SOLVING SYSTEMS OF EQUATIONS FOR POINTS OF INTERSECTION

When any two curves are graphed on the same coordinate plane, any one of three following situations may occur.

* The curves intersect in a finite number of points.
* The curves do not intersect in any points.
* The curves coincide as one curve and intersect as an infinite number of points.

Whether two or more curves intersect or not, is an important topic in mathematics and leads to a study of matrices to determine if a set of problems contains multiple solutions. Over the next few units, we will examine techniques to find solutions to a “System of Equations”. At first, we will examine algebraic and graphing techniques. Later we’ll examine matrix algebra to find our answers.

A “**system of equations**” consists of one or more equations in one or more variables such that the solutions to each equation result in a solution to the system.

In previous mathematics courses, you learned three methods to find a solution to a system of equations. These methods are (1) Method of Variable (or quantity) Substitution, (2) Method of Elimination of Variables, and (3) Graphing methods.

In this unit we will review the first two methods briefly, and then examine graphing methods on the TI-83+ Graphing Calculator.

Method of Variable Substitution

The primary concept behind solving a system is that if the two equations intersect, then for those points of intersection, each equation shares the value of each variable in each equation at that point.

-Systems that intersect are said to be "consistent" at their points of intersection.

In the method of variable substitution, we exploit the concept of variable consistency and allow one variable to exist in both equations simultaneously to determine if there are values where the other variable also exists.

Example #1: Find the solution to the following system:

$$3x + y = 8$$
$$2x - 4y = 5$$

Step #1: Choose either equation and solve for one of the variables.

$$3x + y = 8$$
$$y = 8 - 3x$$

Step #2: Quantify this result, and substitute the expression into the other equation and solve for the remaining variable.

$$2x - 4(8 - 3x) = 5$$
$$2x - 32 + 12x = 5$$
$$14x - 32 = 5$$
$$x = \frac{37}{14}$$

Step #3: Choose either of the original equations and substitute the value of x to find the value for y.

$$3\left(\frac{37}{14}\right)+y=8$$
$$\frac{111}{14}+y=8$$
$$14\left(\frac{111}{14}\right)+14y=14\cdot 8$$
$$111+14y=112$$
$$14y=1$$
$$y=\frac{1}{14}$$

The solution to the system is ordered pair, $\left(\frac{37}{14},\ \frac{1}{14}\right)$.

Recall from previous math courses that original system above involved the equations of two lines. Therefore the coordinates $\left(\frac{37}{14},\ \frac{1}{14}\right)$ represents a point of intersection for those lines. We will graph these systems in another section of this unit.

Example #2: Use variable substitution to solve:

$$4x-3y=12$$
$$-12x+9y=-15$$

Step #1: Solve the second equation for "$3y$".

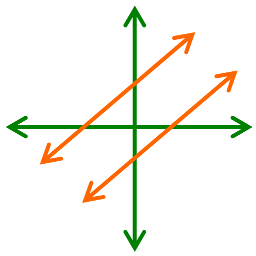
$$9y=12x-15$$
$$3y=4x-5$$

Step #2: Substitute this expression into the first equation.

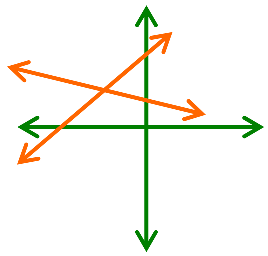
$$4x-(4x-5)=12$$
$$4x-4x+5=12$$
$$5=12$$

This result is false; therefore, there are no solutions to the system. A system with no solutions is called "Independent". Had this process resulted in a true statement without a variable in the result; such as: 5 = 5 or 12 = 12. There would be an infinite number of solutions to the system. We would call the system "Dependent".

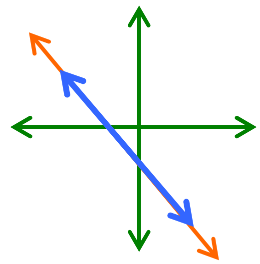
For linear equations, we can demonstrate the above three possibilities ("consistent", "independent", and "dependent") in the following graphs:



Independent



Consistent



Dependent

Method of Elimination of Variables

The idea behind the elimination of variables is to replace one or more of the original equations in the system with an equivalent equation in order to eliminate one variable by adding or subtracting the "two" equations. Elimination of variables is also important as a prelude to the study of matrices and their use in solving a system of equations.

In the second example found in the unit link, "Method of Variable Substitution", we avoided substituting a fractional variable expression into the other equation by substituting for "$3y$" and not simply "y". In many systems however this process is not possible.

In the method of elimination of variables, it is possible to avoid variable fractions and thus simplify the arithmetic. Although the answer to the system may be a fraction, the variables will not be if the steps are performed correctly.

To use the Elimination of Variables method the following rules are employed:

1. The order of equations may be changed:

$$\begin{matrix} 3x+4y=1 & \Rightarrow & 2x-6y=5 \\ 2x-6y=5 & & 3x+4y=1 \end{matrix}$$

2. Either or all equations may be multiplied by any non-variable real number:

$$\begin{matrix} 2x-6y=5 & \Rightarrow & 2(2x-6y=5) & \Rightarrow & 4x-12y=10 \\ 3x+4y=1 & \Rightarrow & 3(3x+4y=1) & \Rightarrow & 9x+12y=3 \end{matrix}$$

3. Any equation may be added to or subtracted from any other equation

$$\begin{array}{l} 4x-12y=10 \\ \underline{9x+12y=3} \\ 13x \qquad =13 \\ \qquad x=1 \end{array}$$

Note: Not all equations may not be added or subtracted to each other all in the same step.

- In a "two-equation" system, addition or subtraction may change only one equation.
- In a "thee-equation" system, only two equations may be changed and not all three in one step. The equation left unchanged must be the equation used to change the other two.
- In a "four-equation" system addition or subtraction may change only three equations.
- In a "five-equation" system, only four may be changed and so on.

Example #1: Find the solution to the following system of equations:

$$\begin{aligned} 3x+2y+4z&=1\\ 2x-2y+5z&=3\\ 5x+\ \ y+5z&=4 \end{aligned}$$

Step #1: Equation #2 will be added to equation #1 and subtracted from equation #3, but equation #2 must remain unchanged.

$$\begin{array}{lcl} 3x+2x+2y-2y+4z+5z=1+3 & & 5x+\ 0+9z=4\\ 2x \quad -2y \quad +5z \quad =3 & \Rightarrow & 2x-2y+5z=3\\ 5x-2x+y-(-2y)+5z-5z=4-3 & & 3x+3y+\ 0=1 \end{array}$$

Step #2: Equation #3 will be added to equation #2.

$$\begin{array}{lcl} 5x+\ 0+9z=4 & & 5x+\ 0+9z=4\\ 2x-2y+5z=3 & \Rightarrow & 5x+\ \ y+5z=4\\ 3x+3y+\ 0=1 & & 3x+3y+\ 0=1 \end{array}$$

Step #3: Equation #1 will be subtracted from equation #2.

$$\begin{array}{lcl} 5x+\ 0+9z=4 & & 5x+\ 0+9z=4\\ 5x+\ \ y+5z=4 & \Rightarrow & 0\ +y-4z=0\\ 3x+3y+\ 0=1 & & 3x+3y+\ 0=1 \end{array}$$

Step #4: Three times equation #2 will be subtracted from equation #3.

$$\begin{array}{lcl} 5x+\ 0+9z=4 & & 5x+\ 0+9z=4\\ 0\ +y-4z=0 & \Rightarrow & 0\ +y-4z=0\\ 3x+3y+\ 0=1 & & 3x+0+12z=1 \end{array}$$

Step #5: Divide equation #3 by 3.

$$\begin{array}{lcl} 5x+\ 0+9z=4 & & 5x+\ 0+9z=4\\ 0\ +y-4z=0 & \Rightarrow & 0\ +y-4z=0\\ 3x+3y+\ 0=1 & & x+0+4z=\frac{1}{3} \end{array}$$

Step #6: Subtract five times equation #3 from equation #1.

$$\begin{aligned} 5x + \ 0 + 9z &= 4 \\ 0 \ + y - 4z &= 0 \\ x + 0 + 4z &= \frac{1}{3} \end{aligned} \quad \Rightarrow \quad \begin{aligned} 0 + \ 0 - 11z &= \frac{7}{3} \\ 0 + y - 4z &= 0 \\ x + 0 + 4z &= \frac{1}{3} \end{aligned}$$

Step #7: Solve equation #1 for z.

$$-11z = \frac{7}{3}$$
$$z = -\frac{7}{33}$$

Step #8: Substitute the value of z into equations #2 and #3 to find the values for x & y.

$$y - 4\left(-\frac{7}{33}\right) = 0 \quad \Rightarrow \quad y = -\frac{28}{33}$$
$$x + 4\left(-\frac{7}{33}\right) = \frac{1}{3} \quad \Rightarrow \quad x = \frac{13}{11}$$

Therefore the solution to the system is: $(x, y, z) = \left(\frac{13}{11}, -\frac{28}{33}, -\frac{7}{33}\right)$

Example #2: Use “Elimination of Variables” to solve the following system:

$$\begin{aligned} 2x + 5y &= 10 \\ 3x - 4y &= 12 \end{aligned}$$

Step #1: Since the y-variables on the two equations are of opposite signs, they will “eliminate” when added, provided their coefficients are the same. The least common multiple (LCM) of their coefficients is ‘20’. Therefore, if we multiply every term in equation #1 by four and every term in equation #2 by five, the coefficients of y will equal 20 in both equations.

$$\begin{aligned} 8x + 20y &= 40 \\ 15x - 20y &= 60 \end{aligned}$$

Step #2: Now add equation #1 to equation #2, and the variable y is eliminated.

$$23x = 100$$

$$x = \frac{100}{23}$$

Step #3: Now substitute the value of x into either of the two original equations to find the value of y.

* Note: To avoid errors, the value of x should be substituted into one of the original equations. Changing equations by this method can result in mistaken solutions if values are substituted into other steps of the process.

$$2\left(\frac{100}{23}\right) + 5y = 10$$

$$y = \frac{6}{23}$$

Therefore, $\left(\frac{100}{23}, \frac{6}{23}\right)$ is the solution to the system.

For the time being we will not solve a system of more than three equations, but will leave these to our study of matrices.

Using a Graphing Calculator to Solve a System of Equations

In the examples in the other unit links for this unit, each system involved a pair of linear equations. However, points of intersection may occur between a circle and a line, between a parabola and a hyperbola, or between two (or more) curves of any shape. In these situations, the algebra involved may be cumbersome. Fortunately, the graphing calculator can be utilized to approximate the values of intersection points. It is important to realize, however, that the answers obtained by using a calculator are only approximations. When exact answers are required, more sophisticated algebraic techniques are often utilized. Before employing the calculator, we will look at one algebraic method commonly used.

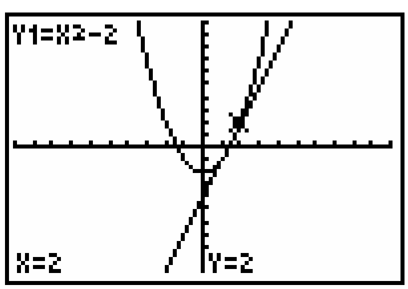
Example #1: Find the points of intersection for: $y = x^2 - 2$ & $y = 3x - 4$

In this system, the first equation is a parabola and the second equation is a line. By using the method of substitution, we can equate both expressions of the variable *x*, and then solve as a quadratic equation.

$$x^2 - 2 = 3x - 4$$
$$x^2 - 3x + 2 = 0$$
$$(x-2)(x-1) = 0$$
$$x - 2 = 0 \quad \text{and} \quad x - 1 = 0$$
$$x = 2 \qquad\qquad x = 1$$

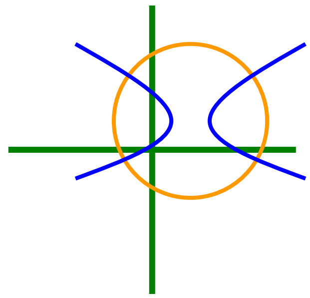
We can then substitute each x into the original equations to find y.
Therefore, $A(2,2)$ and $B(1, -1)$ are the solutions to the system.

On the graphing calculator the two equations appear as:



As we see, attempting to find the points of intersection from the graph alone may not always be conclusive. Therefore we need a combination of techniques to solve many problems. This is important to remember as we investigate the next section.

In the previous example, finding points of intersection was not difficult algebraically because the resulting equation, after substitution, was factorable. However, the possible number of intersection points between a circle and a hyperbola (for example) can be as many as four, as the graph below demonstrates:



To estimate the values of these points, we will employ the calculator and the standard equations of conic sections.

Example #2: Find the points of intersection between:

$$\frac{(x-4)^2}{9}+\frac{(y+2)^2}{7}=1 \qquad \& \qquad y=3x^2-10x-1$$

Horizontal Ellipse　　　　　　Vertical Parabola

(See the unit about conic sections to recall the equations.)

Clearly, substitution will result in excess calculations for these two curves. However, entering these equations into the calculator is also not readily apparent.

Recall that we can graph: $(x-3)^2+y^2=4$

By solving for y and obtaining two equations to enter into the calculator.

We can graph the following two equations:

$$\mathrm{Y}_1=\sqrt{(4-(\mathrm{X}-3)^2)}$$

$$\mathrm{Y}_2=-\sqrt{(4-(\mathrm{X}-3)^2)}$$

In our current example, solving the second equation for y can be tedious. Instead we will derive basic calculator formulas for the conic sections which can be used every time we need to graph a particular conic section.

Example #3: Derive calculator equations from the Standard Equations of the Conic Sections.

Recall that every step in this process needs to be quantified in a set of ( )'s in order for the calculator to graph each curve correctly. We will only demonstrate three derivations for the calculator equations. The remaining equations will be left as an exercise in this unit's assignment.

Goal: Isolate '$\mathrm{Y}_n =$' from each conic's standard equation.

a.) Vertical Parabola:

Standard Equation: $y = ax^2 + bx + c$

Calculator Equation: $\mathrm{Y}_1 = \mathrm{AX}^2 + \mathrm{BX} + \mathrm{C}$

b.) Circle:

Standard Equation: $(x-h)^2 + (y-k)^2 = r^2$

Calculator Equation: Solve for y:

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(y-k)^2 = (r^2 - (x-h)^2)$$

$$y - k = \pm\sqrt{(r^2 - (x-h)^2)}$$

$$y = k \pm \sqrt{(r^2 - (x-h)^2)}$$

Therefore, our calculator equations are:

$$\mathrm{Y}_1 = \mathrm{K} + \sqrt{(\mathrm{R}^2 - (\mathrm{X} - \mathrm{H})^2)}$$

$$\mathrm{Y}_2 = \mathrm{K} - \sqrt{(\mathrm{R}^2 - (\mathrm{X} - \mathrm{H})^2)}$$

c.) Horizontal Ellipse:

Standard Equation: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

Calculator Equation: Solve for y :

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{(y-k)^2}{b^2}=\left(1-\frac{(x-h)^2}{a^2}\right)$$

$$(y-k)^2=\left(b^2\left(1-\frac{(x-h)^2}{a^2}\right)\right)$$

$$y-k=\pm\sqrt{\left(b^2\left(1-\frac{(x-h)^2}{a^2}\right)\right)}$$

$$y=k\pm\sqrt{\left(b^2\left(1-\frac{(x-h)^2}{a^2}\right)\right)}$$

Therefore, our calculator equations are:

$$\mathrm{Y}_1=\mathrm{K}+\sqrt{(\mathrm{B}^2(1-(\mathrm{X}-\mathrm{H})^2/\mathrm{A}^2))}$$

$$\mathrm{Y}_2=\mathrm{K}-\sqrt{(\mathrm{B}^2(1-(\mathrm{X}-\mathrm{H})^2/\mathrm{A}^2))}$$

d.) Vertical Hyperbola:

Standard Equation: $\dfrac{(y-k)^2}{a^2}-\dfrac{(x-h)^2}{b^2}=1$

Calculator Equation: Solve for y :

$$\frac{(y-k)^2}{a^2}=\left(1+\frac{(x-h)^2}{b^2}\right)$$

$$(y-k)^2=\left(a^2\left(1+\frac{(x-h)^2}{b^2}\right)\right)$$

$$y-k=\pm\sqrt{\left(a^2\left(1+\frac{(x-h)^2}{b^2}\right)\right)}$$

$$y=k\pm\sqrt{\left(a^2\left(1+\frac{(x-h)^2}{b^2}\right)\right)}$$

Therefore, our calculator equations are:

$$\mathrm{Y_1=K+\sqrt{(A^2(1+(X-H)^2/B^2))}}$$

$$\mathrm{Y_2=K-\sqrt{(A^2(1+(X-H)^2/B^2))}}$$

Now, let's revisit *Example #2*.

Find the points of intersection between:

$$\frac{(x-4)^2}{9}+\frac{(y+2)^2}{7}=1 \qquad \& \qquad y=3x^2-10x-1$$

Horizontal Ellipse | Vertical Parabola

a.) For the parabola, identify A=3, B=-10, & C=-1
Then enter the following into the calculator:

$$\mathrm{Y_1=3X^2-10X-1}$$

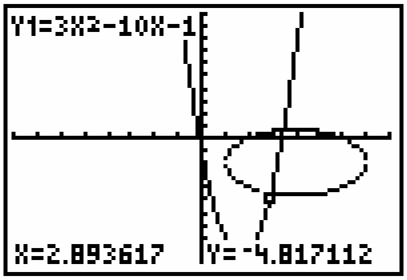
b.) For the ellipse, identify $\mathrm{H=4,\ K=-2,\ A^2=9,\ B^2=7}$

Then enter the following into the calculator:

$$Y_2 = -2 + \sqrt{(7(1-(X-4)^2/9))}$$
$$Y_3 = -2 - \sqrt{(7(1-(X-4)^2/9))}$$

Then press, GRAPH

Once the two equations are graphed, we can press TRACE and cursor to the points of intersection on the curves.



The above graph identifies one point approximately (but not precisely) as:

$(x, y) \approx (2.89, -4.82)$

You can approximate the second point of intersection on your own. (The calculator also has an "Intersect" feature. For our purposes, approximations will satisfy most of our investigations for this course. Consult your calculator's handbook if you wish to use this feature).