7-4 Properties of Logarithms

TEKS 2A.2.A Foundations for functions: use tools including factoring and properties of exponents to simplify expressions and to transform and solve equations.

Objectives

Use properties to simplify logarithmic expressions.

Translate between logarithms in any base.

Who uses this?

Seismologists use properties of logarithms to calculate the energy released by earthquakes. (See Example 6.)

The logarithmic function for pH that you saw in the previous lesson, $\text{pH} = -\log[\text{H}^+]$, can also be expressed in exponential form, as $10^{-\text{pH}} = [\text{H}^+]$. Because logarithms are exponents, you can derive the properties of logarithms from the properties of exponents.

Remember that to *multiply* powers with the same base, you *add* exponents.

$$b^m b^n = b^{m+n}$$

Know it! Note

Product Property of Logarithms

For any positive numbers m, n, and b $(b \neq 1)$,

WORDS	NUMBERS	ALGEBRA
The logarithm of a product is equal to the sum of the logarithms of its factors.	$\log_3 1000 = \log_3(10 \cdot 100)$ $= \log_3 10 + \log_3 100$	$\log_b mn = \log_b m + \log_b n$

Helpful Hint

Think:
$\log j + \log a + \log m$
$= \log jam$

The property above can be used in reverse to write a sum of logarithms (exponents) as a single logarithm, which can often be simplified.

EXAMPLE 1 Adding Logarithms

Express as a single logarithm. Simplify, if possible.

A $\log_4 2 + \log_4 32$

$\log_4(2 \cdot 32)$	*To add the logarithms, multiply the numbers.*
$\log_4 64$	*Simplify.*
3	*Think: $4^? = 64$*

CHECK IT OUT! **Express as a single logarithm. Simplify, if possible.**

1a. $\log_5 625 + \log_5 25$ **1b.** $\log_{\frac{1}{3}} 27 + \log_{\frac{1}{3}} \frac{1}{9}$

Remember that to *divide* powers with the same base, you *subtract* exponents.

$$\frac{b^m}{b^n} = b^{m-n}$$

Because logarithms are exponents, subtracting logarithms with the same base is the same as finding the logarithm of the quotient with that base.

Know it! Note

Quotient Property of Logarithms

For any positive numbers m, n, and $b\ (b \neq 1)$,

WORDS	NUMBERS	ALGEBRA
The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.	$\log_5\left(\frac{16}{2}\right) = \log_5 16 - \log_5 2$	$\log_b \frac{m}{n} = \log_b m - \log_b n$

Caution!

Just as a^5b^3 cannot be simplified, logarithms must have the *same* base to be simplified.

The property above can also be used in reverse.

EXAMPLE 2 Subtracting Logarithms

Express $\log_2 32 - \log_2 4$ as a single logarithm. Simplify, if possible.

$\log_2 32 - \log_2 4$

$\log_2 (32 \div 4)$ — *To subtract the logarithms, divide the numbers.*

$\log_2 8$ — *Simplify.*

3 — *Think: $2^? = 8$*

CHECK IT OUT!

2. Express $\log_7 49 - \log_7 7$ as a single logarithm. Simplify, if possible.

Because you can multiply logarithms, you can also take powers of logarithms.

Know it! Note

Power Property of Logarithms

For any real number p and positive numbers a and $b\ (b \neq 1)$,

WORDS	NUMBERS	ALGEBRA
The logarithm of a power is the product of the exponent and the logarithm of the base.	$\log 10^3$ $\log(10 \cdot 10 \cdot 10)$ $\log 10 + \log 10 + \log 10$ $3\log 10$	$\log_b a^p = p\log_b a$

EXAMPLE 3 Simplifying Logarithms with Exponents

Express as a product. Simplify, if possible.

A $\log_3 81^2$

$2\log_3 81$

$2(4) = 8$ — *Because $3^4 = 81$, $\log_3 81 = 4$.*

B $\log_5 \left(\frac{1}{5}\right)^3$

$3\log_5 \frac{1}{5}$

$3(-1) = -3$ — *$5^{-1} = \frac{1}{5}$*

CHECK IT OUT!

Express as a product. Simplify, if possible.

3a. $\log 10^4$ **3b.** $\log_5 25^2$ **3c.** $\log_2 \left(\frac{1}{2}\right)^5$

Exponential and logarithmic operations undo each other since they are inverse operations.

Know it! Note

Inverse Properties of Logarithms and Exponents

For any base b such that $b > 0$ and $b \neq 1$,

ALGEBRA	EXAMPLE
$\log_b b^x = x$	$\log_{10} 10^7 = 7$
$b^{\log_b x} = x$	$10^{\log_{10} 2} = 2$

EXAMPLE 4 Recognizing Inverses

Simplify each expression.

A $\log_8 8^{3x+1}$

$\log_8 8^{3x+1}$

$3x + 1$

B $\log_5 125$

$\log_5 (5 \cdot 5 \cdot 5)$

$\log_5 5^3$

3

C $2^{\log_2 27}$

$2^{\log_2 27}$

27

CHECK IT OUT! **4a.** Simplify $\log 10^{0.9}$. **4b.** Simplify $2^{\log_2 (8x)}$.

Most calculators calculate logarithms only in base 10 or base e (see Lesson 7-6). You can change a logarithm in one base to a logarithm in another base with the following formula.

Know it! Note

Change of Base Formula

For $a > 0$ and $a \neq 1$ and any base b such that $b > 0$ and $b \neq 1$,

ALGEBRA	EXAMPLE
$\log_b x = \frac{\log_a x}{\log_a b}$	$\log_4 8 = \frac{\log_2 8}{\log_2 4}$

EXAMPLE 5 Changing the Base of a Logarithm

Evaluate $\log_4 8$.

Method 1 Change to base 10.

$$\log_4 8 = \frac{\log 8}{\log 4}$$

$$\approx \frac{0.0903}{0.602}$$ *Use a calculator.*

$$= 1.5$$ *Divide.*

Method 2 Change to base 2, because both 4 and 8 are powers of 2.

$$\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3}{2}$$

$$= 1.5$$

CHECK IT OUT! **5a.** Evaluate $\log_9 27$. **5b.** Evaluate $\log_8 16$.

Logarithmic scales are useful for measuring quantities that have a very wide range of values, such as the intensity (loudness) of a sound or the energy released by an earthquake.

EXAMPLE 6 *Geology Application*

Seismologists use the Richter scale to express the energy, or magnitude, of an earthquake. The Richter magnitude of an earthquake, M, is related to the energy released in ergs E shown by the formula $M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$.

In 1964, an earthquake centered at Prince William Sound, Alaska, registered a magnitude of 9.2 on the Richter scale. Find the energy released by the earthquake.

Helpful Hint

The Richter scale is logarithmic, so an increase of 1 corresponds to a release of 10 times as much energy.

$$9.2 = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$ *Substitute 9.2 for M.*

$$\left(\frac{3}{2}\right)9.2 = \log\left(\frac{E}{10^{11.8}}\right)$$ *Multiply both sides by $\frac{3}{2}$.*

$$13.8 = \log\left(\frac{E}{10^{11.8}}\right)$$ *Simplify.*

$$13.8 = \log E - \log 10^{11.8}$$ *Apply the Quotient Property of Logarithms.*

$$13.8 = \log E - 11.8$$ *Apply the Inverse Properties of Logarithms and Exponents.*

$$25.6 = \log E$$

$$10^{25.6} = E$$ *Given the definition of a logarithm, the logarithm is the exponent.*

$$3.98 \times 10^{25} = E$$ *Use a calculator to evaluate.*

The energy released by an earthquake with a magnitude of 9.2 is 3.98×10^{25} ergs.

CHECK IT OUT!

6. How many times as much energy is released by an earthquake with a magnitude of 9.2 than by an earthquake with a magnitude of 8?

THINK AND DISCUSS

1. Explain how to graph $y = \log_5 x$ on a calculator.

2. Tell how you could find $10^{25.6}$ in Example 6 by applying a law of exponents.

3. Describe what happens when you use the change-of-base formula, $\log_b x = \frac{\log_a x}{\log_a b}$, when $x = a$.

Know it! Note

4. GET ORGANIZED Copy and complete the graphic organizer. Use your own words to show related properties of exponents and logarithms.

Property of Exponents	Property of Logarithms