

# THE LOGIC OF QUANTIFIED STATEMENTS

# Outline

- Intro to predicate logic
- **Predicate**, truth set
- **Quantifiers**, universal, existential statements, universal conditional statements
- Reading & writing quantified statements
- **Negation** of quantified statements
- Converse, Inverse and contrapositive of universal conditional statements
- Statements with multiple quantifiers
- Argument with quantified statements

# So far: Propositional Logic

Propositional logic: statement, compound and simple, logic connectives. Logical equivalence, Valid arguments (rules of inference) ... Extension to Boolean algebra & application to digital logic circuit

<b>Modus Ponens</b>	$p \rightarrow q$ $p$ <ul style="list-style-type: none"> <li>• <math>q</math></li> </ul>	<b>Elimination</b>	<b>a.</b> $p \vee q$ $\sim q$ <ul style="list-style-type: none"> <li>• <math>p</math></li> </ul>	<b>b.</b> $p \vee q$ $\sim p$ <ul style="list-style-type: none"> <li>• <math>q</math></li> </ul>	
<b>Modus Tollens</b>	$p \rightarrow q$ $\sim q$ <ul style="list-style-type: none"> <li>• <math>\sim p</math></li> </ul>	<b>Transitivity</b>	$p \rightarrow q$ $q \rightarrow r$ <ul style="list-style-type: none"> <li>• <math>p \rightarrow r</math></li> </ul>		
<b>Generalization</b>	<b>a.</b> $p$ <ul style="list-style-type: none"> <li>• <math>p \vee q</math></li> </ul>	<b>b.</b> $q$ <ul style="list-style-type: none"> <li>• <math>p \vee q</math></li> </ul>	<b>Proof by Division into Cases</b>	$p \vee q$ $p \rightarrow r$ $q \rightarrow r$ <ul style="list-style-type: none"> <li>• <math>r</math></li> </ul>	
<b>Specialization</b>	<b>a.</b> $p \wedge q$ <ul style="list-style-type: none"> <li>• <math>p</math></li> </ul>	<b>b.</b> $p \wedge q$ <ul style="list-style-type: none"> <li>• <math>q</math></li> </ul>			
<b>Conjunction</b>	$p$ $q$ <ul style="list-style-type: none"> <li>• <math>p \wedge q</math></li> </ul>	<b>Contradiction Rule</b>	$\sim p \rightarrow c$ <ul style="list-style-type: none"> <li>• <math>p</math></li> </ul>		

# Limitation of propositional logic

Is the following a valid argument?

All men are mortal.  
Socrates is a man.  
∴ Socrates is mortal.

Lesson:

In propositional logic, each simple statement is atomic (basic building block). But here we need to analyze different parts of one statement.

Let's try to see if propositional logic can help here...

The form of the argument is:

p  
q  
∴ r

# Why Predicate Logic?

Is the following a valid argument?

All men are mortal.

Socrates is a man.

∴ Socrates is mortal.

How:

In predicate logic, we look inside parts of each statement.

∴ for any  $x$ , if  $x$  “is a man”, then  $x$  “is a mortal”  
Socrates “is a man”  
Socrates “is mortal”

# Predicates

- (in Grammar) “the part of a sentence or clause containing a verb and stating something about the subject”
  - (e.g., went home in “John went home” ).
- In logic, predicates can be obtained by removing some or all nouns from a statement.

# Predicates in a statement

- Predicates can be obtained by removing some or all nouns from a statement.
  - Example: In “**Alice** is a student at **Bedford College**.”:
    1.  $P$  stand for “is a student at Bedford College”, **P** is called predicate symbol  
*need to plug back noun to make a complete sentence:*
      - Original sentence is symbolized as  $P(\text{Alice})$ , which might be true, might be false, but not both.
      - Plugging in (predicate) variable,  $x$ , we get “ $x$  is a student at Bedford College”, i.e.,  $P(x)$ .

# Predicates in a statement

- Predicates can be obtained by removing some or all nouns from a statement.
  - Example: In “**Alice** is a student at **Bedford College**.”:

Q stand for “*is a student at.*” (again Q is predicate symbol)

Plug in nouns to get complete sentence:

- $Q(\text{Alice}, \text{Bedford College})$  is original sentence
- $Q(x, y)$  represents “x is a student at y”



# Predicates are like functions

- When concrete values are substituted in place of predicate variables, a statement results (which has a truth value)
  - $P(x)$  stand for “*x is a student at Bedford College*”,  
 $P(\text{Jack})$  is “Jack is a student at Bedford College”.
  - $Q(x,y)$  stand for “*x is a student at y.*” ,  
 $Q(\text{John Smith}, \text{Fordham University})$  is “John Smith is a student at Fordham University”
  - $P(x)$  stands for “*x is mortal*”, then  $P(\text{Socrates})$  stands for “Socrates is mortal”

# Predicate and its domain

- **Definition**

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

- **Definition**

If  $P(x)$  is a predicate and  $x$  has domain  $D$ , the **truth set** of  $P(x)$  is the set of all elements of  $D$  that make  $P(x)$  true when they are substituted for  $x$ . The truth set of  $P(x)$  is denoted

$$\{x \in D \mid P(x)\}.$$

# Predicate, Truth set

- **Definition**

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

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# Truth Set of a Predicate

- **Definition**

If  $P(x)$  is a predicate and  $x$  has domain  $D$ , the **truth set** of  $P(x)$  is the set of all elements of  $D$  that make  $P(x)$  true when they are substituted for  $x$ . The truth set of  $P(x)$  is denoted

$$\{x \in D \mid P(x)\}.$$

Let  $Q(n)$  be predicate “ $n$  is a factor of 8.”

Find truth set of  $Q(n)$  if the domain of  $n$  is the set  $\mathbf{Z}^+$  of all positive integers

# predicate $\implies$ proposition

Consider predicate “x is divisibly by 5”

- assign specific value to variable x.
  - e.g., if x is 35, then predicate becomes a proposition (“35 is divisible by 5”)
- add **quantifiers**, words that refer to quantities such as “some” or “all” and tell for how many elements a given predicate is true.
  - e.g., **For some** integer x, x is divisible by 5
  - e.g., **For all** integer x, x is divisible by 5
  - e.g., there exists two integer x, such that x is divisible by 5.
  - All above three are now propositions (i.e., they are either true or false)

# Universal Quantifier: $\forall$

Symbol  $\forall$  denotes “for all” , called **universal quantifier**.

Let  $D = \{1, 2, 3, 4, 5\}$ , and consider

$$\forall x \in D, x^2 \geq x.$$

Predicate

The domain of predicate variable (here,  $x$ ) is indicated

- between  $\forall$  symbol and variable name,
- immediately following variable name (see above)

Some other expressions: *for all, for every, for arbitrary, for any, for each, given any.*

# Universal Statement

- **Definition**

Let  $Q(x)$  be a predicate and  $D$  the domain of  $x$ . A **universal statement** is a statement of the form “ $\forall x \in D, Q(x)$ .” It is defined to be true if, and only if,  $Q(x)$  is true for every  $x$  in  $D$ . It is defined to be false if, and only if,  $Q(x)$  is false for at least one  $x$  in  $D$ . A value for  $x$  for which  $Q(x)$  is false is called a **counterexample** to the universal statement.

*True or False?*

**a.** Let  $D = \{1, 2, 3, 4, 5\}$ , and consider the statement

$$\forall x \in D, x^2 \geq x.$$

**b.** Consider the statement

$$\forall x \in \mathbf{R}, x^2 \geq x.$$



# Method of Exhaustion

**Method of exhaustion:** to prove a universal statement to be true, we can show the truth of the predicate separately for each individual element of the domain.

This method can be used when the domain is finite.

# Existential Quantifier: $\exists$

Symbol  $\exists$  denotes “there exists”, “*there is a*”, “*we can find a*”, *there is at least one, for some, and for at least one*

“There is a student in Math 140” can be written as

$\exists$  a person  $p$  **such that**  $p$  is a student in Math 140,

or, more formally,

$\exists p \in P$  **such that**  $p$  is a student in Math 140,

where  $P$  is the set of all people.

The domain of predicate variable (here,  $p$ ) is

- indicated either between  $\exists$  symbol and variable name, or
- immediately following variable name.

# Existential Quantifier: $\exists$

“ $\exists$  integers  $m$  and  $n$  such that  $m + n = m \cdot n$ ,”

$\exists$  symbol refers to both  $m$  and  $n$ .

# Existential statement

- **Definition**

Let  $Q(x)$  be a predicate and  $D$  the domain of  $x$ . An **existential statement** is a statement of the form “ $\exists x \in D$  such that  $Q(x)$ .” It is defined to be true if, and only if,  $Q(x)$  is true for at least one  $x$  in  $D$ . It is false if, and only if,  $Q(x)$  is false for all  $x$  in  $D$ .

# Truth Value of Existential Statements

Consider statement

$$\exists m \in \mathbf{Z}^+ \text{ such that } m^2 = m.$$

Show that this statement is true.

# Exercise: Truth Value of Existential Statements

Let  $E = \{5, 6, 7, 8\}$  and consider statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

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# Formal Versus Informal Language

make sense of  
mathematical concepts that  
are new to you



help to think about a complicated  
problem.



# Formal $\Rightarrow$ Informal Language

Rewrite in a variety of equivalent but *more informal ways*. **Do not use symbol  $\forall$  or  $\exists$ .**

$$\forall x \in \mathbf{R}, x^2 \geq 0.$$

$$\forall x \in \mathbf{R}, x^2 \neq -1.$$

$$\exists m \in \mathbf{Z}^+ \text{ such that } m^2 = m.$$

There is a positive integer whose square is equal to itself.

*Or:* We can find at least one positive integer equal to its own square.

*Or:* Some positive integer equals its own square.

*Or:* Some positive integers equal their own squares.

# Universal Conditional Statements

One of the most important form of statement in mathematics is **universal conditional statement**:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Familiarity with statements of this form is essential if you are to learn to speak mathematics.

# Reading Universal Conditional

Rewrite the following without quantifiers or variables.

$$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4.$$

**Solution:**

If a real number is greater than 2 then its square is greater than 4.

*Or:* Whenever a real number is greater than 2, its square is greater than 4.

*Or:* The square of any real number greater than 2 is greater than 4.

*Or:* The squares of all real numbers greater than 2 are greater than 4.

# Equivalent Forms of Universal and Existential Statements

“ $\forall$  real numbers  $x$ , if  $x$  is an integer then  $x$  is rational”

“ $\forall$  integers  $x$ ,  $x$  is rational”

Both have informal translations “All integers are rational.”

In fact, a statement

$$\forall x \in U, \text{ if } P(x) \text{ then } Q(x)$$

can always be rewritten as

$$\forall x \in D, Q(x)$$

by narrowing  $U$  to be domain  $D$ , where  $D$  is the truth set of  $P(x)$  (consisting of all values of variable  $x$  that make  $P(x)$  true).

# Equivalent Forms of Universal and Existential Statements

Conversely, a statement of the form

$$\forall x \in D, Q(x)$$

can be rewritten as

$$\forall x, \text{ if } x \text{ is in } D \text{ then } Q(x).$$

# Equivalent Forms for Universal Statements

Rewrite the following statement in the two forms “ $\forall x$ , \_\_\_\_\_ if \_\_\_\_\_ then \_\_\_\_\_” and “ $\forall$  \_\_\_\_\_  $x$ , \_\_\_\_\_”:

All squares are rectangles.

# Equivalent Forms of Universal and Existential Statements

Similarly, a statement of the form

“ $\exists x$  such that  $p(x)$  and  $Q(x)$ ”

can be rewritten as

“ $\exists x \in D$  such that  $Q(x)$ ,”

where  $D$  is the set of all  $x$  for which  $P(x)$  is true.

# Equivalent Forms for Existential Statements

A **prime number** is an integer greater than 1 whose only positive integer factors are itself and 1. Consider the statement “There is an integer that is both prime and even.”

Let  $\text{Prime}(n)$  be “ $n$  is prime” and  $\text{Even}(n)$  be “ $n$  is even.” Use the notation  $\text{Prime}(n)$  and  $\text{Even}(n)$  to rewrite this statement in the following two forms:

a.  $\exists n$  such that \_\_\_\_\_  $\wedge$  \_\_\_\_\_ .

b.  $\exists$  \_\_\_\_\_  $n$  such that \_\_\_\_\_ .



# Example 11 – Solution

a.  $\exists n$  such that  $\text{Prime}(n) \wedge \text{Even}(n)$ .

b. Two answers:  $\exists$  a prime number  $n$  such that  $\text{Even}(n)$ .

$\exists$  an even number  $n$  such that  $\text{Prime}(n)$ .

# Implicit Quantification

Mathematical writing contains many examples of implicitly quantified statements.

- Some occur, through the presence of the word *a* or *an*.
- Others occur in cases where the general context of a sentence supplies part of its meaning.

For example, in algebra, the predicate

$$\text{If } x > 2 \text{ then } x^2 > 4$$

is interpreted to mean the same as the statement

$$\forall \text{ real numbers } x, \text{ if } x > 2 \text{ then } x^2 > 4.$$

# Implicit Quantification

Mathematicians often use a double arrow to indicate implicit quantification symbolically.

For instance, they might express the above statement as

$$x > 2 \Rightarrow x^2 > 4.$$

## • Notation

Let  $P(x)$  and  $Q(x)$  be predicates and suppose the common domain of  $x$  is  $D$ .

- The notation  $P(x) \Rightarrow Q(x)$  means that every element in the truth set of  $P(x)$  is in the truth set of  $Q(x)$ , or, equivalently,  $\forall x, P(x) \rightarrow Q(x)$ .
- The notation  $P(x) \Leftrightarrow Q(x)$  means that  $P(x)$  and  $Q(x)$  have identical truth sets, or, equivalently,  $\forall x, P(x) \leftrightarrow Q(x)$ .

# Using $\Rightarrow$ and $\Leftrightarrow$

Let

$Q(n)$  be “ $n$  is a factor of 8,”

$R(n)$  be “ $n$  is a factor of 4,”

$S(n)$  be “ $n < 5$  and  $n \neq 3$ ,”

and suppose the domain of  $n$  is  $\mathbf{Z}^+$ , the set of positive integers. Use the  $\Rightarrow$  and  $\Leftrightarrow$  symbols to indicate true relationships among  $Q(n)$ ,  $R(n)$ , and  $S(n)$ .

# Example 12 – Solution

1. As noted in Example 2, the truth set of  $Q(n)$  is  $\{1, 2, 4, 8\}$  when the domain of  $n$  is  $\mathbf{Z}^+$ . By similar reasoning the truth set of  $R(n)$  is  $\{1, 2, 4\}$ .

Thus it is true that every element in the truth set of  $R(n)$  is in the truth set of  $Q(n)$ , or, equivalently,

$$\forall n \text{ in } \mathbf{Z}^+, R(n) \rightarrow Q(n).$$

So  $R(n) \Rightarrow Q(n)$ , or, equivalently

$n$  is a factor of 4  $\Rightarrow n$  is a factor of 8.

# Example 12 – Solution

2. The truth set of  $S(n)$  is  $\{1, 2, 4\}$ , which is identical to the truth set of  $R(n)$ , or, equivalently,

$$\forall n \text{ in } \mathbf{Z}^+, R(n) \leftrightarrow S(n).$$

So  $R(n) \Leftrightarrow S(n)$ , or, equivalently,

$$n \text{ is a factor of } 4 \Leftrightarrow n < 5 \text{ and } n \neq 3.$$

Moreover, since every element in the truth set of  $S(n)$  is in the truth set of  $Q(n)$ , or, equivalently,

$\forall n \text{ in } \mathbf{Z}^+, S(n) \rightarrow Q(n)$ , then  $S(n) \Rightarrow Q(n)$ , or, equivalently,

$$n < 5 \text{ and } n \neq 3 \Rightarrow n \text{ is a factor of } 8.$$

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# Negations of Quantified Statements

The general form of the negation of a universal statement follows immediately from the definitions of negation and of the truth values for universal and existential statements.

## Theorem 3.2.1 Negation of a Universal Statement

The negation of a statement of the form

$$\forall x \text{ in } D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

Symbolically,  $\sim(\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$



# Negations of Quantified Statements

The negation of a universal statement (“all are”) is logically equivalent to an existential statement (“some are not” or “there is at least one that is not”).

Note that when we speak of **logical equivalence for quantified statements**, we mean that the statements always have identical truth values no matter what predicates are substituted for the predicate symbols and no matter what sets are used for the domains of the predicate variables.

# Negations of Quantified Statements

The general form for the negation of an existential statement follows immediately from the definitions of negation and of the truth values for existential and universal statements.

## Theorem 3.2.2 Negation of an Existential Statement

The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim Q(x).$$

Symbolically,  $\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x).$

**The negation of an existential statement (“some are”) is logically equivalent to a universal statement (“none are” or “all are not”).**

# Negating Quantified Statements

Write formal negations for the following statements:

**a.**  $\forall$  primes  $p$ ,  $p$  is odd.

**b.**  $\exists$  a triangle  $T$  such that the sum of the angles of  $T$  equals  $200^\circ$ .

# Relation among $\forall$ , $\exists$ , $\wedge$ , and $\vee$

The negation of a *for all* statement is a *there exists* statement, and the negation of a *there exists* statement is a *for all* statement.

These facts are analogous to De Morgan's laws, which state that the negation of an *and* statement is an *or* statement and that the negation of an *or* statement is an *and* statement.

$$\sim(\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$$

$$\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x).$$

# Relation among $\forall$ , $\exists$ , $\wedge$ , and $\vee$

If  $Q(x)$  is a predicate and the domain  $D$  of  $x$  is the set  $\{x_1, x_2, \dots, x_n\}$ , then the statements

$$\forall x \in D, Q(x)$$

and

$$Q(x_1) \wedge Q(x_2) \wedge \dots \wedge Q(x_n)$$

are logically equivalent.

By De Morgan's Law ...

# The Relation among $\forall$ , $\exists$ , $\wedge$ , and $\vee$

Similarly, if  $Q(x)$  is a predicate and  $D = \{x_1, x_2, \dots, x_n\}$ , then the statements

$$\exists x \in D \text{ such that } Q(x)$$

and

$$Q(x_1) \vee Q(x_2) \vee \dots \vee Q(x_n)$$

are logically equivalent.

By De Morgan's law:

# Negate Universal Conditional Statements

The form of such negations can be derived from facts that have already been established.

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \rightarrow Q(x)). \quad 3.2.1$$

the negation of an if-then statement is logically equivalent to an *and* statement.

$$\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x). \quad 3.2.2$$

## Negation of a Universal Conditional Statement

$$\sim(\forall x, \text{if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x).$$

# Negate Universal Conditional Statements

Write a formal negation for statement (a) and an informal negation for statement (b).

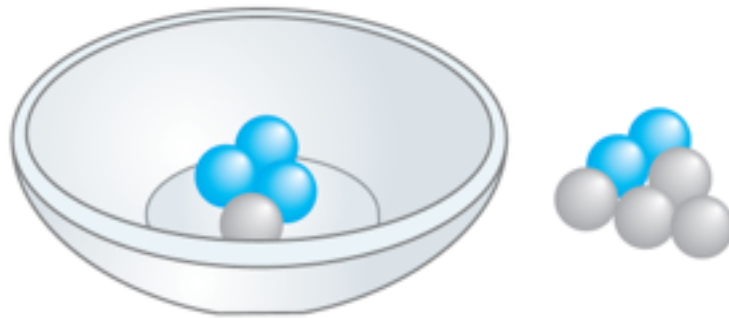
**a.**  $\forall$  people  $p$ , if  $p$  is blond then  $p$  has blue eyes.

**b.** If a computer program has more than 100,000 lines, then it contains a bug.



# Universal Statements

The statement “All the balls in the bowl are blue” would be false (since one of the balls in the bowl is gray).



# Universal Statements

Is the statement true, or false?

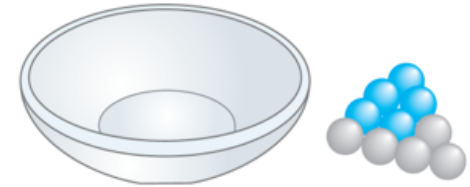
All the balls in the bowl are blue.



Figure 3.2.1(b)

# Vacuous Truth of Universal Statements

Is this statement true or false?



All the balls in the bowl are blue.

The statement is false if, and only if, its negation is true.  
Its negation is:

There exists a ball in the bowl that is not blue.

The negation is false! So the statement is **true** “by default.”

# Vacuous Truth of Universal Statements

A statement of the form

$$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x)$$

is called **vacuously true** or **true by default** if, and only if,  $P(x)$  is false for every  $x$  in  $D$ .

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# Variants of Universal Conditional Statements

A conditional statement has a **contrapositive, a converse, and an inverse.**

**Similarly,**

## • Definition

Consider a statement of the form:  $\forall x \in D$ , if  $P(x)$  then  $Q(x)$ .

1. Its **contrapositive** is the statement:  $\forall x \in D$ , if  $\sim Q(x)$  then  $\sim P(x)$ .
2. Its **converse** is the statement:  $\forall x \in D$ , if  $Q(x)$  then  $P(x)$ .
3. Its **inverse** is the statement:  $\forall x \in D$ , if  $\sim P(x)$  then  $\sim Q(x)$ .

# Example: contrapositive, converse, inverse

Given a universal conditional statement:

*If a real number is greater than 2, then its square is greater than 4.*

Its formal version of this statement is:

$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4.$

**Contrapositive:**  $\forall x \in \mathbf{R}, \text{ if } x^2 \leq 4 \text{ then } x \leq 2.$

*If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.*

# Example: contrapositive, converse, inverse

Given a universal conditional statement:

If a real number is greater than 2, then its square is greater than 4.

Its formal version of this statement is:

$\forall x \in \mathbf{R}$ , if  $x > 2$  then  $x^2 > 4$ .

**Converse:**  $\forall x \in \mathbf{R}$ , if  $x^2 > 4$  then  $x > 2$ .

If the square of a real number is greater than 4, then the number is greater than 2.

**Inverse:**  $\forall x \in \mathbf{R}$ , if  $x \leq 2$  then  $x^2 \leq 4$ .

If a real number is less than or equal to 2, then the square of the number is less than or equal to 4.



# Variants of Universal Conditional Statements

Let  $P(x)$  and  $Q(x)$  be any predicates, let  $D$  be domain of  $x$ ,

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x)$$

and its contrapositive  $\forall x \in D, \text{ if } \sim Q(x) \text{ then } \sim P(x)$ .

- Any particular  $x$  in  $D$  that makes “if  $P(x)$  then  $Q(x)$ ” true also makes “if  $\sim Q(x)$  then  $\sim P(x)$ ” true (by logical equivalence between  $p \rightarrow q$  and  $\sim q \rightarrow \sim p$ ).
- It follows that sentence “If  $P(x)$  then  $Q(x)$ ” is true for all  $x$  in  $D$  **if, and only if**, sentence “If  $\sim Q(x)$  then  $\sim P(x)$ ” is true for all  $x$  in  $D$ .

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \equiv \forall x \in D, \text{ if } \sim Q(x) \text{ then } \sim P(x)$$

# Variants of Universal Conditional Statements

Statement  $\forall x \in \mathbf{R}$ , if  $x > 2$  then  $x^2 > 4$  is true

its converse,  $\forall x \in \mathbf{R}$ , if  $x^2 > 4$  then  $x > 2$ , is *false*. (for instance,  $(-3)^2 = 9 > 4$  but  $-3 \not> 2$ ).

So

$\forall x \in D$ , if  $P(x)$  then  $Q(x) \not\equiv \forall x \in D$ , if  $Q(x)$  then  $P(x)$ .

# Necessary and Sufficient Conditions

Rewrite following statements as **quantified conditional statements**.

- a. Squareness is a sufficient condition for rectangularity.
- b. Being at least 35 years old is a necessary condition for being President of the United States.

**Solution:**

a. A formal version of the statement is

$\forall x$ , if  $x$  is a square, then  $x$  is a rectangle.

# Solution

Or, in informal language:

If a figure is a square, then it is a rectangle.

**b.** Using formal language, you could write the answer as  
 $\forall$  people  $x$ , if  $x$  is younger than 35, then  $x$   
cannot be President of the United States.

Or, by the equivalence between a statement and its  
contrapositive:

$\forall$  people  $x$ , if  $x$  is President of the United States,  
then  $x$  is at least 35 years old.

# Outline

- Intro to predicate logic
- Predicate, truth set
- Quantifiers, universal, existential statements, universal conditional statements
- Reading & writing quantified statements
- Negation of quantified statements
- Converse, Inverse and contrapositive of universal conditional statements
- **Statements with multiple quantifiers**
- Argument with quantified statements

# Statements with Multiple Quantifiers

When a statement contains more than one quantifier, we read **the quantifiers in the order they appear**.

$\forall x$  in set  $D$ ,  $\exists y$  in set  $E$  such that  $x$  and  $y$  satisfy property  $P(x, y)$ .

“for any  $x$  in  $D$ , there exists a  $y$  in  $E$ , so that  $P(x,y)$  is true”

$\exists$  an  $x$  in  $D$  such that  $\forall y$  in  $E$ ,  $x$  and  $y$  satisfy property  $P(x, y)$ .

“there exists a  $x$  in  $D$ , so that for any  $y$  in  $E$ ,  $P(x,y)$  is true”

# Interpreting $\forall \exists$ Statements

To show below statement to be true,

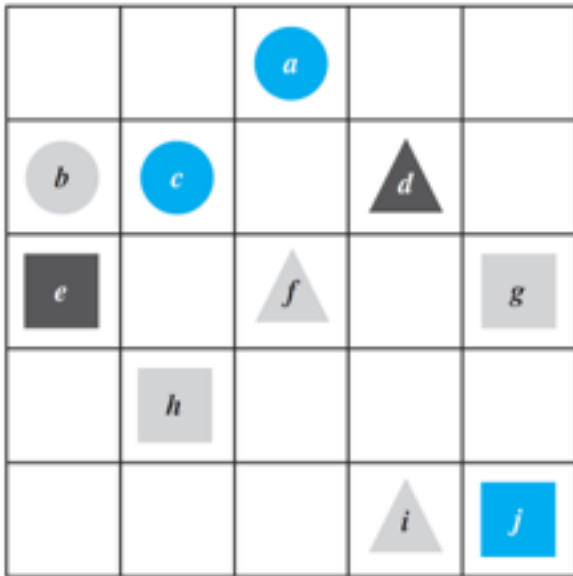
$\forall x$  in set  $D$ ,  $\exists y$  in set  $E$  such that  $x$  and  $y$  satisfy property  $P(x, y)$ .

you must be able to meet following challenge:

1. Imagine that someone is allowed to choose any element whatsoever from  $D$ , and imagine that the person gives you that element. Call it  $x$ .
2. The challenge for you is to find an element  $y$  in  $E$  so that the person's  $x$  and your  $y$ , taken together, satisfy property  $P(x, y)$ .

# Example: Tarski World

Consider Tarski world below, is the statement true?



For all triangles  $x$ , there is a square  $y$  such that  $x$  and  $y$  have the same color.

your challenge is to **allow someone else to pick whatever element  $x$  in  $D$  they wish and then you must find an element  $y$  in  $E$  that “works” for that particular  $x$ .**

Given $x =$	choose $y =$	and check that $y$ is the same color as $x$ .
$d$	$e$	yes •
$f$ or $i$	$h$ or $g$	yes •



# Interpreting $\exists \forall$ Statements

$\exists$  an  $x$  in  $D$  such that  $\forall y$  in  $E$ ,  $x$  and  $y$  satisfy property  $P(x, y)$ .

To show above to be true:

1. you must find one single element (call it  $x$ ) in  $D$  with following property:
2. After you have found your  $x$ , someone is allowed to choose any element whatsoever from  $E$ . The person challenges you by giving you that element. Call it  $y$ .
3. Your job is to show that your  $x$  together with the person's  $y$  satisfy property  $P(x, y)$ .

your job is to find one particular  $x$  in  $D$  that will “work” no matter what  $y$  in  $E$  anyone might choose to challenge you with.

# Multiply-Quantified Statements

A college cafeteria line has four stations:

- salad station offers: green salad, fruit salad
- main course station offers: spaghetti, fish
- dessert station offers: pie, cake
- beverage station offers: milk, soda, coffee

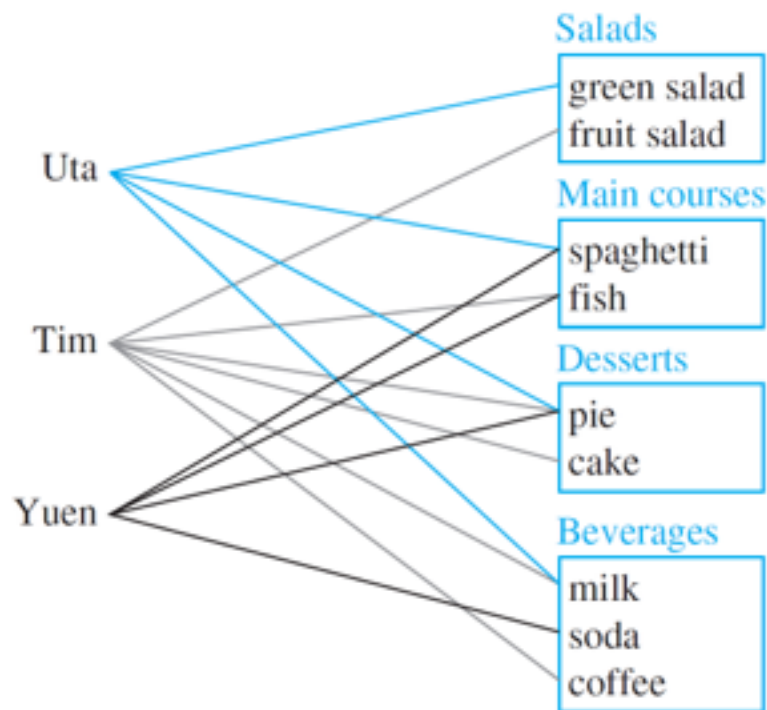
Three students, Uta, Tim, and Yuen, make following choices:

Uta: green salad, spaghetti, pie, milk

Tim: fruit salad, fish, pie, cake, milk, coffee

Yuen: spaghetti, fish, pie, soda

# Interpreting Multiply-Quantified Statements

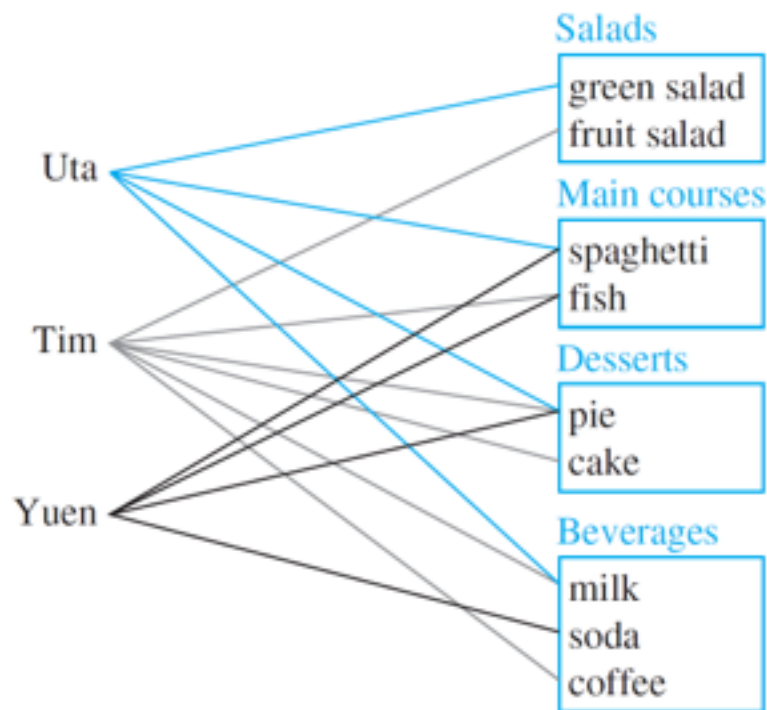


Write each of following statements informally and find its truth value.

$\exists$  an item  $I$  such that  $\forall$  students  $S$ ,  $S$  chose  $I$ .

- . There is an item that was chosen by every student. This is true; every student chose pie.

# Interpreting Multiply-Quantified Statements

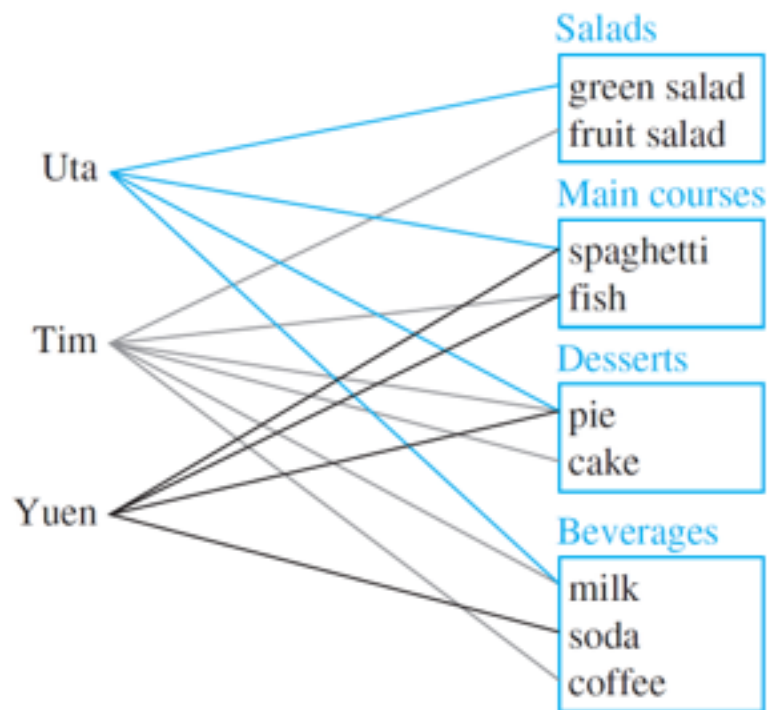


Write each of following statements informally and find its truth value.

$\exists$  a student  $S$  such that  $\forall$  items  $I$ ,  $S$  chose  $I$ .

There is a student who chose every available item. This is false; no student chose all nine items.

# Interpreting Multiply-Quantified Statements

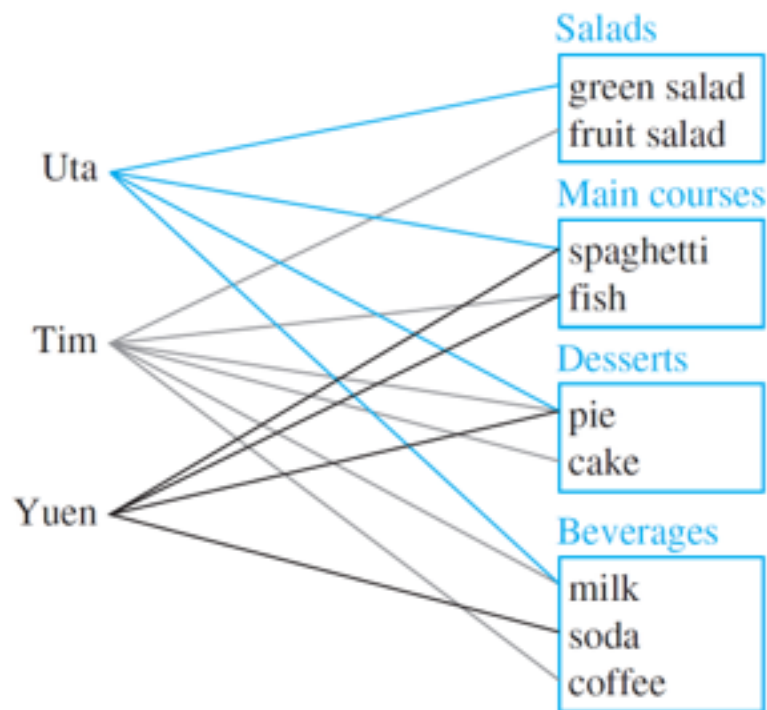


Write each of following statements informally and find its truth value.

$\exists$  a student  $S$  such that  $\forall$  stations  $Z$ ,  $\exists$  an item  $I$  in  $Z$  such that  $S$  chose  $I$ .

There is a student who chose at least one item from every station. This is true; both Uta and Tim chose at least one item from every station.

# Interpreting Multiply-Quantified Statements



Write each of following statements informally and find its truth value.

$\forall$  students  $S$  and  $\forall$  stations  $Z$ ,  $\exists$  an item  $I$  in  $Z$  such that  $S$  chose  $I$ .

Every student chose at least one item from every station. This is false; Yuen did not choose a salad.

# Translate from Informal to Formal Language

The **reciprocal** of a real number  $a$  is a real number  $b$  such that  $ab = 1$ . Rewrite following using quantifiers and variables:

- a. Every nonzero real number has a reciprocal.
- b. There is a real number with no reciprocal.

**Solution:**

- a.  $\forall$  nonzero real numbers  $u$ ,  $\exists$  a real number  $v$  such that  $uv = 1$ .
- b.  $\exists$  a real number  $c$  such that  $\forall$  real numbers  $d$ ,  $cd \neq 1$ .

# Ambiguous Language

Imagine you are visiting a factory that manufactures computer microchips. The factory guide tells you,

There is a person supervising every detail of the production process.

Note that this statement contains informal versions of both the existential quantifier *there is* and the universal quantifier *every*.



# Ambiguous Language

Which of the following best describes its meaning?

- There is one single person who supervises all the details of the production process.
- For any particular production detail, there is a person who supervises that detail, but there might be different supervisors for different details.

# Negate Multiply-Quantified Statements

Recall, we know that

$$\sim(\forall x \text{ in } D, P(x)) \equiv \exists x \text{ in } D \text{ such that } \sim P(x).$$

and

$$\sim(\exists x \text{ in } D \text{ such that } P(x)) \equiv \forall x \text{ in } D, \sim P(x).$$

We want to simplify

$$\sim(\forall x \text{ in } D, \underline{\exists y \text{ in } E \text{ such that } P(x, y)})$$

hint: part underlined is a predicate with variable  $x$ , apply first rule, and then second rule...

# Negate Multiply-Quantified Statements

$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y))$

$\equiv \exists x \text{ in } D \text{ such that } \sim(\exists y \text{ in } E \text{ such that } P(x, y)).$

$\equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y).$

# Negations of Multiply-Quantified Statements

Similarly, can you simplify below:

$$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y))$$

≡

≡

# Negate Multiply-Quantified Statements

These facts can be summarized as follows:

## Negations of Multiply-Quantified Statements

$$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y).$$

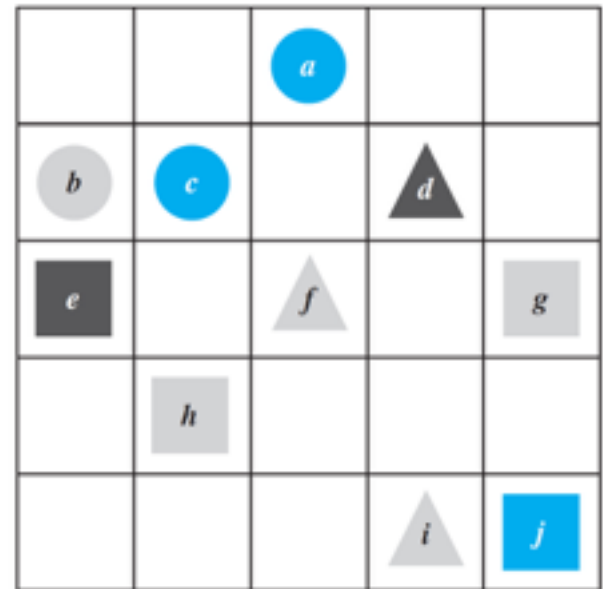
$$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) \equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y).$$

# Example: Negating Statements

Refer to the Tarski world shown below:

Negate following statement and determine which is true: given statement or its negation.

For all squares  $x$ , there is a circle  $y$  such that  $x$  and  $y$  have the same color.

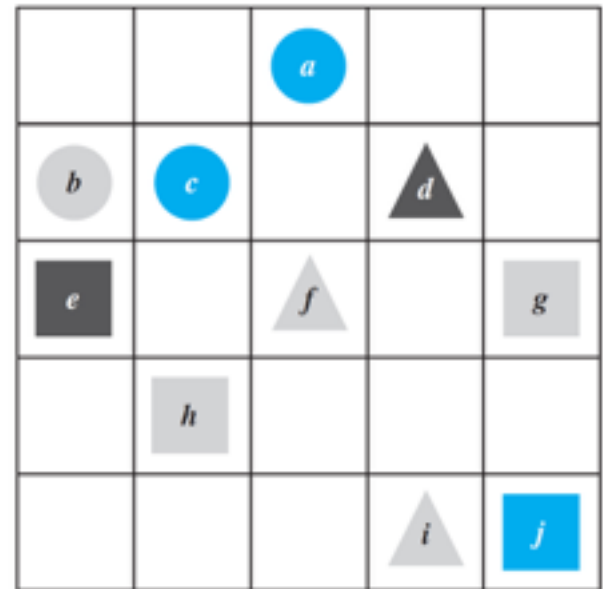


# Example: Negating Statements

Refer to the Tarski world shown below:

Negate following statement and determine which is true: given statement or its negation.

There is a triangle  $x$  such that for all squares  $y$ ,  $x$  is to the right of  $y$ .



## Example 8(a) – Solution

*First version of negation:*  $\exists$  a square  $x$  such that  
 $\sim(\exists$  a circle  $y$  such that  $x$  and  $y$   
have the same color).

*Final version of negation:*  $\exists$  a square  $x$  such that  
 $\forall$  circles  $y$ ,  $x$  and  $y$  do not have  
the same color.

The negation is true. Square  $e$  is black and no circle is black, so there is a square that does not have the same color as any circle.



## Example 8(b) – Solution

*First version of negation:*  $\forall$  triangles  $x, \sim (\forall$  squares  $y, x$  is to the right of  $y)$ .

*Final version of negation:*  $\forall$  triangles  $x, \exists$  a square  $y$  such that  $x$  is not to the right of  $y$ .

The negation is true because no matter what triangle is chosen, it is not to the right of square  $g$  (or square  $j$ ).

# Order of Quantifiers

Consider the following two statements:

$\forall$  people  $x$ ,  $\exists$  a person  $y$  such that  $x$  loves  $y$ .

$\exists$  a person  $y$  such that  $\forall$  people  $x$ ,  $x$  loves  $y$ .

However, the first means that given any person, it is possible to find someone whom that person loves, whereas the second means that there is one amazing individual who is loved by all people.

# Order of Quantifiers

The two sentences illustrate an extremely important property about multiply-quantified statements:

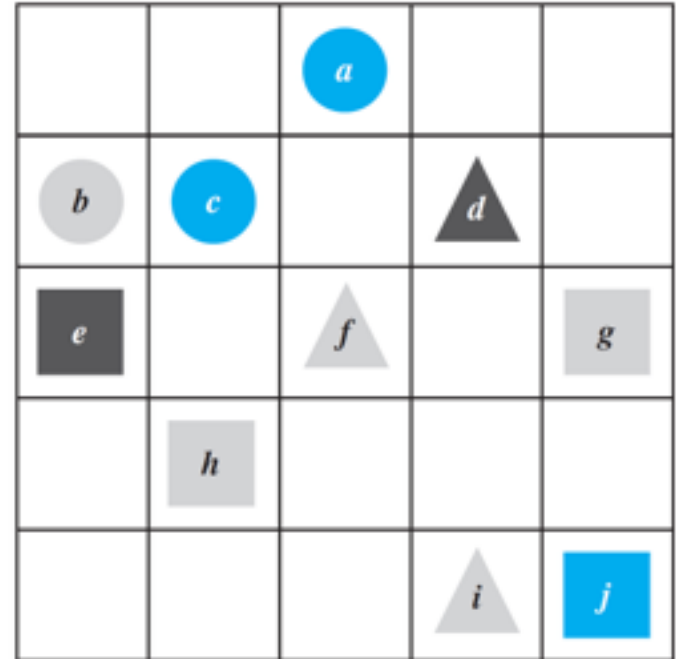
In a statement containing both  $\forall$  and  $\exists$ , changing the order of the quantifiers usually changes the meaning of the statement.

Interestingly, however, if one quantifier immediately follows another quantifier *of the same type*, then the order of the quantifiers does not affect the meaning.

# Example: Quantifier Order

Do following two statements have same truth value?

- a.** For every square  $x$  there is a triangle  $y$  such that  $x$  and  $y$  have different colors.
- b.** There exists a triangle  $y$  such that for every square  $x$ ,  $x$  and  $y$  have different colors.



# Formal Logical Notation

In some areas of computer science, logical statements are expressed in **purely symbolic notation**.

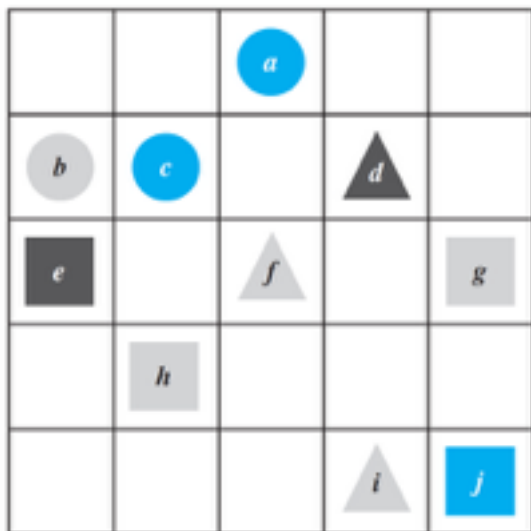
- using predicates to describe all properties of variables and omitting words “*such that*” in existential statements.
- also made use of following:

“ $\forall x \text{ in } D, P(x)$ ” written as “ $\forall x(x \text{ in } D \rightarrow P(x))$ ,”

“ $\exists x \text{ in } D \text{ such that } P(x)$ ” written as “ $\exists x(x \text{ in } D \wedge P(x))$ .”

# Formalizing Statements in a Tarski World

Consider once more the Tarski world:



Let  $\text{Triangle}(x)$  mean “ $x$  is a triangle,”  
 $\text{Circle}(x)$  mean “ $x$  is a circle,”  
 $\text{Square}(x)$  mean “ $x$  is a square”

$\text{Blue}(x)$  mean “ $x$  is blue,”  
 $\text{Gray}(x)$  means “ $x$  is gray,”  
 $\text{Black}(x)$  means “ $x$  is black”;

let  $\text{RightOf}(x, y)$

$\text{Above}(x, y)$ , and  $\text{SameColorAs}(x, y)$  mean “ $x$  is to the right of  $y$ ,” “ $x$  is above  $y$ ,” and “ $x$  has the same color as  $y$ ”; and use the notation  $x = y$  to denote the predicate “ $x$  is equal to  $y$ ”.

Let the common domain  $D$  of all variables be the set of all the objects in the Tarski world.

# Formalizing Statements in a Tarski World

Use formal, logical notation to write each of the following statements, and write a **formal negation** for each statement.

a. For all circles  $x$ ,  $x$  is above  $f$ .

# Example 10(a) – Solution

**For all circles  $x$ ,  $x$  is above  $f$ .**

*Statement:*

$$\forall x(\text{Circle}(x) \rightarrow \text{Above}(x, f)).$$

*Negation:*

$$\sim(\forall x(\text{Circle}(x) \rightarrow \text{Above}(x, f)))$$

$$\equiv \exists x \sim (\text{Circle}(x) \rightarrow \text{Above}(x, f))$$

$$\equiv \exists x(\text{Circle}(x) \wedge \sim \text{Above}(x, f))$$



# Example 10(b) – Solution

There is a square  $x$  such that  $x$  is black.

*Statement:*

$$\exists x(\text{Square}(x) \wedge \text{Black}(x)).$$

*Negation:*

$$\sim(\exists x(\text{Square}(x) \wedge \text{Black}(x)))$$

by the law for negating a  $\exists$  statement

$$\equiv \forall x \sim (\text{Square}(x) \wedge \text{Black}(x))$$

by De Morgan's law

$$\equiv \forall x(\sim \text{Square}(x) \vee \sim \text{Black}(x))$$

# Example 10(c) – Solution

For all circles  $x$ , there is a square  $y$  such that  $x$  and  $y$  have the same color.

*Statement:*

$$\forall x(\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))).$$

*Negation:*

$$\sim(\forall x(\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$$

$$\equiv \exists x \sim (\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$$

$$\equiv \exists x(\text{Circle}(x) \wedge \sim(\exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$$

$$\equiv \exists x(\text{Circle}(x) \wedge \forall y(\sim(\text{Square}(y) \wedge \text{SameColor}(x, y))))$$

$$\equiv \exists x(\text{Circle}(x) \wedge \forall y(\sim\text{Square}(y) \vee \sim\text{SameColor}(x, y)))$$

# Example 10(d) – Solution

There is a square  $x$  such that for all triangles  $y$ ,  $x$  is to right of  $y$ .

*Statement:*

$$\exists x(\text{Square}(x) \wedge \forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))).$$

*Negation:*

$$\begin{aligned} & \sim(\exists x(\text{Square}(x) \wedge \forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y)))) \\ \equiv & \forall x \sim (\text{Square}(x) \wedge \forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))) \\ \equiv & \forall x(\sim\text{Square}(x) \vee \sim(\forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y)))) \\ \equiv & \forall x(\sim\text{Square}(x) \vee \exists y(\sim(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y)))) \\ \equiv & \forall x(\sim\text{Square}(x) \vee \exists y(\text{Triangle}(y) \wedge \sim\text{RightOf}(x, y))) \end{aligned}$$

# Formal Logical Notation

The disadvantage of the fully formal notation is that because it is complex and somewhat remote from intuitive understanding, when we use it, we may make errors that go unrecognized.

The advantage, however, is that operations, such as taking negations, can be made completely mechanical and programmed on a computer.

Also, when we become comfortable with formal manipulations, we can use them to check our intuition, and then we can use our intuition to check our formal manipulations.

# Formal Logical Notation

Formal logical notation is used in branches of computer science such as artificial intelligence, program verification, and automata theory and formal languages.

Taken together, the symbols for quantifiers, variables, predicates, and logical connectives make up what is known as the **language of first-order logic**.

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- Statements with multiple quantifiers
- **Prolog**
- Argument with quantified statements

# Prolog

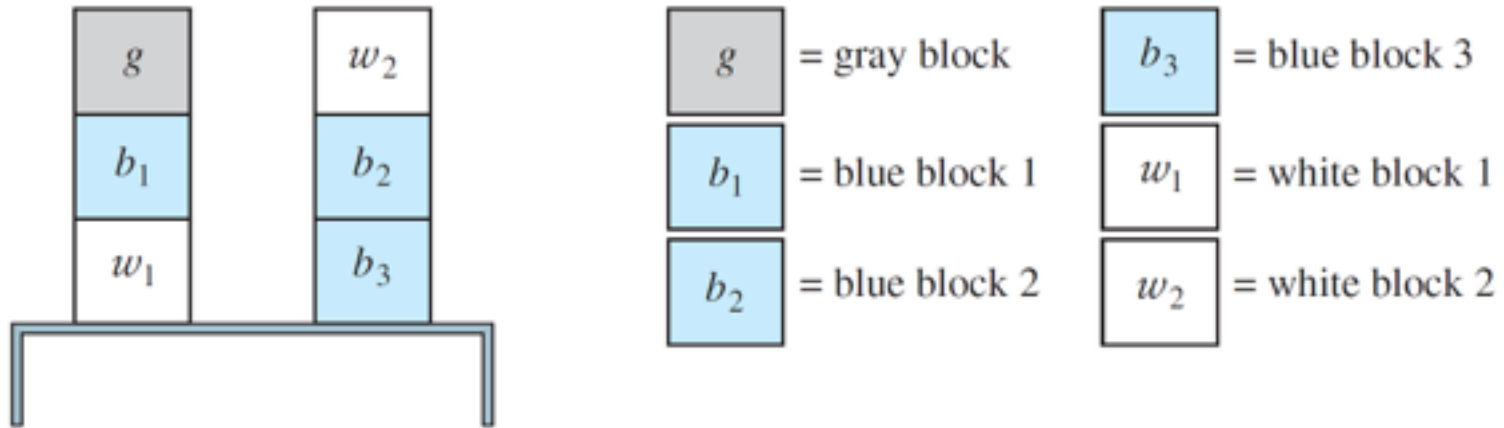
The programming language Prolog (short for *programming in logic*) was developed in France in the 1970s by A. Colmerauer and P. Roussel to help programmers working in field of artificial intelligence.

A simple Prolog program consists of a set of statements describing some situation together with questions about the situation. Built into the language are search and inference techniques needed to answer the questions by deriving the answers from the given statements.

This frees the programmer from the necessity of having to write separate programs to answer each type of question.

# Example: A Prolog Program

Consider following picture, which shows colored blocks stacked on a table.



The following statements in Prolog describe this picture and ask two questions about it.

`isabove(g, b1)`

`color(g, gray)`

`color(b3, blue)`



# Example: A Prolog Program

isabove( $b_1$ ,  $w_1$ )      color( $b_1$ , blue)      color( $w_1$ , white)

isabove( $w_2$ ,  $b_2$ )      color( $b_2$ , blue)      color( $w_2$ , white)

isabove( $b_2$ ,  $b_3$ )      isabove( $X$ ,  $Z$ ) if isabove( $X$ ,  $Y$ ) and  
isabove( $Y$ ,  $Z$ )

?color( $b_1$ , blue)      ?isabove( $X$ ,  $w_1$ )

The statements “isabove( $g$ ,  $b_1$ )” and “color( $g$ , gray)” are to be interpreted as “ $g$  is above  $b_1$ ” and “ $g$  is colored gray”. The statement “isabove( $X$ ,  $Z$ ) if isabove( $X$ ,  $Y$ ) and isabove( $Y$ ,  $Z$ )” is to be interpreted as “For all  $X$ ,  $Y$ , and  $Z$ , if  $X$  is above  $Y$  and  $Y$  is above  $Z$ , then  $X$  is above  $Z$ .”

# Example 11 – A Prolog Program

The program statement

`?color( $b_1$ , blue)`

is a question asking whether block  $b_1$  is colored blue. Prolog answers this by writing

Yes.

The statement

`?isabove( $X$ ,  $w_1$ )`

is a question asking for which blocks  $X$  the predicate “ $X$  is above  $w_1$ ” is true.

# Example 11 – A Prolog Program

Prolog answers by giving a list of all such blocks. In this case, the answer is

$$X = b_1, X = g.$$

Note that Prolog can find the solution  $X = b_1$  by merely searching the original set of given facts. However, Prolog must *infer* the solution  $X = g$  from the following statements:

isabove( $g, b_1$ ),

isabove( $b_1, w_1$ ),

isabove( $X, Z$ ) if isabove( $X, Y$ ) and isabove( $Y, Z$ ).

# Example 11 – A Prolog Program

Write the answers Prolog would give if the following questions were added to the program above.

**a.** ?isabove( $b_2$ ,  $w_1$ )    **b.** ?color( $w_1$ ,  $X$ )    **c.** ?color( $X$ , blue)

## Solution:

**a.** The question means “Is  $b_2$  above  $w_1$ ?”; so the answer is “No.”

**b.** The question means “For what colors  $X$  is the predicate ‘ $w_1$  is colored  $X$ ’ true?”; so the answer is “ $X = \text{white}$ .”

# Example 11 – Solution

- c. The question means “For what blocks is the predicate ‘ $X$  is colored blue’ true?”; so the answer is “ $X = b_1$ ,” “ $X = b_2$ ,” and “ $X = b_3$ .”

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# Universal Instantiation

The rule of *universal instantiation* (in-stan-she-AY-shun):

If some property is true of *everything* in a set, then it is true of *any particular* thing in the set.

*the* fundamental tool of **deductive reasoning**.

# Universal Instantiation

Math. formulas, definitions, and theorems are like general templates that are used over and over in a wide variety of particular situations.

A given theorem says that such and such is true for all things of a certain type.

If, in a given situation, you have a particular object of that type, then by universal instantiation, you conclude that such and such is true for that particular object.

You may repeat this process 10, 20, or more times in a single proof or problem solution.



# Universal Modus Ponens

The rule of **universal instantiation** can be combined with **modus ponens** to obtain a valid form of argument called *universal modus ponens*.

## Universal Modus Ponens

### *Formal Version*

$\forall x$ , if  $P(x)$  then  $Q(x)$ .

$P(a)$  for a particular  $a$ .

- $Q(a)$ .

### *Informal Version*

If  $x$  makes  $P(x)$  true, then  $x$  makes  $Q(x)$  true.

$a$  makes  $P(x)$  true.

- $a$  makes  $Q(x)$  true.

# Recognizing Universal Modus Ponens

Rewrite following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

If an integer is even, then its square is even.

$k$  is a particular integer that is even.

$\therefore k^2$  is even.

Hint:

The major premise of this argument can be rewritten as

$\forall x$ , if  $x$  is an even integer then  $x^2$  is even.

# Use of Universal Modus Ponens in a Proof

Prove that the sum of any two even integers is even.

It makes use of the definition of even integer, namely, that an integer is *even* if, and only if, it equals twice some integer. (Or, more formally:  $\forall$  integer  $x$ ,  $x$  is even if, and only if,  $\exists$  an integer  $k$  such that  $x = 2k$ .)

Suppose  $m$  and  $n$  are particular but arbitrarily chosen even integers. Then  $m = 2r$  for some integer  $r$ ,<sup>(1)</sup> and  $n = 2s$  for some integer  $s$ .<sup>(2)</sup>

# Use of Universal Modus Ponens in a Proof

Hence

$$\begin{aligned}m + n &= 2r + 2s && \text{by substitution} \\ &= 2(r + s)^{(3)} && \text{by factoring out the 2.}\end{aligned}$$

Now  $r + s$  is an integer,<sup>(4)</sup> and so  $2(r + s)$  is even.<sup>(5)</sup>

Thus  $m + n$  is even.

# Use of Universal Modus Ponens in a Proof

The following expansion of the proof shows how each of the numbered steps is justified by arguments that are valid by universal modus ponens.

- (1) If an integer is even, then it equals twice some integer.  
 $m$  is a particular even integer.
  - $m$  equals twice some integer  $r$ .
  
- (2) If an integer is even, then it equals twice some integer.  
 $n$  is a particular even integer.
  - $n$  equals twice some integer  $s$ .

# Use of Universal Modus Ponens in a Proof

(3) If a quantity is an integer, then it is a real number.

$r$  and  $s$  are particular integers.

- $r$  and  $s$  are real numbers.

For all  $a$ ,  $b$ , and  $c$ , if  $a$ ,  $b$ , and  $c$  are real numbers, then  $ab + ac = a(b + c)$ .

2,  $r$ , and  $s$  are particular real numbers.

- $2r + 2s = 2(r + s)$ .

(4) For all  $u$  and  $v$ , if  $u$  and  $v$  are integers, then  $u + v$  is an integer.

$r$  and  $s$  are two particular integers.

- $r + s$  is an integer.

# Use of Universal Modus Ponens in a Proof

(5) If a number equals twice some integer, then that number is even.

$2(r + s)$  equals twice the integer  $r + s$ .

- $2(r + s)$  is even.

# Universal Modus Tollens

*Universal modus tollens*: results from combining universal instantiation with modus tollens.

- heart of proof of contradiction, which is one of the most important methods of mathematical argument.

## Universal Modus Tollens

### *Formal Version*

$\forall x$ , if  $P(x)$  then  $Q(x)$ .

$\sim Q(a)$ , for a particular  $a$ .

- $\sim P(a)$ .

### *Informal Version*

If  $x$  makes  $P(x)$  true, then  $x$  makes  $Q(x)$  true.

$a$  does not make  $Q(x)$  true.

- $a$  does not make  $P(x)$  true.



# Recognizing the Form of Universal Modus Tollens

Rewrite argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All human beings are mortal.

Zeus is not mortal.

- Zeus is not human.

hint:

The major premise can be rewritten as

$\forall x$ , if  $x$  is human then  $x$  is mortal.

# Validity of Arguments with Quantified Statements

An argument is valid if, and only if, the truth of its conclusion follows *necessarily* from the truth of its premises.

- **Definition**

To say that an *argument form* is **valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true. An *argument* is called **valid** if, and only if, its form is valid.

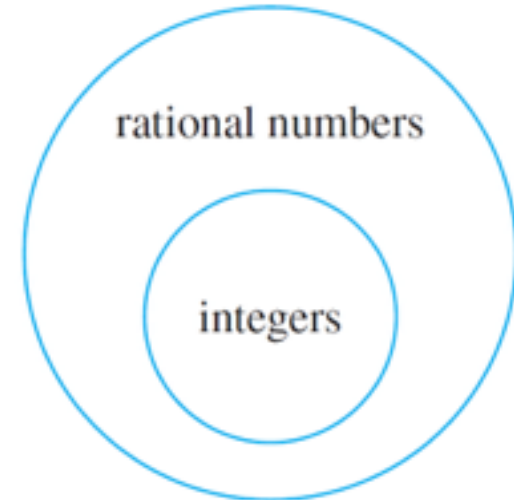
# Using Diagrams to Test for Validity

Consider the statement

All integers are rational numbers.

Or, formally,

$\forall$  integers  $n$ ,  $n$  is a rational number.



Picture the set of all integers and the set of all rational numbers as disks.

The truth of the given statement is represented by placing the integers disk entirely inside the rationals disk

# Using Diagrams to Test for Validity

To test the validity of an argument diagrammatically, represent the truth of both premises with diagrams.

Then analyze the diagrams to see whether they necessarily represent the truth of the conclusion as well.

# Using Diagrams to Show Invalidity

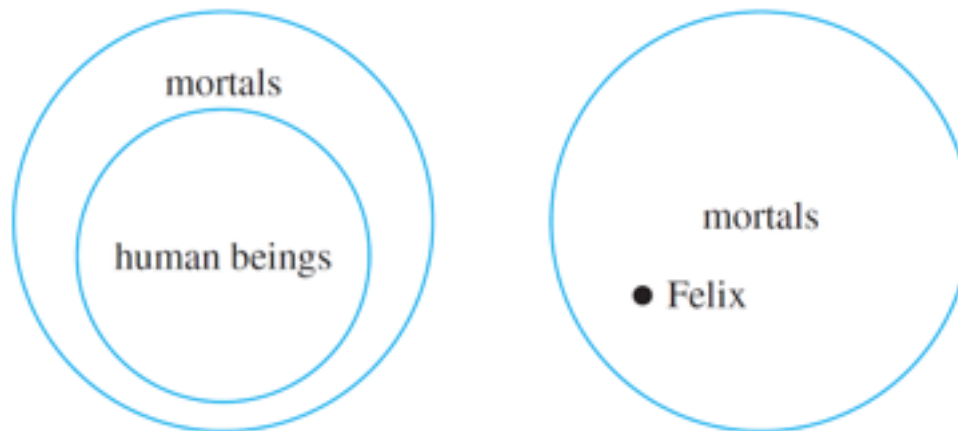
Use a diagram to show the invalidity of the following argument:

All human beings are mortal.

Felix is mortal.

- Felix is a human being.

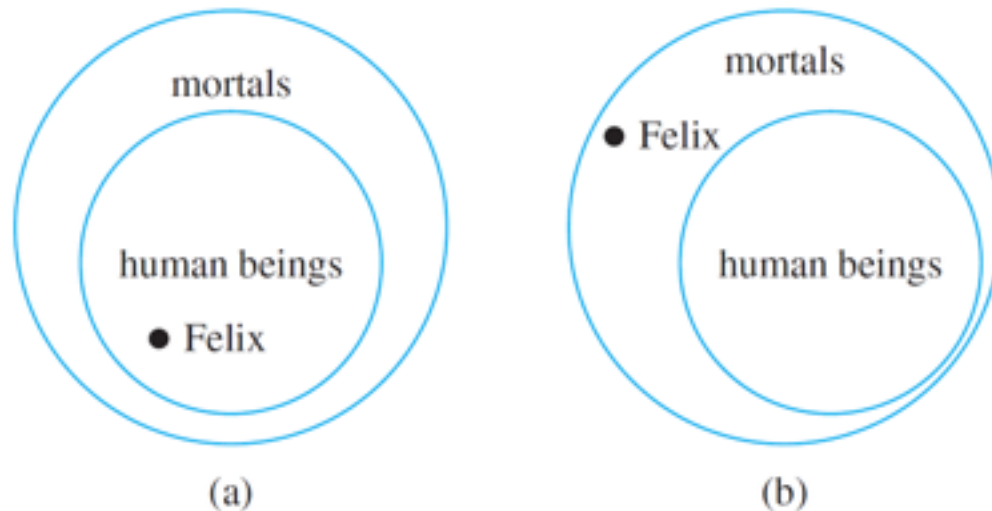
The major and minor premises are represented diagrammatically below:



# Solution

All that is known is that the Felix dot is located *somewhere* inside the mortals disk. Where it is located with respect to the human beings disk cannot be determined.

Two possibilities:



Conclusion “Felix is a human being” is true in the first case but not in the second (Felix might, for example, be a cat). Because the conclusion does not necessarily follow from the premises, the argument is invalid.

# Using Diagrams to Test for Validity

All human beings are mortal.

Felix is mortal.

- Felix is a human being.

This argument would be valid if major premise were replaced by its converse.

We say that this argument exhibits the converse error.

## Converse Error (Quantified Form)

### *Formal Version*

$\forall x$ , if  $P(x)$  then  $Q(x)$ .

$Q(a)$  for a particular  $a$ .

- $P(a)$ . ← invalid conclusion

### *Informal Version*

If  $x$  makes  $P(x)$  true, then  $x$  makes  $Q(x)$  true.

$a$  makes  $Q(x)$  true.

- $a$  makes  $P(x)$  true. ← invalid conclusion

# Using Diagrams to Test for Validity

The following form of argument would be valid if a conditional statement were logically equivalent to its inverse. But it is not, and the argument form is invalid.

We say that it exhibits the inverse error.

## Inverse Error (Quantified Form)

### *Formal Version*

$\forall x$ , if  $P(x)$  then  $Q(x)$ .

$\sim P(a)$ , for a particular  $a$ .

- $\sim Q(a)$ . ← invalid conclusion

### *Informal Version*

If  $x$  makes  $P(x)$  true, then  $x$  makes  $Q(x)$  true.

$a$  does not make  $P(x)$  true.

- $a$  does not make  $Q(x)$  true. ← invalid conclusion



# An Argument with “No”

Use diagrams to test the following argument for validity:

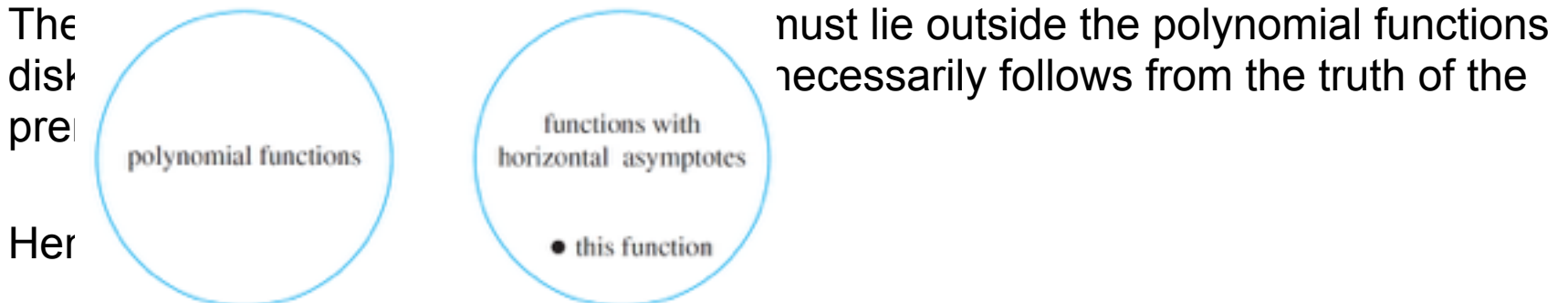
No polynomial functions have horizontal asymptotes.

This function has a horizontal asymptote.

- This function is not a polynomial function.
- •

Represent major premise: two non-overlapping disks

The minor premise is represented by placing a dot labeled “this function” inside the disk for functions with horizontal asymptotes.



# Examine form of argument

An alternative approach to this example is to transform the statement “No polynomial functions have horizontal asymptotes” into the equivalent form “ $\forall x$ , if  $x$  is a polynomial function, then  $x$  does not have a horizontal asymptote.”

If this is done, the argument can be seen to have the form

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$
$$\sim Q(a), \text{ for a particular } a.$$
$$\therefore \sim P(a).$$

where  $P(x)$  is “ $x$  is a polynomial function” and  $Q(x)$  is “ $x$  does not have a horizontal asymptote.”

This is valid by universal modus tollens.

# Creating Additional Forms of Argument

Universal modus ponens and modus tollens were obtained by combining universal instantiation with modus ponens and modus tollens.

In the same way, additional forms of arguments involving universally quantified statements can be obtained by combining universal instantiation with other of the valid argument forms discussed earlier.

# Creating Additional Forms of Argument

Consider the following argument:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \therefore p \rightarrow r \end{array}$$

This argument form can be combined with universal instantiation to obtain the following valid argument form.

## Universal Transitivity

*Formal Version*

- $\forall x P(x) \rightarrow Q(x).$
- $\forall x Q(x) \rightarrow R(x).$
- $\forall x P(x) \rightarrow R(x).$

*Informal Version*

- Any  $x$  that makes  $P(x)$  true makes  $Q(x)$  true.
- Any  $x$  that makes  $Q(x)$  true makes  $R(x)$  true.
- Any  $x$  that makes  $P(x)$  true makes  $R(x)$  true.

# Outline

- Intro to predicate logic
- **Predicate**, truth set
- **Quantifiers**, universal, existential statements, universal conditional statements
- Reading & writing quantified statements
- **Negation** of quantified statements
- Converse, Inverse and contrapositive of universal conditional statements
- Statements with multiple quantifiers
- Argument with quantified statements

# Remark on Converse and Inverse Errors

A variation of converse error is a very useful reasoning tool, provided that it is used with caution.

It is the type of reasoning that is used by doctors to make medical diagnoses and by auto mechanics to repair cars.

It is the type of reasoning used to generate explanations for phenomena. It goes like this: If a statement of the form

For all  $x$ , if  $P(x)$  then  $Q(x)$

is true, and if

$Q(a)$  is true, for a particular  $a$ ,

then check out the statement  $P(a)$ ; it just might be true.

# Example

For instance, suppose a doctor knows that

For all  $x$ , if  $x$  has pneumonia, then  $x$  has a fever and chills, coughs deeply, and feels exceptionally tired and miserable.

And suppose the doctor also knows that

John has a fever and chills, coughs deeply, and feels exceptionally tired and miserable.

On the basis of these data, the doctor concludes that a diagnosis of pneumonia is a strong possibility, though not a certainty.

# Remark on Converse and Inverse Errors

The doctor will probably attempt to gain further support for this diagnosis through laboratory testing that is specifically designed to detect pneumonia.

Note that the closer a set of symptoms comes to being a necessary and sufficient condition for an illness, the more nearly certain the doctor can be of his or her diagnosis.

This form of reasoning has been named **abduction** by researchers working in artificial intelligence. It is used in certain computer programs, called expert systems, that attempt to duplicate the functioning of an expert in some field of knowledge.