both of the variables in the data set. We did this already when we treated the years 1900–2000 as 0–100. Such a transformation of a data set is a **re-expression** of the data.



[–1, 10] by [–5, 30]
(a)



[–100, 1500] by [–1000, 12 000]
(b)

FIGURE 3.30 Scatter plots of the planetary data from (a) Table 3.20 and (b) Table 2.10.

Recall from Section 2.2 that Kepler's Third Law states that the square of the orbit period T for each planet is proportional to the cube of its average distance a from the Sun. If we re-express the Kepler planetary data in Table 2.10 using Earth-based units, the constant of proportion becomes 1 and the "is proportional to" in Kepler's Third Law becomes "equals." We can do this by dividing the "average distance" column by 149.6 Gm/AU and the "period of orbit" column by 365.2 days/yr. The re-expressed data are shown in Table 3.20.

Table 3.20 Average Distances and Orbit Periods for the Six Innermost Planets

Planet	Average Distance from Sun (AU)	Period of Orbit (yr)
Mercury	0.3870	0.2410
Venus	0.7233	0.6161
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.86
Saturn	9.539	29.46

Source: Re-expression of data from: Shupe, et al., National Geographic Atlas of the World (rev. 6th ed.). Washington, DC: National Geographic Society, 1992, plate 116.

Notice that the pattern in the scatter plot of these re-expressed data, shown in Figure 3.30a, is essentially the same as the pattern in the plot of the original data, shown in Figure 3.30b. What we have done is to make the numerical values of the data more convenient and to guarantee that our plot contains the ordered pair (1, 1) for Earth, which could potentially simplify our model. What we have *not* done and still wish to do is to clarify the relationship between the variables a (distance from the Sun) and T (orbit period).

Logarithms can be used to re-express data and help us clarify relationships and uncover hidden patterns. For the planetary data, if we plot ($\ln a$, $\ln T$) pairs instead of (a, T) pairs, the pattern is much clearer. In Example 7, we carry out this re-expression of the data and then use an algebraic *tour de force* to obtain Kepler's Third Law.

EXAMPLE 7 Establishing Kepler's Third Law Using Logarithmic Re-expression

Re-express the (a, T) data pairs in Table 3.20 as ($\ln a$, $\ln T$) pairs. Find a linear regression model for the ($\ln a$, $\ln T$) pairs. Rewrite the linear regression in terms of a and T, and rewrite the equation in a form with no logs or fractional exponents.

SOLUTION

Model

We use grapher list operations to obtain the ($\ln a$, $\ln T$) pairs (see Figure 3.31a on the next page). We make a scatter plot of the re-expressed data in Figure 3.31b on the next page. The ($\ln a$, $\ln T$) pairs appear to lie along a straight line.

continued

FOLLOW-UP

Ask students how the quotient rule for logarithms can be derived from the product rule and the power rule.

ASSIGNMENT GUIDE

Day 1: Ex. 3–48, multiples of 3
Day 2: Ex. 51, 52, 53, 55, 63, 65, 66, 67, 70, 72

COOPERATIVE LEARNING

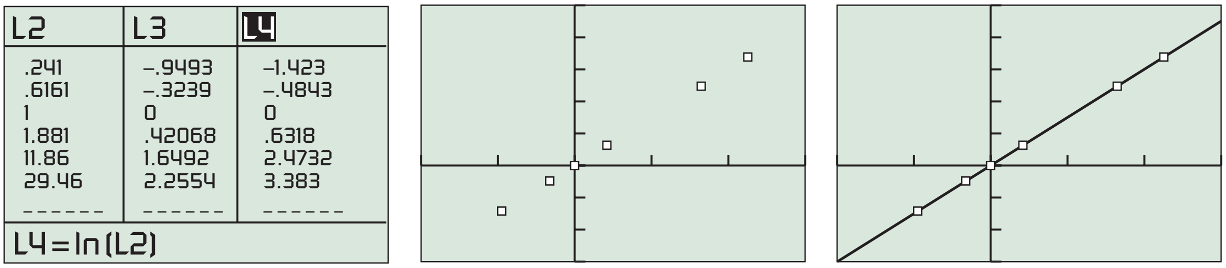
Group Activity: Ex. 69

NOTES ON EXERCISES

Ex. 1–36 provide practice in applying the properties of logarithms.
Ex. 57–62 provide practice for standardized tests.
Ex. 63–65 require the use of regression models.

ONGOING ASSESSMENT

Self-Assessment: Ex. 1, 9, 13, 23, 37, 39, 65
Embedded Assessment: Ex. 55, 70



[–2, 3] by [–3, 5] [–2, 3] by [–3, 5]

(a) (b) (c)

FIGURE 3.31 Scatter plot and graphs for Example 7.

We let $y = \ln T$ and $x = \ln a$. Then using linear regression, we obtain the following model:

$$y = 1.49950x + 0.00070 \approx 1.5x.$$

Figure 3.31c shows the scatter plot for the $(x, y) = (\ln a, \ln T)$ pairs together with a graph of $y = 1.5x$. You can see that the line fits the re-expressed data remarkably well.

Remodel

Returning to the original variables a and T, we obtain:

$$\ln T = 1.5 \cdot \ln a \qquad y = 1.5x$$

$$\frac{\ln T}{\ln a} = 1.5 \qquad \text{Divide by } \ln a.$$

$$\log_a T = \frac{3}{2} \qquad \text{Change of base}$$

$$T = a^{3/2} \qquad \text{Switch to exponential form.}$$

$$T^2 = a^3 \qquad \text{Square both sides.}$$

Interpret

This is Kepler's Third Law!

Now try Exercise 65.

QUICK REVIEW 3.4 (*For help, go to Sections A.1 and 3.3.*)

In Exercises 1–4, evaluate the expression without using a calculator.

1. $\log 10^2$ 2

2. $\ln e^3$ 3

3. $\ln e^{-2}$ -2

4. $\log 10^{-3}$ -3

In Exercises 5–10, simplify the expression.

5. $\dfrac{x^5y^{-2}}{x^2y^{-4}}$ x^3y^2

6. $\dfrac{u^{-3}v^7}{u^{-2}v^2}$ v^5/u

7. $(x^6y^{-2})^{1/2}$ $|x|^3/|y|$

8. $(x^{-8}y^{12})^{3/4}$ $|y|^9/x^6$

9. $\dfrac{(u^2v^{-4})^{1/2}}{(27u^6v^{-6})^{1/3}}$ $1/(3|u|)$

10. $\dfrac{(x^{-2}y^3)^{-2}}{(x^3y^{-2})^{-3}}$ x^{13}/y^{12}

SECTION 3.4 EXERCISES

In Exercises 1–12, assuming x and y are positive, use properties of logarithms to write the expression as a sum or difference of logarithms or multiples of logarithms.

1. $\ln 8x$ $3 \ln 2 + \ln x$

2. $\ln 9y$ $2 \ln 3 + \ln y$

3. $\log \dfrac{3}{x}$ $\log 3 - \log x$

4. $\log \dfrac{2}{y}$ $\log 2 - \log y$

5. $\log_2 y^5$ $5 \log_2 y$

6. $\log_2 x^{-2}$ $-2 \log_2 x$

7. $\log x^3y^2$ $3 \log x + 2 \log y$

8. $\log xy^3$ $\log x + 3 \log y$

9. $\ln \dfrac{x^2}{y^3}$ $2 \ln x - 3 \ln y$

10. $\log 1000x^4$ $3 + 4 \log x$

11. $\log \sqrt[4]{\dfrac{x}{y}}$ $\dfrac{1}{4}\log x - \dfrac{1}{4}\log y$

12. $\ln \dfrac{\sqrt[3]{x}}{\sqrt[3]{y}}$ $\dfrac{1}{3}\ln x - \dfrac{1}{3}\ln y$

In Exercises 13–22, assuming x, y, and z are positive, use properties of logarithms to write the expression as a single logarithm.

13. $\log x + \log y$ $\log xy$

14. $\log x + \log 5$ $\log 5x$

15. $\ln y - \ln 3$ $\ln (y/3)$

16. $\ln x - \ln y$ $\ln (x/y)$

17. $\dfrac{1}{3} \log x$ $\log \sqrt[3]{x}$

18. $\dfrac{1}{5} \log z$ $\log \sqrt[5]{z}$

19. $2 \ln x + 3 \ln y$ $\ln (x^2y^3)$

20. $4 \log y - \log z$ $\log (y^4/z)$

21. $4 \log (xy) - 3 \log (yz)$ $\log (x^4y/z^3)$

22. $3 \ln (x^3y) + 2 \ln (yz^2)$ $\log (x^9y^5z^4)$

In Exercises 23–28, use the change-of-base formula and your calculator to evaluate the logarithm.

23. $\log_2 7$ 2.8074

24. $\log_5 19$ 1.8295

25. $\log_8 175$ 2.4837

26. $\log_{12} 259$ 2.2362

27. $\log_{0.5} 12$ -3.5850

28. $\log_{0.2} 29$ -2.0922

In Exercises 29–32, write the expression using only natural logarithms.

29. $\log_3 x$ $\ln x/\ln 3$

30. $\log_7 x$ $\ln x/\ln 7$

31. $\log_2 (a + b)$ $\ln (a + b)/\ln 2$

32. $\log_5 (c - d)$ $\ln (c - d)/\ln 5$

In Exercises 33–36, write the expression using only common logarithms.

33. $\log_2 x$ $\log x/\log 2$

34. $\log_4 x$ $\log x/\log 4$

35. $\log_{1/2} (x + y)$ $-\log (x + y)/\log 2$

36. $\log_{1/3} (x - y)$ $-\log (x - y)/\log 3$

37. Prove the quotient rule of logarithms.

38. Prove the power rule of logarithms.

In Exercises 39–42, describe how to transform the graph of $g(x) = \ln x$ into the graph of the given function. Sketch the graph by hand and support with a grapher.

39. $f(x) = \log_4 x$

40. $f(x) = \log_7 x$

41. $f(x) = \log_{1/3} x$

42. $f(x) = \log_{1/5} x$

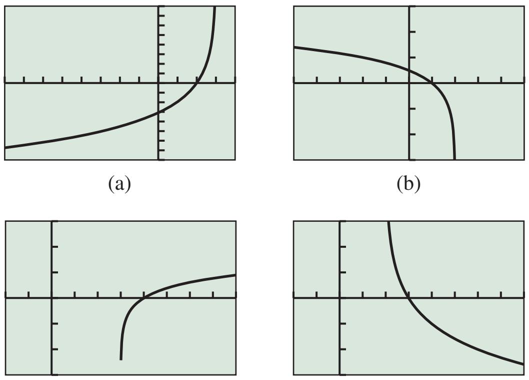
In Exercises 43–46, match the function with its graph. Identify the window dimensions, Xscl, and Yscl of the graph.

43. $f(x) = \log_4 (2 - x)$

44. $f(x) = \log_6 (x - 3)$

45. $f(x) = \log_{0.5} (x - 2)$

46. $f(x) = \log_{0.7} (3 - x)$



(a) (b) (c) (d)

In Exercises 47–50, graph the function, and analyze it for domain, range, continuity, increasing or decreasing behavior, asymptotes, and end behavior.

47. $f(x) = \log_2 (8x)$

48. $f(x) = \log_{1/3} (9x)$

49. $f(x) = \log (x^2)$

50. $f(x) = \ln (x^3)$

51. Sound Intensity Compute the sound intensity level in decibels for each sound listed in Table 3.21.

Table 3.21 Approximate Intensities for Selected Sounds

Sound	Intensity (Watts/m^2)
(a) Hearing threshold	10^{-12} 0
(b) Rustling leaves	10^{-11} 10
(c) Conversation	10^{-6} 60
(d) School cafeteria	10^{-4} 80
(e) Jack hammer	10^{-2} 100
(f) Pain threshold	1 120

Sources: J. J. Dwyer, College Physics (Belmont, CA: Wadsworth, 1984), and E. Connally et al., Functions Modeling Change (New York: Wiley, 2000).

52. Earthquake Intensity The **Richter scale** magnitude R of an earthquake is based on the features of the associated seismic wave and is measured by

$$R = \log(a/T) + B,$$

where a is the amplitude in μm (micrometers), T is the period in seconds, and B accounts for the weakening of the seismic wave due to the distance from the epicenter. Compute the earthquake magnitude R for each set of values.

(a) $a = 250$, $T = 2$, and $B = 4.25$ ≈ 6.3469

(b) $a = 300$, $T = 4$, and $B = 3.5$ ≈ 5.3751

53. Light Intensity in Lake Erie The relationship between intensity I of light (in lumens) at a depth of x feet in Lake Erie is given by

$$\log \frac{I}{12} = -0.00235x.$$

What is the intensity at a depth of 40 ft? $\approx$ 9.6645 lumens

54. Light Intensity in Lake Superior The relationship between intensity I of light (in lumens) at a depth of x feet in Lake Superior is given by

$$\log \frac{I}{12} = -0.0125x.$$

What is the intensity at a depth of 10 ft? $\approx$ 8.9987 lumens

55. Writing to Learn Use the change-of-base formula to explain how we know that the graph of $f(x) = \log_3 x$ can be obtained by applying a transformation to the graph of $g(x) = \ln x$.

56. Writing to Learn Use the change-of-base formula to explain how the graph of $f(x) = \log_{0.8} x$ can be obtained by applying transformations to the graph of $g(x) = \log x$.

55. vertical stretch by a factor of ≈ 0.9102

56. reflect across the x-axis; vertical stretch by a factor of ≈ 10.32

Standardized Test Questions

57. True or False The logarithm of the product of two positive numbers is the sum of the logarithms of the numbers. Justify your answer. True, by the product rule for logarithms

58. True or False The logarithm of a positive number is positive. Justify your answer. False. For example, $\log 0.01 = -2$.

In Exercises 59–62, solve the problem without using a calculator.

59. Multiple Choice $\log 12 =$ B

(A) $3 \log 4$ **(B)** $\log 3 + \log 4$

(C) $4 \log 3$ **(D)** $\log 3 \cdot \log 4$

(E) $2 \log 6$

60. Multiple Choice $\log_9 64 =$ C

(A) $5 \log_3 2$ **(B)** $(\log_3 8)^2$

(C) $(\ln 64)/(\ln 9)$ **(D)** $2 \log_9 32$

(E) $(\log 64)/9$

61. Multiple Choice $\ln x^5 =$ A

(A) $5 \ln x$ **(B)** $2 \ln x^3$

(C) $x \ln 5$ **(d)** $3 \ln x^2$

(E) $\ln x^2 \cdot \ln x^3$

62. Multiple Choice $\log_{1/2} x^2 =$ E

(A) $-2 \log_2 x$ **(B)** $2 \log_2 x$

(C) $-0.5 \log_2 x$ **(D)** $0.5 \log_2 x$

(E) $-2 \log_2 |x|$

Explorations

63. (a) Compute the power regression model for the following data. $2.75x^{5.0}$

x	4	6.5	8.5	10
y	2816	31,908	122,019	275,000

(b) Predict the y value associated with $x = 7.1$ using the power regression model. 49,616

(c) Re-express the data in terms of their natural logarithms and make a scatter plot of $(\ln x, \ln y)$.

(d) Compute the linear regression model $(\ln y) = a(\ln x) + b$ for $(\ln x, \ln y)$. $(\ln y) = 5.00 (\ln x) + 1.01$

(e) Confirm that $y = e^b \cdot x^a$ is the power regression model found in (a).

64. (a) Compute the power regression model for the following data. $y = 8.095x^{-0.113}$

x	2	3	4.8	7.7
y	7.48	7.14	6.81	6.41

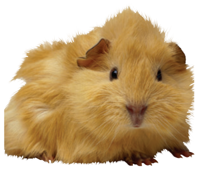
(b) Predict the y value associated with $x = 9.2$ using the power regression model. 6.30

(c) Re-express the data in terms of their natural logarithms and make a scatter plot of $(\ln x, \ln y)$.

(d) Compute the linear regression model $(\ln y) = a(\ln x) + b$ for $(\ln x, \ln y)$. $-0.113(\ln x) + 2.091$

(e) Confirm that $y = e^b \cdot x^a$ is the power regression model found in (a).

65. Keeping Warm—Revisited Recall from Exercise 55 of Section 2.2 that scientists have found the pulse rate r of mammals to be a power function of their body weight w.



(a) Re-express the data in Table 3.22 in terms of their *common* logarithms and make a scatter plot of $(\log w, \log r)$.

(b) Compute the linear regression model $(\log r) = a(\log w) + b$ for $(\log w, \log r)$.

(c) Superimpose the regression curve on the scatter plot.

(d) Use the regression equation to predict the pulse rate for a 450-kg horse. Is the result close to the 38 beats/min reported by A. J. Clark in 1927?

(e) Writing to Learn Why can we use either common or natural logarithms to re-express data that fit a power regression model?

Table 3.22 Weight and Pulse Rate of Selected Mammals

Mammal	Body weight (kg)	Pulse rate (beats/min)
Rat	0.2	420
Guinea pig	0.3	300
Rabbit	2	205
Small dog	5	120
Large dog	30	85
Sheep	50	70
Human	70	72

Source: A. J. Clark, Comparative Physiology of the Heart (New York: Macmillan, 1927).

66. Let $a = \log 2$ and $b = \log 3$. Then, for example $\log 6 = a + b$. List the common logs of all the positive integers less than 100 that can be expressed in terms of a and b, writing equations such as $\log 6 = a + b$ for each case. 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, 81, 96

Extending the Ideas

67. Solve $\ln x > \sqrt[3]{x}$. $(6.41, 93.35)$

68. Solve $1.2^x \le \log_{1.2} x$. $[1.26, 14.77]$

69. Group Activity Work in groups of three. Have each group member graph and compare the domains for one pair of functions.

(a) $f(x) = 2 \ln x + \ln (x - 3)$ and $g(x) = \ln x^2(x - 3)$

(b) $f(x) = \ln (x + 5) - \ln (x - 5)$ and $g(x) = \ln \dfrac{x + 5}{x - 5}$

(c) $f(x) = \log (x + 3)^2$ and $g(x) = 2 \log (x + 3)$

Writing to Learn After discussing your findings, write a brief group report that includes your overall conclusions and insights.

70. Prove the change-of-base formula for logarithms.

71. Prove that $f(x) = \log x/\ln x$ is a constant function with restricted domain by finding the exact value of the constant $\log x/\ln x$ expressed as a common logarithm.

72. Graph $f(x) = \ln (\ln (x))$, and analyze it for domain, range, continuity, increasing or decreasing behavior, symmetry, asymptotes, end behavior, and invertibility.

69. (a) Domain of f and g: $(3, \infty)$ (b) Domain of f and g: $(5, \infty)$
(c) Domain of f: $(-\infty, -3) \cup (-3, \infty)$; Domain of g: $(-3, \infty)$; Answers will vary.

70. Given $a, b, x > 0$, $a \ne 1$, $b \ne 1$, $\log_a x = \log_a b^{\log_b x} = \log_b x \cdot \log_a b$, which yields the desired formula: $\log_b x = \dfrac{\log_a x}{\log_a b}$.

71. $\dfrac{\log x}{\ln x} = \dfrac{\log x}{\log x / \log e} = \log e$, $x > 0$, $x \ne 1$

3.5 Equation Solving and Modeling

What you'll learn about

- Solving Exponential Equations
- Solving Logarithmic Equations
- Orders of Magnitude and Logarithmic Models
- Newton's Law of Cooling
- Logarithmic Re-expression

. . . and why

The Richter scale, pH, and Newton's Law of Cooling are among the most important uses of logarithmic and exponential functions.

OBJECTIVE

Students will be able to apply the properties of logarithms to solve exponential and logarithmic equations algebraically and solve application problems using these equations.

MOTIVATE

Ask students to use a grapher to graph $y = \log x^2$ and $y = 2 \log x$, and to comment on any differences they see. Do the results contradict the power rule for logarithms? **(No)**

LESSON GUIDE

Day 1: Solving Exponential Equations; Solving Logarithmic Equations; Orders of Magnitude and Logarithmic Models
Day 2: Newton's Law of Cooling; Logarithmic Re-expression

Solving Exponential Equations

Some logarithmic equations can be solved by changing to exponential form, as we saw in Example 5 of Section 3.3. For other equations, the properties of exponents or the properties of logarithms are used. A property of both exponential and logarithmic functions that is often helpful for solving equations is that they are one-to-one functions.

One-to-One Properties

For any exponential function $f(x) = b^x$,

- If $b^u = b^v$, then $u = v$.

For any logarithmic function $f(x) = \log_b x$,

- If $\log_b u = \log_b v$, then $u = v$.

Example 1 shows how the one-to-oneness of exponential functions can be used. Examples 3 and 4 use the one-to-one property of logarithms.

EXAMPLE 1 Solving an Exponential Equation Algebraically

Solve $20(1/2)^{x/3} = 5$.

SOLUTION

$$20\left(\frac{1}{2}\right)^{x/3} = 5$$

$$\left(\frac{1}{2}\right)^{x/3} = \frac{1}{4} \quad \text{Divide by 20.}$$

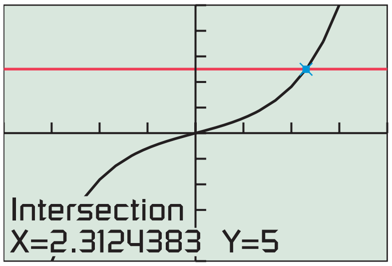
$$\left(\frac{1}{2}\right)^{x/3} = \left(\frac{1}{2}\right)^2 \quad \frac{1}{4} = \left(\frac{1}{2}\right)^2$$

$$\frac{x}{3} = 2 \quad \text{One-to-one property}$$

$$x = 6 \quad \text{Multiply by 3.}$$

Now try Exercise 1.

The equation in Example 2 involves a difference of two exponential functions, which makes it difficult to solve algebraically. So we start with a graphical approach.



[–4, 4] by [–10, 10]

FIGURE 3.32 $y = (e^x - e^{-x})/2$ and $y = 5$. (Example 2)

EXAMPLE 2 Solving an Exponential Equation

Solve $(e^x - e^{-x})/2 = 5$.

SOLUTION

Solve Graphically Figure 3.32 shows that the graphs of $y = (e^x - e^{-x})/2$ and $y = 5$ intersect when $x \approx 2.31$.

Confirm Algebraically The algebraic approach involves some ingenuity. If we multiply each side of the original equation by $2e^x$ and rearrange the terms we can obtain a quadratic equation in e^x:

$$\frac{e^x - e^{-x}}{2} = 5$$

$$e^{2x} - e^0 = 10e^x \qquad \text{Multiply by } 2e^x.$$

$$(e^x)^2 - 10(e^x) - 1 = 0 \qquad \text{Subtract } 10e^x.$$

If we let $w = e^x$, this equation becomes $w^2 - 10w - 1 = 0$, and the quadratic formula gives

$$w = e^x = \frac{10 \pm \sqrt{104}}{2} = 5 \pm \sqrt{26}.$$

Because e^x is always positive, we reject the possibility that e^x has the negative value $5 - \sqrt{26}$. Therefore,

$$e^x = 5 + \sqrt{26}$$

$$x = \ln(5 + \sqrt{26}) \qquad \text{Convert to logarithmic form.}$$

$$x = 2.312... \approx 2.31 \qquad \text{Approximate with a grapher.}$$

Now try Exercise 31.

A CINCH?

You may recognize the left-hand side of the equation in Example 2 as the *hyperbolic sine function* that was introduced in Exercise 59 of Section 3.2. This function is often used in calculus. We write $\sinh(x) = (e^x - e^{-x})/2$. "Sinh" is pronounced as if spelled "cinch."

TEACHING NOTE

Students have probably seen extraneous solutions in previous mathematics courses, but they may not have seen examples where solutions are missed. However, this can also happen with simpler equations. For example, if one attempts to solve $2x = x^2$ by dividing both sides by x, then 0 is deleted from the domain and is missed as a solution.

Solving Logarithmic Equations

When logarithmic equations are solved algebraically, it is important to keep track of the domain of each expression in the equation as it is being solved. A particular algebraic method may introduce extraneous solutions or worse yet *lose* some valid solutions, as illustrated in Example 3.

EXAMPLE 3 Solving a Logarithmic Equation

Solve $\log x^2 = 2$.

SOLUTION

Method 1 Use the one-to-one property of logarithms.

$$\log x^2 = 2$$

$$\log x^2 = \log 10^2 \qquad y = \log 10^y$$

$$x^2 = 10^2 \qquad \text{One-to-one property}$$

$$x^2 = 100 \qquad 10^2 = 100$$

$$x = 10 \quad \text{or} \quad x = -10$$

continued

Method 2 Change the equation from logarithmic to exponential form.

$$\begin{aligned} \log x^2 &= 2 & \\ x^2 &= 10^2 & &\text{Change to exponential form.} \\ x^2 &= 100 & &10^2 = 100 \\ x = 10 \quad &\text{or} \quad x = -10 & \end{aligned}$$

Method 3 (Incorrect) Use the power rule of logarithms.

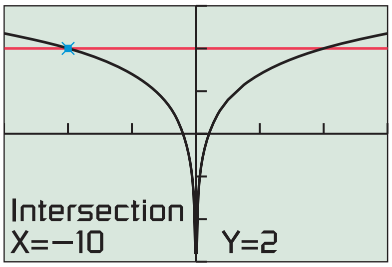
$$\begin{aligned} \log x^2 &= 2 & \\ 2 \log x &= 2 & &\text{Power rule, incorrectly applied} \\ \log x &= 1 & &\text{Divide by 2.} \\ x &= 10 & &\text{Change to exponential form.} \end{aligned}$$

Support Graphically

Figure 3.33 shows that the graphs of $f(x) = \log x^2$ and $y = 2$ intersect when $x = -10$. From the symmetry of the graphs due to f being an even function, we can see that $x = 10$ is also a solution.

Interpret

Method 1 and 2 are correct. Method 3 fails because the domain of $\log x^2$ is all nonzero real numbers, but the domain of $\log x$ is only the positive real numbers. The correct solution includes both 10 and -10 because both of these x-values make the original equation true. ***Now try Exercise 25.***



[–15, 15] by [–3, 3]

FIGURE 3.33 Graphs of $f(x) = \log x^2$ and $y = 2$. (Example 3)

Method 3 above violates an easily overlooked condition of the power rule $\log_b R^c = c \log_a R$, namely, that the rule holds *if* R is *positive*. In the expression $\log x^2$, x plays the role of R, and x can be -10, which is *not* positive. Because algebraic manipulation of a logarithmic equation can produce expressions with different domains, a graphical solution is often less prone to error.

EXAMPLE 4 Solving a Logarithmic Equation

Solve $\ln(3x - 2) + \ln(x - 1) = 2 \ln x$.

SOLUTION

To use the x-intercept method, we rewrite the equation as

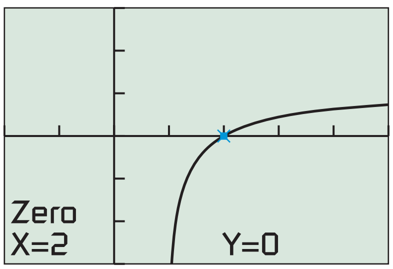
$$\ln(3x - 2) + \ln(x - 1) - 2 \ln x = 0,$$

and graph

$$f(x) = \ln(3x - 2) + \ln(x - 1) - 2 \ln x,$$

as shown in Figure 3.34. The x-intercept is $x = 2$, which is the solution to the equation.

Now try Exercise 35.



[–2, 5] by [–3, 3]

FIGURE 3.34 The zero of $f(x) = \ln(3x - 2) + \ln(x - 1) - 2 \ln x$ is $x = 2$. (Example 4)

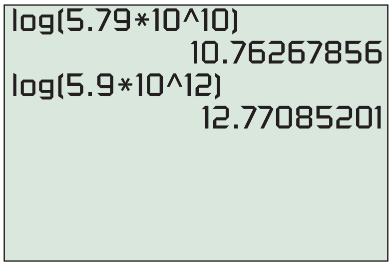


FIGURE 3.35 Pluto is two orders of magnitude farther from the Sun than Mercury.

Orders of Magnitude and Logarithmic Models

When comparing quantities, their sizes sometimes span a wide range. This is why scientific notation was developed.

For instance, the planet Mercury is 57.9 billion meters from the Sun; whereas Pluto is 5900 billion meters from the Sun, roughly 100 times farther! In scientific notation, Mercury is 5.79×10^{10} m from the Sun, and Pluto is 5.9×10^{12} m from the Sun. So Pluto's distance is 2 powers of ten greater than Mercury's distance. From Figure 3.35, we see that the difference in the common logs of these two distances is about 2. The common logarithm of a positive quantity is its **order of magnitude**. So we say, Pluto's distance from the Sun is 2 orders of magnitude greater than Mercury's.

Orders of magnitude can be used to compare any like quantities:

- A kilometer is 3 orders of magnitude longer than a meter.
- A dollar is 2 orders of magnitude greater than a penny.
- A horse weighing 400 kg is 4 orders of magnitude heavier than a mouse weighing 40 g.
- New York City with 8 million people is 6 orders of magnitude bigger than Earmuff Junction with a population of 8.

TEACHING NOTE

Orders of magnitude are widely used in science and engineering, but are usually ignored in mathematics. The concept should be new to most students but easy to motivate.

EXPLORATION EXTENSIONS

Have the students calculate $10^{1.60205991}$, $10^{2.60205991}$, and $10^{3.60205991}$ and discuss their findings.

EXPLORATION 1 Comparing Scientific Notation and Common Logarithms

1. Using a calculator compute $\log(4 \cdot 10)$, $\log(4 \cdot 10^2)$, $\log(4 \cdot 10^3)$, . . . , $\log(4 \cdot 10^{10})$.
2. What is the pattern in the integer parts of these numbers?
3. What is the pattern of their decimal parts?
4. How many orders of magnitude greater is $4 \cdot 10^{10}$ than $4 \cdot 10$? 9

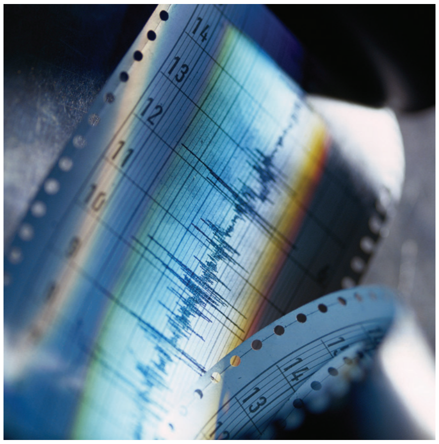
Orders of magnitude have many applications. For a sound or noise, the *bel*, mentioned in Section 3.3, measures the order of magnitude of its intensity compared to the threshold of hearing. For instance, a sound of 3 bels or 30 dB (decibels) has a sound intensity 3 orders of magnitude above the threshold of hearing.

Orders of magnitude are also used to compare the severity of earthquakes and the acidity of chemical solutions. We now turn our attention to these two applications.

As mentioned in Exercise 52 of Section 3.4, the *Richter scale* magnitude R of an earthquake is

$$R = \log \frac{a}{T} + B,$$

where a is the amplitude in micrometers (μm) of the vertical ground motion at the receiving station, T is the period of the associated seismic wave in seconds, and B accounts for the weakening of the seismic wave with increasing distance from the epicenter of the earthquake.



EXAMPLE 5 Comparing Earthquake Intensities

How many times more severe was the 2001 earthquake in Gujarat, India ($R_1 = 7.9$) than the 1999 earthquake in Athens, Greece ($R_2 = 5.9$)?

SOLUTION

Model

The severity of an earthquake is measured by the associated amplitude. Let a_1 be the amplitude for the Gujarat earthquake and a_2 be the amplitude for the Athens earthquake. Then

$$R_1 = \log\frac{a_1}{T} + B = 7.9$$

$$R_2 = \log\frac{a_2}{T} + B = 5.9$$

Solve Algebraically We seek the ratio of severities a_1/a_2:

$$\left(\log\frac{a_1}{T} + B\right) - \left(\log\frac{a_2}{T} + B\right) = R_1 - R_2$$

$$\log\frac{a_1}{T} - \log\frac{a_2}{T} = 7.9 - 5.9 \qquad B - B = 0$$

$$\log\frac{a_1}{a_2} = 2 \qquad \text{Quotient rule}$$

$$\frac{a_1}{a_2} = 10^2 = 100$$

Interpret

A Richter scale difference of 2 corresponds to an amplitude ratio of 2 powers of 10, or $10^2 = 100$. So the Gujarat quake was 100 times as severe as the Athens quake.

Now try Exercise 45.

In chemistry, the acidity of a water-based solution is measured by the concentration of hydrogen ions in the solution (in moles per liter). The hydrogen-ion concentration is written $[H^+]$. Because such concentrations usually involve *negative* powers of ten, *negative* orders of magnitude are used to compare acidity levels. The measure of acidity used is **pH**, the opposite of the common log of the hydrogen-ion concentration:

$$\text{pH} = -\log [H^+]$$

More acidic solutions have higher hydrogen-ion concentrations and lower pH values.

EXAMPLE 6 Comparing Chemical Acidity

Some especially sour vinegar has a pH of 2.4, and a box of Leg and Sickle baking soda has a pH of 8.4.

(a) What are their hydrogen-ion concentrations?

(b) How many times greater is the hydrogen-ion concentration of the vinegar than that of the baking soda?

(c) By how many orders of magnitude do the concentrations differ?

SOLUTION

(a)

$$\begin{aligned} \text{Vinegar:}\quad -\log [H^+] &= 2.4 \\ \log [H^+] &= -2.4 \\ [H^+] &= 10^{-2.4} \approx 3.98 \times 10^{-3} \text{ moles per liter} \\ \text{Baking soda:}\quad -\log [H^+] &= 8.4 \\ \log [H^+] &= -8.4 \\ [H^+] &= 10^{-8.4} \approx 3.98 \times 10^{-9} \text{ moles per liter} \end{aligned}$$

(b)

$$\frac{[H^+] \text{ of vinegar}}{[H^+] \text{ of baking soda}} = \frac{10^{-2.4}}{10^{-8.4}} = 10^{(-2.4)-(-8.4)} = 10^6$$

(c) The hydrogen-ion concentration of the vinegar is 6 orders of magnitude greater than that of the Leg and Sickle baking soda, exactly the difference in their pH values.

Now try Exercise 47.

Newton's Law of Cooling

An object that has been heated will cool to the temperature of the medium in which it is placed, such as the surrounding air or water. The temperature T of the object at time t can be modeled by

$$T(t) = T_m + (T_0 - T_m)e^{-kt}$$

for an appropriate value of k, where

$$T_m = \text{the temperature of the surrounding medium,}$$

$$T_0 = \text{initial temperature of the object.}$$

This model assumes that the surrounding medium, although taking heat from the object, essentially maintains a constant temperature. In honor of English mathematician and physicist Isaac Newton (1643–1727), this model is called **Newton's Law of Cooling**.

EXAMPLE 7 Applying Newton's Law of Cooling

A hard-boiled egg at temperature 96°C is placed in 16°C water to cool. Four minutes later the temperature of the egg is 45°C. Use Newton's Law of Cooling to determine when the egg will be 20°C.

SOLUTION

Model Because $T_0 = 96$ and $T_m = 16$, $T_0 - T_m = 80$ and

$$T(t) = T_m + (T_0 - T_m)e^{-kt} = 16 + 80e^{-kt}.$$

To find the value of k we use the fact that $T = 45$ when $t = 4$.

$$45 = 16 + 80e^{-4k}$$

$$\frac{29}{80} = e^{-4k} \qquad \text{Subtract 16, then divide by 80.}$$

$$\ln \frac{29}{80} = -4k \qquad \text{Change to logarithmic form.}$$

$$k = -\frac{\ln(29/80)}{4} \qquad \text{Divide by } -4.$$

$$k = 0.253\ldots$$

We save this k value because it is part of our model. (See Figure 3.36.)

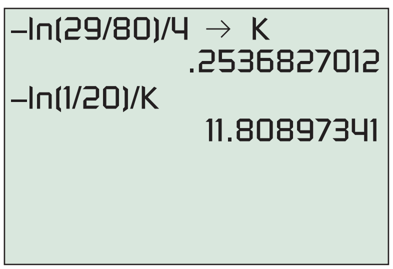


FIGURE 3.36 Storing and using the constant k.

continued

Table 3.23 Temperature Data from a CBL™ Experiment

Time t	Temp T	$T - T_m$
2	64.8	60.3
5	49.0	44.5
10	31.4	26.9
15	22.0	17.5
20	16.5	12.0
25	14.2	9.7
30	12.0	7.5

Solve Algebraically To find t when $T = 20°C$, we solve the equation:

$$20 = 16 + 80e^{-kt}$$

$$\frac{4}{80} = e^{-kt} \qquad \text{Subtract 16, then divide by 80.}$$

$$\ln \frac{4}{80} = -kt \qquad \text{Change to logarithmic form.}$$

$$t = -\frac{\ln(4/80)}{k} \approx 11.81 \qquad \text{See Figure 3.36.}$$

Interpret The temperature of the egg will be 20°C after about 11.81 min (11 min 49 sec).

Now try Exercise 49.

We can rewrite Newton's Law of Cooling in the following form:

$$T(t) - T_m = (T_0 - T_m)e^{-kt}$$

We use this form of Newton's Law of Cooling when modeling temperature using data gathered from an actual experiment. Because the difference $T - T_m$ is an exponential function of time t, we can use exponential regression on $T - T_m$ versus t to obtain a model, as illustrated in Example 8.

EXAMPLE 8 Modeling with Newton's Law of Cooling

In an experiment, a temperature probe connected to a Calculator-Based-Laboratory™ device was removed from a cup of hot coffee and placed in a glass of cold water. The first two columns of Table 3.23 show the resulting data for time t (in seconds since the probe was placed in the water) and temperature T (in °C). In the third column, the temperature data have been *re-expressed* by subtracting the temperature of the water, which was 4.5°C.

(a) Estimate the temperature of the coffee.

(b) Estimate the time when the temperature probe reading was 40°C.

SOLUTION

Model Figure 3.37a shows a scatter plot of the re-expressed temperature data. Using exponential regression, we obtain the following model:

$$T(t) - 4.5 = 61.656 \times 0.92770^t$$

Figure 3.37b shows the graph of this model with the scatter plot of the data. You can see that the curve fits the data fairly well.

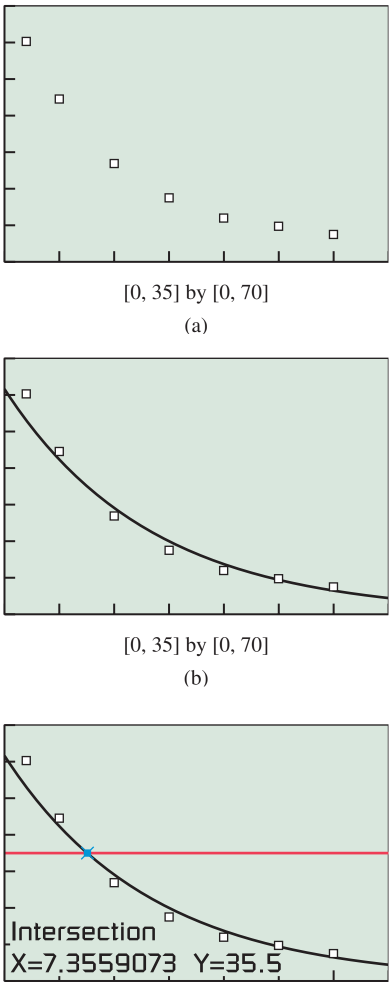
(a) Solve Algebraically From the model we see that $T_0 - T_m \approx 61.656$. So

$$T_0 \approx 61.656 + T_m = 61.656 + 4.5 \approx 66.16$$

(b) Solve Graphically Figure 3.37c shows that the graph of $T(t) - 4.5 = 61.656 \times 0.92770^t$ intersects $y = 40 - 4.5 = 35.5$ when $t \approx 7.36$.

Interpret The temperature of the coffee was roughly 66.2°C, and the probe reading was 40°C about 7.4 sec after it was placed in the water.

Now try Exercise 51.



[0, 35] by [0, 70]
(a)

[0, 35] by [0, 70]
(b)

[0, 35] by [0, 70]
(c)

FIGURE 3.37 Scatter plot and graphs for Example 8.

Logarithmic Re-expression

In Example 7 of Section 3.4 we learned that data pairs (x, y) that fit a power model have a linear relationship when re-expressed as $(\ln x, \ln y)$ pairs. We now illustrate that data pairs (x, y) that fit a logarithmic or exponential regression model can also be *linearized* through *logarithmic re-expression*.

Regression Models Related by Logarithmic Re-Expression

- **Linear regression**: $y = ax + b$
- **Natural logarithmic regression**: $y = a + b \ln x$
- **Exponential regression**: $y = a \cdot b^x$
- **Power regression**: $y = a \cdot x^b$

When we examine a scatter plot of data pairs (x, y), we should ask whether one of these four regression models could be the best choice. If the data plot appears to be linear, a linear regression may be the best choice. But when it is visually evident that the data plot is not linear, the best choice may be a natural logarithmic, exponential, or power regression.

Knowing the shapes of logarithmic, exponential, and power function graphs helps us choose an appropriate model. In addition, it is often helpful to re-express the (x, y) data pairs as $(\ln x, y)$, $(x, \ln y)$, or $(\ln x, \ln y)$ and create scatter plots of the re-expressed data. If any of the scatter plots appear to be linear, then we have a likely choice for an appropriate model. See page 329.

The three regression models can be justified algebraically. We give the justification for exponential regression, and leave the other two justifications as exercises.

$$v = ax + b$$

$$\ln y = ax + b \qquad v = \ln y$$

$$y = e^{ax+b} \qquad \text{Change to exponential form.}$$

$$y = e^{ax} \cdot e^b \qquad \text{Use the laws of exponents.}$$

$$y = e^b \cdot (e^a)^x$$

$$y = c \cdot d^x \qquad \text{Let } c = e^b \text{ and } d = e^a.$$

Example 9 illustrates how a combination of knowledge about the shapes of logarithmic, exponential, and power function graphs is used in combination with logarithmic re-expression to choose a curve of best fit.

FOLLOW-UP

Ask students to explain how extraneous solutions might be introduced by replacing

$\ln x - \ln(x - 2)$ with $\ln\left(\dfrac{x}{x-2}\right)$

in an equation.

ASSIGNMENT GUIDE

Day 1: Ex. 3–48, multiples of 3
Day 2: Ex. 49, 51, 53, 55, 56, 67, 69, 70, 74, 76

COOPERATIVE LEARNING

Group Activity: Ex. 67

NOTES ON EXERCISES

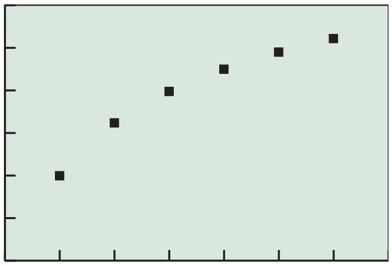
Ex. 25–38 give students a choice of methods to solve the equations. Ex. 59–64 provide practice for standardized tests.

ONGOING ASSESSMENT

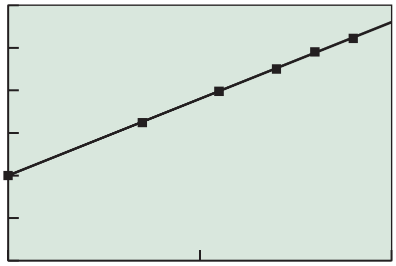
Self-Assessment: Ex. 1, 25, 29, 35, 45, 47, 49, 51, 55
Embedded Assessment: Ex. 54, 68

Three Types of Logarithmic Re-expression

1. Natural Logarithmic Regression Re-expressed: $(x, y) \rightarrow (\ln x, y)$



[0, 7] by [0, 30]
(x, y) data
(a)

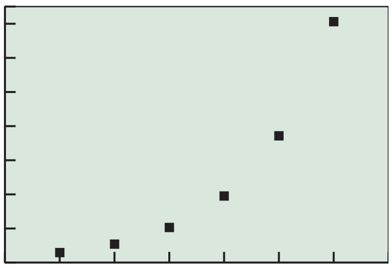


[0, 2] by [0, 30]
$(\ln x, y) = (u, y)$ data with
linear regression model
$y = au + b$
(b)

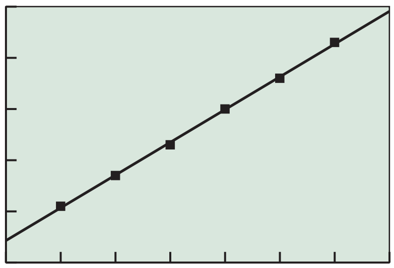
Conclusion:

$y = a \ln x + b$ is the logarithmic regression model for the (x, y) data.

2. Exponential Regression Re-expressed: $(x, y) \rightarrow (x, \ln y)$



[0, 7] by [0, 75]
(x, y) data
(a)

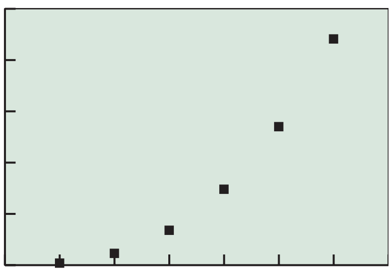


[0, 7] by [0, 5]
$(x, \ln y) = (x, v)$ data with
linear regression model
$v = ax + b$
(b)

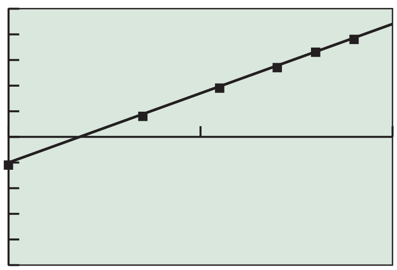
Conclusion:

$y = c(d^x)$, where $c = e^b$ and $d = e^a$, is the exponential regression model for the (x, y) data.

3. Power Regression Re-expressed: $(x, y) \rightarrow (\ln x, \ln y)$



[0, 7] by [0, 50]
(x, y) data
(a)



[0, 2] by [–5, 5]
$(\ln x, \ln y) = (u, v)$ data with
linear regression model
$v = au + b$
(b)

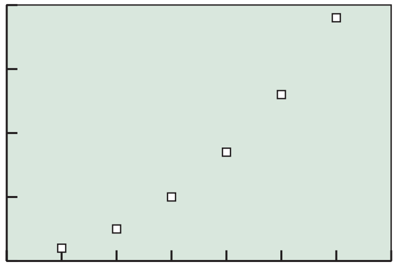
Conclusion:

$y = c(x^a)$, where $c = e^b$, is the power regression model for the (x, y) data.

EXAMPLE 9 Selecting a Regression Model

Decide whether these data can be best modeled by logarithmic, exponential, or power regression. Find the appropriate regression model.

x	1	2	3	4	5	6
y	2	5	10	17	26	38

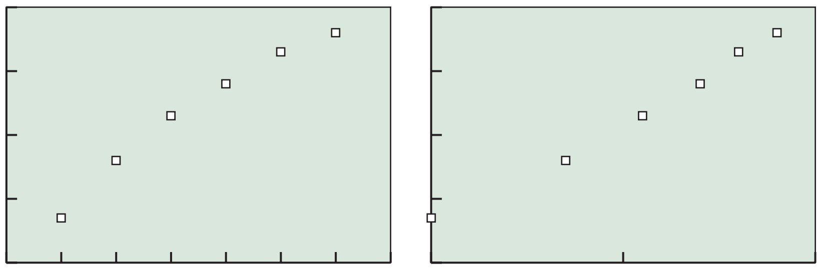


[0, 7] by [0, 40]

FIGURE 3.38 A scatter plot of the original data of Example 9.

SOLUTION The shape of the data plot in Figure 3.38 suggests that the data could be modeled by an exponential or power function.

Figure 3.39a shows the $(x, \ln y)$ plot, and Figure 3.39b shows the $(\ln x, \ln y)$ plot. Of these two plots, the $(\ln x, \ln y)$ plot appears to be more linear, so we find the power regression model for the original data.



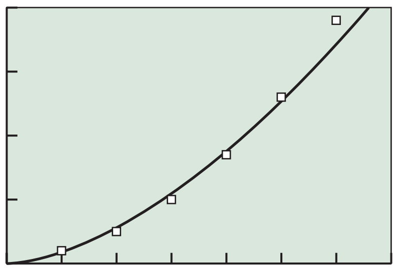
[0, 7] by [0, 4]
$(x, \ln y)$
(a)

[0, 2] by [0, 4]
$(\ln x, \ln y)$
(b)

FIGURE 3.39 Two logarithmic re-expressions of the data of Example 9.

Figure 3.40 shows the scatter plot of the original (x, y) data with the graph of the power regression model $y = 1.7910x^{1.6472}$ superimposed.

Now try Exercise 55.



[0, 7] by [0, 40]

FIGURE 3.40 A power regression model fits the data of Example 9.

QUICK REVIEW 3.5 *(For help, go to Sections P.1 and 1.4.)*

In Exercises 1–4, prove that each function in the given pair is the inverse of the other.

1. $f(x) = e^{2x}$ and $g(x) = \ln (x^{1/2})$

2. $f(x) = 10^{x/2}$ and $g(x) = \log x^2, x > 0$

3. $f(x) = (1/3) \ln x$ and $g(x) = e^{3x}$

4. $f(x) = 3 \log x^2, x > 0$ and $g(x) = 10^{x/6}$

In Exercises 5 and 6, write the number in scientific notation.

5. The mean distance from Jupiter to the Sun is about 778,300,000 km. 7.783×10^8 km

6. An atomic nucleus has a diameter of about 0.000000000000001 m. 1×10^{-15} m

In Exercises 7 and 8, write the number in decimal form.

7. Avogadro's number is about 6.02×10^{23}.

8. The atomic mass unit is about 1.66×10^{-27} kg.

In Exercises 9 and 10, use scientific notation to simplify the expression; leave your answer in scientific notation.

9. (186,000)(31,000,000) 5.766×10^{12}

10. $\dfrac{0.0000008}{0.000005}$ 1.6×10^{-1}

SECTION 3.5 EXERCISES

In Exercises 1–10, find the exact solution algebraically, and check it by substituting into the original equation.

1. $36\left(\frac{1}{3}\right)^{x/5} = 4$ 10

2. $32\left(\frac{1}{4}\right)^{x/3} = 2$ 6

3. $2 \cdot 5^{x/4} = 250$ 12

4. $3 \cdot 4^{x/2} = 96$ 5

5. $2(10^{-x/3}) = 20$ −3

6. $3(5^{-x/4}) = 15$ −4

7. $\log x = 4$ 10,000

8. $\log_2 x = 5$ 32

9. $\log_4 (x - 5) = -1$ 5.25

10. $\log_4 (1 - x) = 1$ −3

In Exercises 11–18, solve each equation algebraically. Obtain a numerical approximation for your solution and check it by substituting into the original equation.

11. $1.06^x = 4.1$ ≈ 24.2151

12. $0.98^x = 1.6$ ≈ -23.2644

13. $50e^{0.035x} = 200$ ≈ 39.6084

14. $80e^{0.045x} = 240$ ≈ 24.4136

15. $3 + 2e^{-x} = 6$ ≈ -0.4055

16. $7 - 3e^{-x} = 2$ ≈ -0.5108

17. $3 \ln (x - 3) + 4 = 5$

18. $3 - \log (x + 2) = 5$ −1.99

In Exercises 19–24, state the domain of each function. Then match the function with its graph. (Each graph shown has a window of $[-4.7, 4.7]$ by $[-3.1, 3.1]$.)

19. $f(x) = \log [x(x + 1)]$

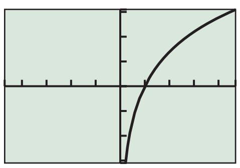
20. $g(x) = \log x + \log (x + 1)$

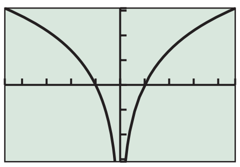
21. $f(x) = \ln \dfrac{x}{x + 1}$

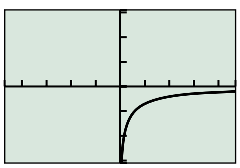
22. $g(x) = \ln x - \ln (x + 1)$

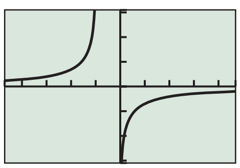
23. $f(x) = 2 \ln x$

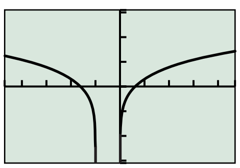
24. $g(x) = \ln x^2$

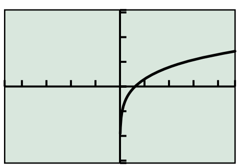

(a)


(b)


(c)


(d)


(e)


(f)

In Exercises 25–38, solve each equation by the method of your choice. Support your solution by a second method.

25. $\log x^2 = 6$ $x = 1000$ or $x = -1000$

26. $\ln x^2 = 4$ $x = e^2 \approx 7.389$ or $x = -e^2 \approx -7.389$

27. $\log x^4 = 2$ $\pm\sqrt{10}$

28. $\ln x^6 = 12$ $\pm e^2$

29. $\dfrac{2^x - 2^{-x}}{3} = 4$

30. $\dfrac{2^x + 2^{-x}}{2} = 3$

31. $\dfrac{e^x + e^{-x}}{2} = 4$

32. $2e^{2x} + 5e^x - 3 = 0$

33. $\dfrac{500}{1 + 25e^{0.3x}} = 200$

34. $\dfrac{400}{1 + 95e^{-0.6x}} = 150$

35. $\dfrac{1}{2} \ln (x + 3) - \ln x = 0$

36. $\log x - \dfrac{1}{2} \log (x + 4) = 1$

37. $\ln (x - 3) + \ln (x + 4) = 3 \ln 2$ 4

38. $\log (x - 2) + \log (x + 5) = 2 \log 3$ $x \approx 3.1098$

In Exercises 39–44, determine how many orders of magnitude the quantities differ.

39. A \$100 bill and a dime 3

40. A canary weighing 20 g and a hen weighing 2 kg 2

41. An earthquake rated 7 on the Richter scale and one rated 5.5. 1.5

42. Lemon juice with pH = 2.3 and beer with pH = 4.1 1.8

43. The sound intensities of a riveter at 95 dB and ordinary conversation at 65 dB 3

44. The sound intensities of city traffic at 70 dB and rustling leaves at 10 dB 6

45. Comparing Earthquakes How many times more severe was the 1978 Mexico City earthquake ($R = 7.9$) than the 1994 Los Angeles earthquake ($R = 6.6$)? about 20 times greater

46. Comparing Earthquakes How many times more severe was the 1995 Kobe, Japan, earthquake ($R = 7.2$) than the 1994 Los Angeles earthquake ($R = 6.6$)? about 4 times greater

47. Chemical Acidity The pH of carbonated water is 3.9 and the pH of household ammonia is 11.9.

(a) What are their hydrogen-ion concentrations?

(b) How many times greater is the hydrogen-ion concentration of the carbonated water than that of the ammonia?

(c) By how many orders of magnitude do the concentrations differ? 8

48. Chemical Acidity Stomach acid has a pH of about 2.0, and blood has a pH of 7.4.

(a) What are their hydrogen-ion concentrations?

(b) How many times greater is the hydrogen-ion concentration of the stomach acid than that of the blood?

(c) By how many orders of magnitude do the concentrations differ? 5.4

49. Newton's Law of Cooling A cup of coffee has cooled from 92°C to 50°C after 12 min in a room at 22°C. How long will the cup take to cool to 30°C? ≈ 28.41 min

50. Newton's Law of Cooling A cake is removed from an oven at 350°F and cools to 120°F after 20 min in a room at 65°F. How long will the cake take to cool to 90°F?

51. Newton's Law of Cooling Experiment A thermometer is removed from a cup of coffee and placed in water with a temperature (T_m) of 10°C. The data in Table 3.24 were collected over the next 30 sec.

Table 3.24 Experimental Data

Time t	Temp T	$T - T_m$
2	80.47	70.47
5	69.39	59.39
10	49.66	39.66
15	35.26	25.26
20	28.15	18.15
25	23.56	13.56
30	20.62	10.62

(a) Draw a scatter plot of the data $T - T_m$.

(b) Find an exponential regression equation for the $T - T_m$ data. Superimpose its graph on the scatter plot.

(c) Estimate the thermometer reading when it was removed from the coffee. 89.47°C

52. Newton's Law of Cooling Experiment A thermometer was removed from a cup of hot chocolate and placed in water with temperature $T_m = 0$°C. The data in Table 3.25 were collected over the next 30 sec.

(a) Draw a scatter plot of the data $T - T_m$.

(b) Find an exponential regression equation for the $T - T_m$ data. Superimpose its graph on the scatter plot.

(c) Estimate the thermometer reading when it was removed from the hot chocolate. 79.96°C

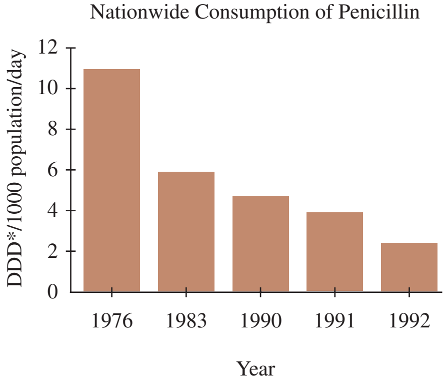
Table 3.25 Experimental Data

Time t	Temp T	$T - T_m$
2	74.68	74.68
5	61.99	61.99
10	34.89	34.89
15	21.95	21.95
20	15.36	15.36
25	11.89	11.89
30	10.02	10.02

53. Penicillin Use The use of penicillin became so widespread in the 1980s in Hungary that it became practically useless against common sinus and ear infections. Now the use of more effective antibiotics has caused a decline in penicillin resistance. The bar graph shows the use of penicillin in Hungary for selected years.

(a) From the bar graph we read the data pairs to be approximately (1, 11), (8, 6), (15, 4.8), (16, 4), and (17, 2.5), using $t = 1$ for 1976, $t = 8$ for 1983, and so on. Complete a scatter plot for these data.

(b) Writing to Learn Discuss whether the bar graph shown or the scatter plot that you completed best represents the data and why.



*Defined Daily Dose
Source: Science, vol. 264, April 15, 1994, American Association for the Advancement of Science.

54. Writing to Learn Which regression model would you use for the data in Exercise 53? Discuss various options, and explain why you chose the model you did. Support your writing with tables and graphs as needed.

Writing to Learn In Exercises 55–58, tables of (x, y) data pairs are given. Determine whether a linear, logarithmic, exponential, or power regression equation is the best model for the data. Explain your choice. Support your writing with tables and graphs as needed.

55.

x	1	2	3	4
y	3	4.4	5.2	5.8

56.

x	1	2	3	4
y	6	18	54	162

57.

x	1	2	3	4
y	3	6	12	24

58.

x	1	2	3	4
y	5	7	9	11

Standardized Test Questions

59. True or False The order of magnitude of a positive number is its natural logarithm. Justify your answer.

60. True or False According to Newton's Law of Cooling, an object will approach the temperature of the medium that surrounds it. Justify your answer.

In Exercises 61–64, solve the problem without using a calculator.

61. Multiple Choice Solve $2^{3x-1} = 32$. B

(A) $x = 1$ **(B)** $x = 2$ **(C)** $x = 4$

(D) $x = 11$ **(E)** $x = 13$

62. Multiple Choice Solve $\ln x = -1$. B

(A) $x = -1$ **(B)** $x = 1/e$ **(C)** $x = 1$

(D) $x = e$ **(E)** No solution is possible.

63. Multiple Choice How many times more severe was the 2001 earthquake in Arequipa, Peru ($R_1 = 8.1$) than the 1998 double earthquake in Takhar province, Afghanistan ($R_2 = 6.1$)? E

(A) 2 **(B)** 6.1 **(C)** 8.1

(D) 14.2 **(E)** 100

64. Multiple Choice Newton's Law of Cooling is A

(A) an exponential model. **(B)** a linear model.

(C) a logarithmic model. **(D)** a logistic model.

(E) a power model.

Explorations

In Exercises 65 and 66, use the data in Table 3.26. Determine whether a linear, logarithmic, exponential, power, or logistic regression equation is the best model for the data. Explain your choice. Support your writing with tables and graphs as needed.

Table 3.26 Populations of Two U.S. States (in thousands)

Year	Alaska	Hawaii
1900	63.6	154
1910	64.4	192
1920	55.0	256
1930	59.2	368
1940	72.5	423
1950	128.6	500
1960	226.2	633
1970	302.6	770
1980	401.9	965
1990	550.0	1108
2000	626.9	1212

Source: U.S. Census Bureau.

65. Writing to Learn Modeling Population Which regression equation is the best model for Alaska's population? logistic regression

66. Writing to Learn Modeling Population Which regression equation is the best model for Hawaii's population? logistic regression

67. Group Activity Normal Distribution The function

$$f(x) = k \cdot e^{-cx^2},$$

where c and k are positive constants, is a bell-shaped curve that is useful in probability and statistics.

(a) Graph f for $c = 1$ and $k = 0.1, 0.5, 1, 2, 10$. Explain the effect of changing k. As k increases, the bell curve stretches vertically.

(b) Graph f for $k = 1$ and $c = 0.1, 0.5, 1, 2, 10$. Explain the effect of changing c. As c increases, the bell curve compresses horizontally.

Extending the Ideas

68. Writing to Learn Prove if $u/v = 10^n$ for $u > 0$ and $v > 0$, then $\log u - \log v = n$. Explain how this result relates to powers of ten and orders of magnitude.

69. Potential Energy The potential energy E (the energy stored for use at a later time) between two ions in a certain molecular structure is modeled by the function

$$E = -\frac{5.6}{r} + 10e^{-r/3}$$

where r is the distance separating the nuclei.

(a) Writing to Learn Graph this function in the window $[-10, 10]$ by $[-10, 30]$, and explain which portion of the graph does not represent this potential energy situation.

(b) Identify a viewing window that shows that portion of the graph (with $r \le 10$) which represents this situation, and find the maximum value for E.

70. In Example 8, the Newton's Law of Cooling model was

$$T(t) - T_m = (T_0 - T_m)e^{-kt} = 61.656 \times 0.92770^t$$

Determine the value of k. $k \approx 0.075$

71. Justify the conclusion made about natural logarithmic regression on page 329.

72. Justify the conclusion made about power regression on page 329.

In Exercises 73–78, solve the equation or inequality.

73. $e^x + x = 5$ $x \approx 1.3066$

74. $e^{2x} - 8x + 1 = 0$ $x \approx 0.4073$ or $x \approx 0.9333$

75. $e^x < 5 + \ln x$ $0 < x < 1.7115$ (approx.)

76. $\ln |x| - e^{2x} \ge 3$ $x \le -20.0855$ (approx.)

77. $2 \log x - 4 \log 3 > 0$ $x > 9$

78. $2 \log (x + 1) - 2 \log 6 < 0$ $-1 < x < 5$

3.6 Mathematics of Finance

What you'll learn about

- Interest Compounded Annually
- Interest Compounded k Times per Year
- Interest Compounded Continuously
- Annual Percentage Yield
- Annuities—Future Value
- Loans and Mortgages—Present Value

. . . and why

The mathematics of finance is the science of letting your money work for you—valuable information indeed!

OBJECTIVE

Students will be able to use exponential functions and equations to solve business and finance applications related to compound interest and annuities.

MOTIVATE

Ask. . .
How might you determine the interest rate necessary to double your money within 8 years? **(Solve an equation such as $2 = (1 + r)^t$ or $2 = e^{rt}$.)**

TEACHING NOTE

Have students compare Table 3.27 with the table on page 290.

LESSON GUIDE

Day 1: Interest Compounded Annually; Interest Compounded k Times per Year; Interest Compounded Continuously
Day 2: Annual Percentage Yield; Annuities—Future Value; Loans and Mortgages—Present Value

Interest Compounded Annually

In business, as the saying goes, "time is money." We must pay interest for the use of property or money over time. When we borrow money, we pay interest, and when we loan money, we receive interest. When we invest in a savings account, we are actually lending money to the bank.

Suppose a principal of P dollars is invested in an account bearing an interest rate r expressed in decimal form and calculated at the end of each year. If A_n represents the total amount in the account at the end of n years, then the value of the investment follows the growth pattern shown in Table 3.27.

Table 3.27 Interest Computed Annually

Time in years	Amount in the account
0	$A_0 = P =$ principal
1	$A_1 = P + P \cdot r = P(1 + r)$
2	$A_2 = A_1 \cdot (1 + r) = P(1 + r)^2$
3	$A_3 = A_2 \cdot (1 + r) = P(1 + r)^3$
⋮	⋮
n	$A = A_n = P(1 + r)^n$

Notice that this is the constant percentage growth pattern studied in Section 3.2, and so the value of an investment is an exponential function of time. We call interest computed in this way **compound interest** because the interest becomes part of the investment, so that interest is earned on the interest itself.

Interest Compounded Annually

If a principal P is invested at a fixed annual interest rate r, calculated at the end of each year, then the value of the investment after n years is

$$A = P(1 + r)^n,$$

where r is expressed as a decimal.

EXAMPLE 1 Compounding Annually

Suppose Quan Li invests \$500 at 7% interest compounded annually. Find the value of her investment 10 years later.

SOLUTION Letting $P = 500$, $r = 0.07$, and $n = 10$,

$A = 500(1 + 0.07)^{10} = 983.575. \ldots$ Rounding to the nearest cent, we see that the value of Quan Li's investment after 10 years is \$983.58.

Now try Exercise 1.

NOTES ON EXAMPLES

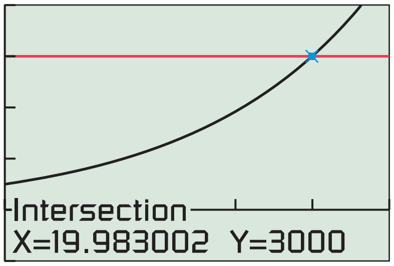
To illustrate the growth that leads to the answer in Example 1, enter 500 on the computation screen of your grapher (key in the number followed by ENTER or EXE). The grapher should return the answer of 500. Then key in ANS * 1.07 and press ENTER or EXE ten times, one press for each year of growth. Each time you press the ENTER or EXE key, a new line of text should appear. The screen should eventually show a list of values that culminates in 983.575. . . . This progression of values is a geometric sequence with first term 500 and common ratio 1.07.

TEACHING NOTE

The ideas of business finance are important in our society. In this section, you may need to teach financial concepts as well as mathematical ideas. This section provides many useful real-world applications.

TEACHING NOTE

Developing the concepts of compound interest and continuous interest can be enhanced by using the large-screen display of a grapher, along with its recursive, replay, and editing capabilities. Begin by setting the decimal display to two places to show answers in dollars and cents.



[0, 25] by [–1000, 4000]

FIGURE 3.41 Graph for Example 3.

NOTES ON EXAMPLES

In Examples 3 and 4, when we "confirm algebraically," the last step is actually numerical, so the check is neither purely algebraic nor completely rigorous.

Interest Compounded *k* Times per Year

Suppose a principal P is invested at an annual interest rate r compounded k times a year for t years. Then r/k is the interest rate per compounding period, and kt is the number of compounding periods. The amount A in the account after t years is

$$A = P\left(1 + \frac{r}{k}\right)^{kt}.$$

EXAMPLE 2 Compounding Monthly

Suppose Roberto invests \$500 at 9% annual interest *compounded monthly*, that is, compounded 12 times a year. Find the value of his investment 5 years later.

SOLUTION Letting $P = 500$, $r = 0.09$, $k = 12$, and $t = 5$,

$$A = 500\left(1 + \frac{0.09}{12}\right)^{12(5)} = 782.840\ldots.$$

So the value of Roberto's investment after 5 years is \$782.84.

Now try Exercise 5.

The problems in Examples 1 and 2 required that we calculate A. Examples 3 and 4 illustrate situations that require us to determine the values of other variables in the compound interest formula.

EXAMPLE 3 Finding the Time Period of an Investment

Judy has \$500 to invest at 9% annual interest compounded monthly. How long will it take for her investment to grow to \$3000?

SOLUTION

Model Let $P = 500$, $r = 0.09$, $k = 12$, and $A = 3000$ in the equation

$$A = P\left(1 + \frac{r}{k}\right)^{kt},$$

and solve for t.

Solve Graphically For

$$3000 = 500\left(1 + \frac{0.09}{12}\right)^{12t},$$

we let

$$f(t) = 500\left(1 + \frac{0.09}{12}\right)^{12t} \quad \text{and} \quad y = 3000,$$

and then find the point of intersection of the graphs. Figure 3.41 shows that this occurs at $t \approx 19.98$.

continued

Confirm Algebraically

$$3000 = 500(1 + 0.09/12)^{12t}$$

$$6 = 1.0075^{12t} \qquad \text{Divide by 500.}$$

$$\ln 6 = \ln (1.0075^{12t}) \qquad \text{Apply ln to each side.}$$

$$\ln 6 = 12t \ln (1.0075) \qquad \text{Power rule}$$

$$t = \frac{\ln 6}{12 \ln 1.0075} \qquad \text{Divide by 12 ln 1.0075.}$$

$$= 19.983\ldots \qquad \text{Calculate.}$$

Interpret So it will take Judy 20 years for the value of the investment to reach (and slightly exceed) $3000. *Now try Exercise 21.*

EXAMPLE 4 Finding an Interest Rate

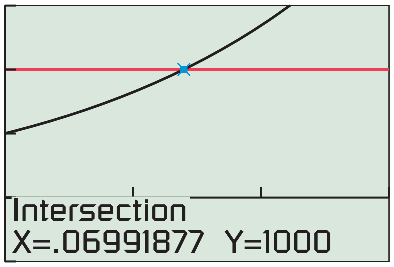
Stephen has $500 to invest. What annual interest rate *compounded quarterly* (four times per year) is required to double his money in 10 years?

SOLUTION

Model Letting $P = 500$, $k = 4$, $t = 10$, and $A = 1000$ yields the equation

$$1000 = 500\left(1 + \frac{r}{4}\right)^{4(10)}$$

that we solve for r.



[0, 0.15] by [–500, 1500]

FIGURE 3.42 Graph for Example 4.

Solve Graphically Figure 3.42 shows that $f(r) = 500(1 + r/4)^{40}$ and $y = 1000$ intersect at $r \approx 0.0699$, or $r = 6.99\%$.

Interpret Stephen's investment of $500 will double in 10 years at an annual interest rate of 6.99% compounded quarterly. *Now try Exercise 25.*

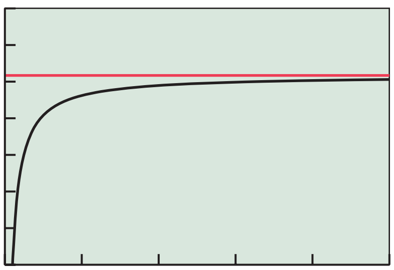
Interest Compounded Continuously

In Exploration 1, $1000 is invested for 1 year at a 10% interest rate. We investigate the value of the investment at the end of 1 year as the number of compounding periods k increases. In other words, we determine the "limiting" value of the expression $1000(1 + 0.1/k)^k$ as k assumes larger and larger integer values.

EXPLORATION 1 **Increasing the Number of Compounding Periods Boundlessly**

Let $A = 1000\left(1 + \frac{0.1}{k}\right)^k$.

1. Complete a table of values of A for $k = 10, 20, \ldots, 100$. What pattern do you observe?
2. Figure 3.43 shows the graphs of the function $A(k) = 1000(1 + 0.1/k)^k$ and the horizontal line $y = 1000e^{0.1}$. Interpret the meanings of these graphs.



[0, 50] by [1100, 1107]

FIGURE 3.43 Graph for Exploration 1.

EXPLORATION EXTENSIONS

Repeat the Exploration for an interest rate of 5%. Use an appropriate function for A, and use $y = 1000e^{0.05}$.

Recall from Section 3.1 that $e = \lim_{x\to\infty}(1 + 1/x)^x$. Therefore, for a fixed interest rate r, if we let $x = k/r$,

$$\lim_{k\to\infty}\left(1 + \frac{r}{k}\right)^{k/r} = e.$$

We do not know enough about limits yet, but with some calculus, it can be proved that $\lim_{k\to\infty} P(1 + r/k)^{kt} = Pe^{rt}$. So $A = Pe^{rt}$ is the formula used when interest is **compounded continuously**. In nearly any situation, one of the following two formulas can be used to compute compound interest:

Compound Interest—Value of an Investment

Suppose a principal P is invested at a fixed annual interest rate r. The value of the investment after t years is

- $A = P\left(1 + \frac{r}{k}\right)^{kt}$ when interest compounds k times per year,
- $A = Pe^{rt}$ when interest compounds continuously.

EXAMPLE 5 Compounding Continuously

Suppose LaTasha invests \$100 at 8% annual interest compounded continuously. Find the value of her investment at the end of each of the years 1, 2, . . . , 7.

SOLUTION Substituting into the formula for continuous compounding, we obtain $A(t) = 100e^{0.08t}$. Figure 3.44 shows the values of $y_1 = A(x) = 100e^{0.08x}$ for $x = 1, 2, \ldots, 7$. For example, the value of her investment is \$149.18 at the end of 5 years, and \$175.07 at the end of 7 years. *Now try Exercise 9.*

X	Y1	
1	108.33	
2	117.35	
3	127.12	
4	137.71	
5	149.18	
6	161.61	
7	175.07	

Y1 = 100e^(0.08X)

FIGURE 3.44 Table of values for Example 5.

Annual Percentage Yield

With so many different interest rates and methods of compounding it is sometimes difficult for a consumer to compare two different options. For example, would you prefer an investment earning 8.75% annual interest compounded quarterly or one earning 8.7% compounded monthly?

A common basis for comparing investments is the **annual percentage yield (APY)**—the percentage rate that, compounded annually, would yield the same return as the given interest rate with the given compounding period.

EXAMPLE 6 Computing Annual Percentage Yield (APY)

Ursula invests \$2000 with Crab Key Bank at 5.15% annual interest compounded quarterly. What is the equivalent APY?

SOLUTION Let $x =$ the equivalent APY. The value of the investment at the end of 1 year using this rate is $A = 2000(1 + x)$. Thus, we have

$$2000(1 + x) = 2000\left(1 + \frac{0.0515}{4}\right)^4$$

$$(1 + x) = \left(1 + \frac{0.0515}{4}\right)^4 \quad \text{Divide by 2000.}$$

$$x = \left(1 + \frac{0.0515}{4}\right)^4 - 1 \quad \text{Subtract 1.}$$

$$\approx 0.0525 \quad \text{Calculate.}$$

continued

TEACHING NOTE

In Exercise 57, students should discover that $APY = (1 + (r/k))^k - 1$. You may wish to give them this formula or have them derive it by generalizing Examples 6 and 7. The formula is included in the Chapter Review.

The annual percentage yield is 5.25%. In other words, Ursula's $2000 invested at 5.15% compounded quarterly for 1 year earns the same interest and yields the same value as $2000 invested elsewhere paying 5.25% interest once at the end of the year.

Now try Exercise 41.

Example 6 shows that the APY does not depend on the principal P because both sides of the equation were divided by $P = 2000$. So we can assume that $P = 1$ when comparing investments.

EXAMPLE 7 Comparing Annual Percentage Yields (APYs)

Which investment is more attractive, one that pays 8.75% compounded quarterly or another that pays 8.7% compounded monthly?

SOLUTION

Let

$$r_1 = \text{the APY for the 8.75\% rate,}$$

$$r_2 = \text{the APY for the 8.7\% rate.}$$

$$1 + r_1 = \left(1 + \frac{0.0875}{4}\right)^4 \qquad 1 + r_2 = \left(1 + \frac{0.087}{12}\right)^{12}$$

$$r_1 = \left(1 + \frac{0.0875}{4}\right)^4 - 1 \qquad r_2 = \left(1 + \frac{0.087}{12}\right)^{12} - 1$$

$$\approx 0.09041 \qquad \approx 0.09055$$

The 8.7% rate compounded monthly is more attractive because its APY is 9.055% compared with 9.041% for the 8.75% rate compounded quarterly.

Now try Exercise 45.

Annuities—Future Value

So far, in all of the investment situations we have considered, the investor has made a single *lump-sum* deposit. But suppose an investor makes regular deposits monthly, quarterly, or yearly—the same amount each time. This is an *annuity* situation.

An **annuity** is a sequence of equal periodic payments. The annuity is **ordinary** if deposits are made at the end of each period at the same time the interest is posted in the account. Figure 3.45 represents this situation graphically. We will consider only ordinary annuities in this textbook.

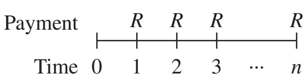


FIGURE 3.45 Payments into an ordinary annuity.

Let's consider an example. Suppose Sarah makes quarterly $500 payments at the end of each quarter into a retirement account that pays 8% interest compounded quarterly. How much will be in Sarah's account at the end of the first year? Notice the pattern.

End of Quarter 1:

$$\$500 = \$500$$

End of Quarter 2:

$$\$500 + \$500(1.02) = \$1010$$

End of Quarter 3:

$$\$500 + \$500(1.02) + \$500(1.02)^2 = \$1530.20$$

End of the year:

$$\$500 + \$500(1.02) + \$500(1.02)^2 + \$500(1.02)^3 \approx \$2060.80$$

Thus the total value of the investment returned from an annuity consists of all the periodic payments together with all the interest. This value is called the **future value** of the annuity because it is typically calculated when projecting into the future.

Future Value of an Annuity

The future value FV of an annuity consisting of n equal periodic payments of R dollars at an interest rate i per compounding period (payment interval) is

$$FV = R\frac{(1+i)^n - 1}{i}.$$

TEACHING NOTES

Show students how an investment in an annuity can accumulate over a 40-year period if they were to begin making monthly deposits of $50 the first month after graduating from high school. Use several different interest rates.

You may wish to point out that in the future and present value formulas, $i = r/k$ and $n = kt$ using the notation of compound interest.

NOTES ON EXAMPLES

You can challenge your students using Example 8. Have them keep FV and R constant and try to solve for i (the quarterly interest rate) using an algebraic method. Then have them graph FV as a function of i by letting the variable x replace i in the formula.

EXAMPLE 8 Calculating the Value of an Annuity

At the end of each quarter year, Emily makes a $500 payment into the Lanaghan Mutual Fund. If her investments earn 7.88% annual interest compounded quarterly, what will be the value of Emily's annuity in 20 years?

SOLUTION Let $R = 500$, $i = 0.0788/4$, $n = 20(4) = 80$. Then,

$$FV = R\frac{(1+i)^n - 1}{i}$$

$$FV = 500 \cdot \frac{(1 + 0.0788/4)^{80} - 1}{0.0788/4}$$

$$FV = 95{,}483.389...$$

So the value of Emily's annuity in 20 years will be $95,483.39.

Now try Exercise 13.

Loans and Mortgages—Present Value

An annuity is a sequence of equal period payments. The net amount of money put into an annuity is its **present value**. The net amount returned from the annuity is its future value. The periodic and equal payments on a loan or mortgage actually constitute an annuity.

How does the bank determine what the periodic payments should be? It considers what would happen to the present value of an investment with interest compounding over the term of the loan and compares the result to the future value of the loan repayment annuity.

We illustrate this reasoning by assuming that a bank lends you a present value $PV =$ $50,000 at 6% to purchase a house with the expectation that you will make a mortgage payment each month (at the monthly interest rate of $0.06/12 = 0.005$).

- The future value of an investment at 6% compounded monthly for n months is

$$PV(1+i)^n = 50{,}000(1 + 0.005)^n.$$

TEACHING NOTE

Have students investigate the current rates of interest available for financing a new automobile. Let students determine what auto they want to purchase and how much they would need to finance through a loan. Students should use the formulas from this section to calculate the monthly payment necessary to amortize the loan over a fixed number of months.

Sample loans for houses may also be investigated. It is interesting to calculate the interest paid for a 20 year or 30 year loan on $100,000.

FOLLOW-UP

Ask students how the interest rate affects the present and future values of an annuity. (**A higher interest rate gives a lower present value and a higher future value.**)

ASSIGNMENT GUIDE

Day 1: Ex. 1–12, 21–39, multiples of 3
Day 2: Ex. 13, 15, 17, 19, 45, 46, 48, 50, 52, 53, 55, 58, 60, 68, 69

COOPERATIVE LEARNING

Group Activity: Ex. 59

NOTES ON EXERCISES

The exercises in this section should be interesting to students because they deal with real-life financial situations. Students can apply these methods to their own financial planning.
Ex. 55–56 illustrate the results of making accelerated payments on a mortgage.
Ex. 61–66 provide practice for standardized tests.
Ex. 67 is a follow-up to Example 9.

ONGOING ASSESSMENT

Self-Assessment: Ex. 1, 5, 9, 13, 17, 21, 25, 41, 45
Embedded Assessment: Ex. 55, 57

• The future value of an annuity of R dollars (the loan payments) is

$$R\frac{(1+i)^n-1}{i} = R\frac{(1+0.005)^n-1}{0.005}.$$

To find R, we would solve the equation

$$50{,}000(1+0.005)^n = R\frac{(1+0.005)^n-1}{0.005}.$$

In general, the monthly payments of R dollars for a loan of PV dollars must satisfy the equation

$$PV(1+i)^n = R\frac{(1+i)^n-1}{i}.$$

Dividing both sides by $(1+i)^n$ leads to the following formula for the present value of an annuity.

Present Value of an Annuity

The present value PV of an annuity consisting of n equal payments of R dollars earning an interest rate i per period (payment interval) is

$$PV = R\frac{1-(1+i)^{-n}}{i}.$$

The annual interest rate charged on consumer loans is the **annual percentage rate (APR)**. The APY for the lender is higher than the APR. See Exercise 58.

EXAMPLE 9 Calculating Loan Payments

Carlos purchases a new pickup truck for $18,500. What are the monthly payments for a 4-year loan with a $2000 down payment if the annual interest rate (APR) is 2.9%?

SOLUTION

Model The down payment is $2000, so the amount borrowed is $16,500. Since APR = 2.9%, $i = 0.029/12$ and the monthly payment is the solution to

$$16{,}500 = R\frac{1-(1+0.029/12)^{-4(12)}}{0.029/12}.$$

Solve Algebraically

$$R\left[1-\left(1+\frac{0.029}{12}\right)^{-4(12)}\right] = 16{,}500\left(\frac{0.029}{12}\right)$$

$$R = \frac{16{,}500(0.029/12)}{1-(1+0.029/12)^{-48}}$$

$$= 364.487\ldots$$

Interpret Carlos will have to pay $364.49 per month for 47 months, and slightly less the last month. *Now try Exercise 19.*

QUICK REVIEW 3.6

1. Find 3.5% of 200. 7
2. Find 2.5% of 150. 3.75
3. What is one-fourth of 7.25%? 1.8125%
4. What is one-twelfth of 6.5%? ≈ 0.5417%
5. 78 is what percent of 120? 65%
6. 28 is what percent of 80? 35%
7. 48 is 32% of what number? 150
8. 176.4 is 84% of what number? 210
9. How much does Jane have at the end of 1 year if she invests $300 at 5% simple interest? $315
10. How much does Reggie have at the end of 1 year if he invests $500 at 4.5% simple interest? $522.50

SECTION 3.6 EXERCISES

In Exercises 1–4, find the amount A accumulated after investing a principal P for t years at an interest rate r compounded annually.

1. $P = \$1500, r = 7\%, t = 6$ $2251.10
2. $P = \$3200, r = 8\%, t = 4$ $4353.56
3. $P = \$12{,}000, r = 7.5\%, t = 7$ $19,908.59
4. $P = \$15{,}500, r = 9.5\%, t = 12$ $46,057.58

In Exercises 5–8, find the amount A accumulated after investing a principal P for t years at an interest rate r compounded k times per year.

5. $P = \$1500, r = 7\%, t = 5, k = 4$ $2122.17
6. $P = \$3500, r = 5\%, t = 10, k = 4$ $5752.67
7. $P = \$40{,}500, r = 3.8\%, t = 20, k = 12$ $86,496.26
8. $P = \$25{,}300, r = 4.5\%, t = 25, k = 12$ $77,765.69

In Exercises 9–12, find the amount A accumulated after investing a principal P for t years at interest rate r compounded continuously.

9. $P = \$1250, r = 5.4\%, t = 6$ $1728.31
10. $P = \$3350, r = 6.2\%, t = 8$ $5501.17
11. $P = \$21{,}000, r = 3.7\%, t = 10$ $30,402.43
12. $P = \$8{,}875, r = 4.4\%, t = 25$ $26,661.97

In Exercises 13–15, find the future value FV accumulated in an annuity after investing periodic payments R for t years at an annual interest rate r, with payments made and interest credited k times per year.

13. $R = \$500, r = 7\%, t = 6, k = 4$ $14,755.51
14. $R = \$300, r = 6\%, t = 12, k = 4$ $20,869.57
15. $R = \$450, r = 5.25\%, t = 10, k = 12$ $70,819.63
16. $R = \$610, r = 6.5\%, t = 25, k = 12$ $456,790.28

In Exercises 17 and 18, find the present value PV of a loan with an annual interest rate r and periodic payments R for a term of t years, with payments made and interest charged 12 times per year.

17. $r = 4.7\%, R = \$815.37, t = 5$ $43,523.31
18. $r = 6.5\%, R = \$1856.82, t = 30$ $293,769.01

In Exercises 19 and 20, find the periodic payment R of a loan with present value PV and an annual interest rate r for a term of t years, with payments made and interest charged 12 times per year.

19. $PV = \$18{,}000, r = 5.4\%, t = 6$ $293.24
20. $PV = \$154{,}000, r = 7.2\%, t = 15$ $1401.47
21. **Finding Time** If John invests $2300 in a savings account with a 9% interest rate compounded quarterly, how long will it take until John's account has a balance of $4150?
22. **Finding Time** If Joelle invests $8000 into a retirement account with a 9% interest rate compounded monthly, how long will it take until this single payment has grown in her account to $16,000?
23. **Trust Officer** Megan is the trust officer for an estate. If she invests $15,000 into an account that carries an interest rate of 8% compounded monthly, how long will it be until the account has a value of $45,000 for Megan's client?
24. **Chief Financial Officer** Willis is the financial officer of a private university with the responsibility for managing an endowment. If he invests $1.5 million at an interest rate of 8% compounded quarterly, how long will it be until the account exceeds $3.75 million?
25. **Finding the Interest Rate** What interest rate compounded daily (365 days/year) is required for a $22,000 investment to grow to $36,500 in 5 years? ≈ 10.13%
26. **Finding the Interest Rate** What interest rate compounded monthly is required for an $8500 investment to triple in 5 years? ≈ 22.17%
27. **Pension Officer** Jack is an actuary working for a corporate pension fund. He needs to have $14.6 million grow to $22 million in 6 years. What interest rate compounded annually does he need for this investment? 7.07%
28. **Bank President** The president of a bank has $18 million in his bank's investment portfolio that he wants to grow to $25 million in 8 years. What interest rate compounded annually does he need for this investment? ≈ 4.19%

29. Doubling Your Money Determine how much time is required for an investment to double in value if interest is earned at the rate of 5.75% compounded quarterly.

30. Tripling Your Money Determine how much time is required for an investment to triple in value if interest is earned at the rate of 6.25% compounded monthly.

In Exercises 31–34, complete the table about continuous compounding.

	Initial Investment	APR	Time to Double	Amount in 15 years
31.	\$12,500	9%	?	?
32.	\$32,500	8%	?	?
33.	\$ 9,500	?	4 years	?
34.	\$16,800	?	6 years	?

In Exercises 35–40, complete the table about doubling time of an investment.

	APR	Compounding Periods	Time to Double
35.	4%	Quarterly	?
36.	8%	Quarterly	?
37.	7%	Annually	?
38.	7%	Quarterly	?
39.	7%	Monthly	?
40.	7%	Continuously	?

In Exercises 41–44, find the annual percentage yield (APY) for the investment.

41. \$3000 at 6% compounded quarterly ≈ 6.14%

42. \$8000 at 5.75% compounded daily ≈ 5.92%

43. *P* dollars at 6.3% compounded continuously ≈ 6.50%

44. *P* dollars at 4.7% compounded monthly ≈ 4.80%

45. Comparing Investments Which investment is more attractive, 5% compounded monthly or 5.1% compounded quarterly? 5.1% quarterly

46. Comparing Investments Which investment is more attractive, $5\frac{1}{8}\%$ compounded annually or 5% compounded continuously? 5% continuously

In Exercises 47–50, payments are made and interest is credited at the end of each month.

47. An IRA Account Amy contributes \$50 per month into the Lincoln National Bond Fund that earns 7.26% annual interest. What is the value of Amy's investment after 25 years? \$42,211.46

48. An IRA Account Andrew contributes \$50 per month into the Hoffbrau Fund that earns 15.5% annual interest. What is the value of his investment after 20 years? \$80,367.73

49. An Investment Annuity Jolinda contributes to the Celebrity Retirement Fund that earns 12.4% annual interest. What should her monthly payments be if she wants to accumulate \$250,000 in 20 years? \$239.41 per month

50. An Investment Annuity Diego contributes to a Commercial National money market account that earns 4.5% annual interest. What should his monthly payments be if he wants to accumulate \$120,000 in 30 years? \$158.02

51. Car Loan Payment What is Kim's monthly payment for a 4-year \$9000 car loan with an APR of 7.95% from Century Bank? \$219.51 per month

52. Car Loan Payment What is Ericka's monthly payment for a 3-year \$4500 car loan with an APR of 10.25% from County Savings Bank? \$145.73 per month

53. House Mortgage Payment Gendo obtains a 30-year \$86,000 house loan with an APR of 8.75% from National City Bank. What is her monthly payment? \$676.56

54. House Mortgage Payment Roberta obtains a 25-year \$100,000 house loan with an APR of 9.25% from NBD Bank. What is her monthly payment? \$856.38 per month

55. Mortgage Payment Planning An \$86,000 mortgage for 30 years at 12% APR requires monthly payments of \$884.61. Suppose you decided to make monthly payments of \$1050.00.

(a) When would the mortgage be completely paid?

(b) How much do you save with the greater payments compared with the original plan? \$137,859.60

56. Mortgage Payment Planning Suppose you make payments of \$884.61 for the \$86,000 mortgage in Exercise 53 for 10 years and then make payments of \$1050 until the loan is paid.

(a) When will the mortgage be completely paid under these circumstances? 22 years 2 months

(b) How much do you save with the greater payments compared with the original plan? \$59,006.40

57. Writing to Learn Explain why computing the APY for an investment does not depend on the actual amount being invested. Give a formula for the APY on a \$1 investment at annual rate *r* compounded *k* times a year. How do you extend the result to a \$1000 investment?

58. Writing to Learn Give reasons why banks might not announce their APY on a loan they would make to you at a given APR. What is the bank's APY on a loan that they make at 4.5% APR?

59. Group Activity Work in groups of three or four. Consider population growth of humans or other animals, bacterial growth, radioactive decay, and compounded interest. Explain how these problem situations are similar and how they are different. Give examples to support your point of view.

60. Simple Interest versus Compounding Annually Steve purchases a $1000 certificate of deposit and will earn 6% each year. The interest will be mailed to him, so he will not earn interest on his interest.

(a) Writing to Learn Explain why after t years, the total amount of interest he receives from his investment plus the original $1000 is given by

$$f(t) = 1000(1 + 0.06t).$$

(b) Steve invests another $1000 at 6% compounded annually. Make a table that compares the value of the two investments for $t = 1, 2, \dots, 10$ years.

Standardized Test Questions

61. True or False If $100 is invested at 5% annual interest for 1 year, there is no limit to the final value of the investment if it is compounded sufficiently often. Justify your answer.

62. True or False The total interest paid on a 15-year mortgage is less than half of the total interest paid on a 30-year mortgage with the same loan amount and APR. Justify your answer.

In Exercises 63–66, you may use a graphing calculator to solve the problem.

63. Multiple Choice What is the total value after 6 years of an initial investment of $2250 that earns 7% interest compounded quarterly? B

(A) $3376.64 **(B)** $3412.00 **(C)** $3424.41

(D) $3472.16 **(E)** $3472.27

64. Multiple Choice The annual percentage yield of an account paying 6% compounded monthly is C

(A) 6.03%. **(B)** 6.12%. **(C)** 6.17%.

(D) 6.20%. **(E)** 6.24%.

65. Multiple Choice Mary Jo deposits $300 each month into her retirement account that pays 4.5% APR (0.375% per month). Use the formula $FV = R((1 + i)^n - 1)/i$ to find the value of her annuity after 20 years. E

(A) $71,625.00

(B) $72,000.00

(C) $72,375.20

(D) $73,453.62

(E) $116,437.31

66. Multiple Choice To finance their home, Mr. and Mrs. Dass have agreed to a $120,000 mortgage loan at 7.25% APR. Use the formula $PV = R(1 - (1 + i)^{-n})/i$ to determine their monthly payments if the loan has a term of 15 years. A

(A) $1095.44

(B) $1145.44

(C) $1195.44

(D) $1245.44

(E) $1295.44

Explorations

67. Loan Payoff Use the information about Carlos's truck loan in Example 9 to make a spreadsheet of the payment schedule. The first few lines of the spreadsheet should look like the following table:

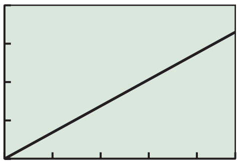
Month No.	Payment	Interest	Principal	Balance
0				$16,500.00
1	$364.49	$39.88	$324.61	$16,175.39
2	$364.49	$39.09	$325.40	$15,849.99

To create the spreadsheet successfully, however, you need to use formulas for many of the cells, as shown in boldface type in the following sample:

Month No.	Payment	Interest	Principal	Balance
0				$16,500.00
=A2+1	$364.49	**=round(E2*2.9%/12,2)**	**=B3–C3**	**=E2–D3**
=A3+1	$364.49	**=round(E3*2.9%/12,2)**	**=B4–C4**	**=E3–D4**

Continue the spreadsheet using copy-and-paste techniques, and determine the amount of the 48th and final payment so that the final balance is $0.00. $364.38

68. Writing to Learn Loan Payoff Which of the following graphs is an accurate graph of the loan balance as a function of time, based on Carlos's truck loan in Example 9 and Exercise 67? Explain your choice based on increasing or decreasing behavior and other analytical characteristics. Would you expect the graph of loan balance versus time for a 30-year mortgage loan at twice the interest rate to have the same shape or a different shape as the one for the truck loan? Explain.

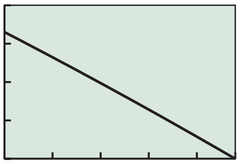


[0, 48] by [0, 20 000]

(a)

[0, 48] by [0, 20 000]

(b)



[0, 48] by [0, 20 000]

(c)

68. (c). Graph (c) shows the loan balance decreasing at a fairly steady rate over time. The graph of loan balance versus time for a 30-year mortgage at double the interest rate would start off nearly horizontal and get more steeply decreasing over time.

Extending the Ideas

69. The function

$$f(x) = 100\frac{(1 + 0.08/12)^x - 1}{0.08/12}$$

describes the future value of a certain annuity.

(a) What is the annual interest rate? 8%

(b) How many payments per year are there? 12

(c) What is the amount of each payment? \$100

70. The function

$$f(x) = 200\frac{1 - (1 + 0.08/12)^{-x}}{0.08/12}$$

describes the present value of a certain annuity.

(a) What is the annual interest rate? 8%

(b) How many payments per year are there? 12

(c) What is the amount of each payment? \$200

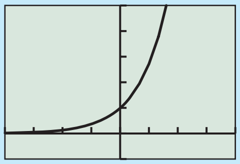
CHAPTER 3 Key Ideas

PROPERTIES, THEOREMS, AND FORMULAS

PROCEDURES

GALLERY OF FUNCTIONS

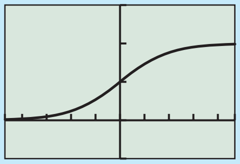
Exponential



[–4, 4] by [–1, 5]

$f(x) = e^x$

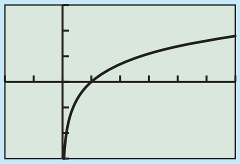
Basic Logistic



[–4.7, 4.7] by [–0.5, 1.5]

$$f(x) = \frac{1}{1 + e^{-x}}$$

Natural Logarithmic



[–2, 6] by [–3, 3]

$f(x) = \ln x$

CHAPTER 3 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

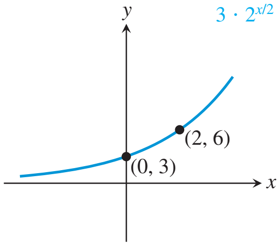
In Exercises 1 and 2, compute the exact value of the function for the given x value without using a calculator.

1. $f(x) = -3 \cdot 4^x$ for $x = \frac{1}{3}$ $-3\sqrt[3]{4}$

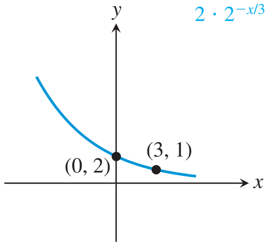
2. $f(x) = 6 \cdot 3^x$ for $x = -\frac{3}{2}$ $\frac{2}{\sqrt{3}}$

In Exercises 3 and 4, determine a formula for the exponential function whose graph is shown in the figure.

3.



4.



In Exercises 5–10, describe how to transform the graph of f into the graph of $g(x) = 2^x$ or $h(x) = e^x$. Sketch the graph by hand and support your answer with a grapher.

5. $f(x) = 4^{-x} + 3$ **6.** $f(x) = -4^{-x}$

7. $f(x) = -8^{-x} - 3$ **8.** $f(x) = 8^{-x} + 3$

9. $f(x) = e^{2x-3}$ **10.** $f(x) = e^{3x-4}$

In Exercises 11 and 12, find the y-intercept and the horizontal asymptotes.

11. $f(x) = \dfrac{100}{5 + 3e^{-0.05x}}$ **12.** $f(x) = \dfrac{50}{5 + 2e^{-0.04x}}$

In Exercises 13 and 14, state whether the function is an exponential growth function or an exponential decay function, and describe its end behavior using limits.

13. $f(x) = e^{4-x} + 2$ **14.** $f(x) = 2(5^{x-3}) + 1$

In Exercises 15–18, graph the function, and analyze it for domain, range, continuity, increasing or decreasing behavior, symmetry, boundedness, extrema, asymptotes, and end behavior.

15. $f(x) = e^{3-x} + 1$ **16.** $g(x) = 3(4^{x+1}) - 2$

17. $f(x) = \dfrac{6}{1 + 3 \cdot 0.4^x}$ **18.** $g(x) = \dfrac{100}{4 + 2e^{-0.01x}}$

In Exercises 19–22, find the exponential function that satisfies the given conditions.

19. Initial value = 24, increasing at a rate of 5.3% per day

20. Initial population = 67,000, increasing at a rate of 1.67% per year

21. Initial height = 18 cm, doubling every 3 weeks

22. Initial mass = 117 g, halving once every 262 hours

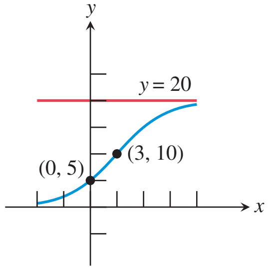
In Exercises 23 and 24, find the logistic function that satisfies the given conditions.

23. Initial value = 12, limit to growth = 30, passing through (2, 20). $f(x) \approx 30/(1 + 1.5e^{-0.55x})$

24. Initial height = 6, limit to growth = 20, passing through (3, 15). $f(x) \approx 20/(1 + 2.33e^{-0.65x})$

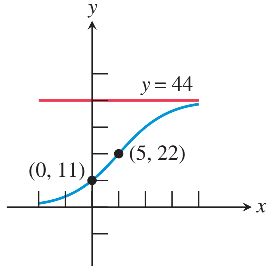
In Exercises 25 and 26, determine a formula for the logistic function whose graph is shown in the figure.

25.



In

26.



Exercises 27–30, evaluate the logarithmic expression without using a calculator.

27. $\log_2 32$ 5 **28.** $\log_3 81$ 4

29. $\log \sqrt[3]{10}$ 1/3 **30.** $\ln \dfrac{1}{\sqrt{e^7}}$ −7/2

20. $f(x) = 67{,}000 \cdot (1.0167)^x$

In Exercises 31–34, rewrite the equation in exponential form.

31. $\log_3 x = 5$ $3^5 = x$ **32.** $\log_2 x = y$ $2^y = x$

33. $\ln \dfrac{x}{y} = -2$ $y = xe^2$ **34.** $\log \dfrac{a}{b} = -3$ $b = 1000a$

In Exercises 35–38, describe how to transform the graph of $y = \log_2 x$ into the graph of the given function. Sketch the graph by hand and support with a grapher.

35. $f(x) = \log_2 (x + 4)$ **36.** $g(x) = \log_2 (4 - x)$

37. $h(x) = -\log_2 (x - 1) + 2$ **38.** $h(x) = -\log_2 (x + 1) + 4$

In Exercises 39–42, graph the function, and analyze it for domain, range, continuity, increasing or decreasing behavior, symmetry, boundedness, extrema, asymptotes, and end behavior.

39. $f(x) = x \ln x$ **40.** $f(x) = x^2 \ln x$

41. $f(x) = x^2 \ln |x|$ **42.** $f(x) = \dfrac{\ln x}{x}$

In Exercises 43–54, solve the equation.

43. $10^x = 4$ $\log 4 \approx 0.6021$ **44.** $e^x = 0.25$ $\ln 0.25 \approx -1.3863$

45. $1.05^x = 3$ ≈ 22.5171 **46.** $\ln x = 5.4$ $e^{5.4} \approx 221.4064$

47. $\log x = -7$ 0.0000001 **48.** $3^{x-3} = 5$ ≈ 4.4650

49. $3 \log_2 x + 1 = 7$ 4 **50.** $2 \log_3 x - 3 = 4$ ≈ 46.7654

51. $\dfrac{3^x - 3^{-x}}{2} = 5$ ≈ 2.1049 **52.** $\dfrac{50}{4 + e^{2x}} = 11$ ≈ -0.3031

53. $\log (x + 2) + \log (x - 1) = 4$ ≈ 99.5112

54. $\ln (3x + 4) - \ln (2x + 1) = 5$ ≈ -0.4915

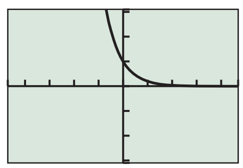
In Exercises 55 and 56, write the expression using only natural logarithms.

55. $\log_2 x$ $\ln x/\ln 2$ **56.** $\log_{1/6} (6x^2)$

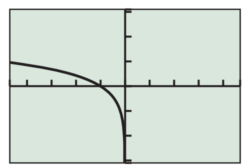
In Exercises 57 and 58, write the expression using only common logarithms.

57. $\log_5 x$ $\log x/\log 5$ **58.** $\log_{1/2} (4x^3)$

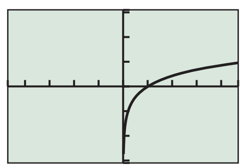
In Exercises 59–62, match the function with its graph. All graphs are drawn in the window $[-4.7, 4.7]$ by $[-3.1, 3.1]$.



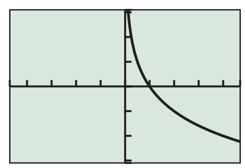
[–4.7, 4.7] by [–3.1, 3.1]
(a)



[–4.7, 4.7] by [–3.1, 3.1]
(b)



[–4.7, 4.7] by [–3.1, 3.1]
(c)



[–4.7, 4.7] by [–3.1, 3.1]
(d)

59. $f(x) = \log_5 x$ (c) **60.** $f(x) = \log_{0.5} x$ (d)

61. $f(x) = \log_5 (-x)$ (b) **62.** $f(x) = 5^{-x}$ (a)

63. Compound Interest Find the amount A accumulated after investing a principal $P = \$450$ for 3 years at an interest rate of 4.6% compounded annually. \$515.00

64. Compound Interest Find the amount A accumulated after investing a principal $P = \$4800$ for 17 years at an interest rate 6.2% compounded quarterly. \$13,660.81

65. Compound Interest Find the amount A accumulated after investing a principal P for t years at interest rate r compounded continuously. Pe^{rt}

66. Future Value Find the future value FV accumulated in an annuity after investing periodic payments R for t years at an annual interest rate r, with payments made and interest credited k times per year.

67. Present Value Find the present value PV of a loan with an annual interest rate $r = 5.5\%$ and periodic payments $R = \$550$ for a term of $t = 5$ years, with payments made and interest charged 12 times per year. \$28,794.06

68. Present Value Find the present value PV of a loan with an annual interest rate $r = 7.25\%$ and periodic payments $R = \$953$ for a term of $t = 15$ years, with payments made and interest charged 26 times per year. \$226,396.22

In Exercises 69 and 70, determine the value of k so that the graph of f passes through the given point.

69. $f(x) = 20e^{-kx}$, (3, 50) **70.** $f(x) = 20e^{-kx}$, (1, 30)

In Exercises 71 and 72, use the data in Table 3.28.

Table 3.28 Populations of Two U.S. States (in millions)

Year	Georgia	Illinois
1900	2.2	4.8
1910	2.6	5.6
1920	2.9	6.5
1930	2.9	7.6
1940	3.1	7.9
1950	3.4	8.7
1960	3.9	10.1
1970	4.6	11.1
1980	5.5	11.4
1990	6.5	11.4
2000	8.2	12.4

Source: U.S. Census Bureau as reported in the World Almanac and Book of Facts 2005.

71. Modeling Population Find an exponential regression model for Georgia's population, and use it to predict the population in 2005.

72. Modeling Population Find a logistic regression model for Illinois's population, and use it to predict the population in 2010.

73. Drug Absorption A drug is administered intravenously for pain. The function $f(t) = 90 - 52 \ln (1 + t)$, where $0 \le t \le 4$, gives the amount of the drug in the body after t hours.

(a) What was the initial ($t = 0$) number of units of drug administered? 90 units

(b) How much is present after 2 hr? 32.8722 units

(c) Draw the graph of f.

74. Population Decrease The population of Metroville is 123,000 and is decreasing by 2.4% each year.

(a) Write a function that models the population as a function of time t. $P(t) = 123{,}000(0.976)^t$

(b) Predict when the population will be 90,000. 12.86 years

75. Population Decrease The population of Preston is 89,000 and is decreasing by 1.8% each year.

(a) Write a function that models the population as a function of time t. $P(t) = 89{,}000(0.982)^t$

(b) Predict when the population will be 50,000. 31.74 years

76. Spread of Flu The number P of students infected with flu at Northridge High School t days after exposure is modeled by

$$P(t) = \frac{300}{1 + e^{4-t}}.$$

(a) What was the initial ($t = 0$) number of students infected with the flu? 5 or 6 students

(b) How many students were infected after 3 days? 80 or 81 students

(c) When will 100 students be infected? Sometime on the fourth day

(d) What would be the maximum number of students infected? 300

77. Rabbit Population The number of rabbits in Elkgrove doubles every month. There are 20 rabbits present initially.

(a) Express the number of rabbits as a function of the time t.

(b) How many rabbits were present after 1 year? after 5 years?

(c) When will there be 10,000 rabbits? $\approx$ 8.9658 months

78. Guppy Population The number of guppies in Susan's aquarium doubles every day. There are four guppies initially.

(a) Express the number of guppies as a function of time t.

(b) How many guppies were present after 4 days? after 1 week? 64; 512

(c) When will there be 2000 guppies? $\approx$ 8.9658 days

79. Radioactive Decay The half-life of a certain radioactive substance is 1.5 sec. The initial amount of substance is S_0 grams.

(a) Express the amount of substance S remaining as a function of time t. $S(t) = S_0 \cdot (1/2)^{t/1.5}$

(b) How much of the substance is left after 1.5 sec? after 3 sec?

(c) Determine S_0 if there was 1 g left after 1 min.

80. Radioactive Decay The half-life of a certain radioactive substance is 2.5 sec. The initial amount of substance is S_0 grams.

(a) Express the amount of substance S remaining as a function of time t. $S(t) = S_0 \cdot (1/2)^{t/2.5}$

(b) How much of the substance is left after 2.5 sec? after 7.5 sec?

(c) Determine S_0 if there was 1 g left after 1 min. 16,777.216 kg

79. (b) $S_0/2$; $S_0/4$ **80. (b)** $S_0/2$; $S_0/8$

81. Richter Scale Afghanistan suffered two major earthquakes in 1998. The one on February 4 had a Richter magnitude of 6.1, causing about 2300 deaths, and the one on May 30 measured 6.9 on the Richter scale, killing about 4700 people. How many times more powerful was the deadlier quake? 6.31

82. Chemical Acidity The pH of seawater is 7.6, and the pH of milk of magnesia is 10.5.

(a) What are their hydrogen-ion concentrations?

(b) How many times greater is the hydrogen-ion concentration of the seawater than that of milk of magnesia? 794.33

(c) By how many orders of magnitude do the concentrations differ? 2.9

83. Annuity Finding Time If Joenita invests \$1500 into a retirement account with an 8% interest rate compounded quarterly, how long will it take this single payment to grow to \$3750? 11.75 years

84. Annuity Finding Time If Juan invests \$12,500 into a retirement account with a 9% interest rate compounded continuously, how long will it take this single payment to triple in value? ≈ 12.2068 years

85. Monthly Payments The time t in months that it takes to pay off a \$60,000 loan at 9% annual interest with monthly payments of x dollars is given by

$$t = 133.83 \ln\left(\frac{x}{x - 450}\right).$$

Estimate the length (term) of the \$60,000 loan if the monthly payments are \$700. 137.7940 — about 11 years 6 months

86. Monthly Payments Using the equation in Exercise 85, estimate the length (term) of the \$60,000 loan if the monthly payments are \$500. about 25 years 9 months

87. Finding APY Find the annual percentage yield for an investment with an interest rate of 8.25% compounded monthly. ≈ 8.57%

88. Finding APY Find the annual percentage yield that can be used to advertise an account that pays interest at 7.20% compounded countinuously. ≈ 7.47%

89. Light Absorption The Beer-Lambert law of absorption applied to Lake Superior states that the light intensity I (in lumens) at a depth of x feet satisfies the equation

$$\log \frac{I}{12} = -0.0125x.$$

Find the light intensity at a depth of 25 ft. ≈ 5.84 lumens

90. For what values of b is $\log_b x$ a vertical stretch of $y = \ln x$? A vertical shrink of $y = \ln x$? $e^{-1} < b < e$; $0 < b < e^{-1}$ or $b > e$

91. For what values of b is $\log_b x$ a vertical stretch of $y = \log x$? A vertical shrink of $y = \log x$?

92. If $f(x) = ab^x$, $a > 0$, $b > 0$, prove that $g(x) = \ln f(x)$ is a linear function. Find its slope and y-intercept.

93. Spread of Flu The number of students infected with flu after t days at Springfield High School is modeled by the function

$$P(t) = \frac{1600}{1 + 99e^{-0.4t}}.$$

(a) What was the initial number of infected students? 16

(b) When will 800 students be infected? about $11\frac{1}{2}$ days

(c) The school will close when 400 of the 1600 student body are infected. When would the school close?

94. Population of Deer The population P of deer after t years in Briggs State Park is modeled by the function

$$P(t) = \frac{1200}{1 + 99e^{-0.4t}}.$$

(a) What was the inital population of deer? 12 deer

(b) When will there be 1000 deer? about $15\frac{1}{2}$ years

(c) What is the maximum number of deer planned for the park? 1200

95. Newton's Law of Cooling A cup of coffee cooled from 96°C to 65°C after 8 min in a room at 20°C. When will it cool to 25°C? ≈ 41.54 minutes

96. Newton's Law of Cooling A cake is removed from an oven at 220°F and cools to 150°F after 35 min in a room at 75°F. When will it cool to 95°F? ≈ 105.17 minutes

97. The function

$$f(x) = 100\frac{(1 + 0.09/4)^x - 1}{0.09/4}$$

describes the future value of a certain annuity.

(a) What is the annual interest rate? 9%

(b) How many payments per year are there? 4

(c) What is the amount of each payment? \$100

98. The function

$$g(x) = 200\frac{1 - (1 + 0.11/4)^{-x}}{0.11/4}$$

describes the present value of a certain annuity.

(a) What is the annual interest rate? 11%

(b) How many payments per year are there? 4

(c) What is the amount of each payment? \$200

99. Simple Interest versus Compounding Continuously Grace purchases a \$1000 certificate of deposit that will earn 5% each year. The interest will be mailed to her, so she will not earn interest on her interest.

(a) Show that after t years, the total amount of interest she receives from her investment plus the original \$1000 is given by

$$f(t) = 1000(1 + 0.05t).$$

(b) Grace invests another \$1000 at 5% compounded continuously. Make a table that compares the values of the two investments for $t = 1, 2, \ldots, 10$ years.

CHAPTER 3 Project

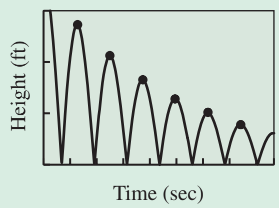
Analyzing a Bouncing Ball

When a ball bounces up and down on a flat surface, the maximum height of the ball decreases with each bounce. Each rebound is a percentage of the previous height. For most balls, the percentage is a constant. In this project, you will use a motion detection device to collect height data for a ball bouncing underneath a motion detector, then find a mathematical model that describes the maximum bounce height as a function of bounce number.

Collecting the Data

Set up the Calculator Based Laboratory (CBL™) system with a motion detector or a Calculator Based Ranger (CBR™) system to collect ball bounce data using a ball bounce program for the CBL or the Ball Bounce Application for the CBR. See the CBL/CBR guidebook for specific setup instruction.

Hold the ball at least 2 feet below the detector and release it so that it bounces straight up and down beneath the detector. These programs convert distance versus time data to height from the ground versus time. The graph shows a plot of sample data collected with a racquetball and CBR. The data table below shows each maximum height collected.



[0, 4.25] by [0, 3]

Bounce Number	Maximum Height (feet)
0	2.7188
1	2.1426
2	1.6565
3	1.2640
4	0.98309
5	0.77783

EXPLORATIONS

1. If you collected motion data using a CBL or CBR, a plot of height versus time should be shown on your graphing calculator or computer screen. Trace to the maximum height for each bounce and record your data in a table and use other lists in your calculator to enter this data. If you don't have access to a CBL/CBR, enter the data given in the table into your graphing calculator/computer.
2. Bounce height 1 is what percentage of bounce height 0? Calculate the percentage return for each bounce. The numbers should be fairly constant.
3. Create a scatter plot for maximum height versus bounce number.
4. For bounce 1, the height is predicted by multiplying bounce height 0, or H, by the percentage P. The second height is predicted by multiplying this height HP by P which gives the HP^2. Explain why $y = HP^x$ is the appropriate model for this data, where x is the bounce number.
5. Enter this equation into your calculator using your values for H and P. How does the model fit your data?
6. Use the statistical features of the calculator to find the exponential regression for this data. Compare it to the equation that you used as a model. $y \approx 2.733 \cdot 0.776^x$
7. How would your data and equation change if you used a different type of ball?
8. What factors would change the H value and what factors affect the P value?
9. Rewrite your equation using base e instead of using P as the base for the exponential equation.
10. What do you predict the graph of ln (bounce height) versus bounce number to look like? Linear
11. Plot ln (bounce height) versus bounce number. Calculate the linear regression and use the concept of logarithmic re-expression to explain how the slope and y-intercept are related to P and H.