3 Sums and Integrals

Definite integrals are limits of sums. We will therefore begin our study of integrals by reviewing finite sums and the relation between sums and integrals. This will allow you to understand approximate values of integrals even when you can't evaluate the integral analytically (another instance of gaining number sense!). The first topic, finite sums, is very elementary but I don't know any good references so I'm including a reasonably complete treatment.

3.1 Finite sums

The preparatory homework for this sections deals with the nuts and bolts of writing finite sums. If given a sum such as $\displaystyle\sum_{n=5}^{19} \frac{3}{n-2}$ you should easily be able to tell what explicit sum it represents: how many terms, what are the first few and the last, how would you write it using an equation with ... and so forth. The above sum, for example, contains 15 terms and could be written as $\displaystyle\frac{3}{3} + \frac{3}{4} + \cdots + \frac{3}{17}$.

It is a little harder going the other way, writing a sum in Sigma notation when you are given its terms. One reason is that there is more than one way to do this. For example there is no reason why the index in the previous sum should go from 5 to 19. There have to be fifteen terms but why not write it with the index going from 1 to 15? Then it would look like

$$\sum_{n=1}^{15} \frac{3}{n+2} .$$

Another natural choice is to let the index run from 0 to 14:

$$\sum_{n=0}^{14} \frac{3}{n+3} .$$

All three of these formulas represent the exact same sum.

Another difficulty is that you need to know tricks to represent certain patterns with formulas. Really this is not a difficulty with smmations as much as with writing a formula to represent the general term a_n of a given sequence. Realize that these problems are inherently the same: writing the n^{th} term of a sequence as a function of n and writing the summand in a summation as a function of its index. The preparatory homework starts off with sequence writing and then has you do some summations as well.

Here are some tricks to write certain patterns. The term $(-1)^n$ bounces back and forth between $+1$ and -1, starting with -1 when $n = 1$ (or starting with $+1$ if your sum has a term for $n = 0$). You can incorporate this in a sum as a multiplicative factor and it will change the sign of every second term. Thus for example, to write the sum $1 - 2 + 3 - 4 + \cdots - 100$ you can write

$$\sum_{n=1}^{100} (-1)^{n+1} \cdot n\,.$$

Note that we used $(-1)^{n+1}$ rather than $(-1)^n$ so as to start off with a positive rather than a negative term.

When the sum has a pattern that takes a couple of steps to repeat, the greatest integer function can be useful. For example, $1+1+1+2+2+2+3+3+3+\cdots+10+10+10$ can be written as $\displaystyle\sum_{n=1}^{30} \left\lfloor \frac{n+2}{3} \right\rfloor$.

Sequences and sums can use definitions by cases just the way functions do. Suppose you want to define a sequence with an opposite sign on every third term, such as $-1, -1, 1, -1 - 1, 1, \ldots$. You can do this by cases as follows.

$$a_n = \begin{cases} -1 & n \text{ is not a multiple of } 3 \\ 1 & n \text{ is a multiple of } 3 \end{cases}$$

Although you will not be required to know this, you can use sophisticated tricks to avoid this kind of definition by cases. One way[1] is to use the greatest integer function:

$$a_n = (-1)^{\lfloor 2(n-1)/3 \rfloor}\,.$$

Notational observations: A sequence denoted $a_1, a_2, a_3, \ldots$ could just as easily be written as a function $a(1), a(2), a(3), \ldots$. The value of a term a_n is a function of the index n and there is no difference whether we write n as a subscript or as an argument.

[1] Another way is to use complex numbers, but you'll have to ask me about that separately if you're curious.

Series you can explicitly sum

We will learn to sum three kinds of series: arithmetic (accent on the third syllable) series, geometric series and telescoping series.

Arithmetic series

An arithmetic series is a sum in which the terms increase or decrease by the same amount (additively) each time. You can always write these in the form $a_n = A + dn$ where A is the initial term and d is how much each term increases over the one before (it could be negative if the terms decrease). Here you should start the sum at $n = 0$ or else use the term $A + (d - 1)n$. The standard trick for summing these is to pair up the first and last, the second and second-to-last, and so on, recognizing that each pair sums to twice the average and therefore that the sum is the number of terms times the average term. Here is an example in a particular case and then the general formula.

Example: Evaluate $\displaystyle\sum_{n=13}^{29} n$. There are 17 terms and the average is 21, which can be computed by averaging the first and last terms: $(13 + 29)/2 = 21$. Therefore, the sum is equal to $17 \times 21 = 357$.

General case: Evaluate $\displaystyle\sum_{n=0}^{M} A + dn$. There are $M+1$ terms and the average is $A+(dM/2)$. Therefore the sum is equal to $(M + 1)(A + (dM/2)) = A(M + 1) + dM(M + 1)/2$.

Geometric series

A geometric series is a sum in which the terms increase or decrease by the same multiplicative factor each time. You can always write these in the form $a_n = A \cdot r^n$ where A is the initial term and r is the factor by which the term increases each time. If the terms decrease then r will be less than 1. If they alternate in sign, r will be negative. Also, again, A will be the initial term only if one starts with the $n = 0$ term or changes the summand to $A \cdot r^{n-1}$.

The standard trick for summing these is to notice that the sum and r times the sum are very similar. I'll explain with an example.

EXAMPLE: Evaluate $\displaystyle\sum_{n=1}^{10} 7 \cdot 4^{n-1}$.

To do this we let S denote the value of the sum. We then evaluate $S - 4S$ (because $r = 4$). I have written this out so you can see the cancellation better.

$$\begin{aligned} S - 4S &= 7 + 28 + 112 + \cdots + 7 \cdot 4^9 \\ &\; - \; (28 + 112 + \cdots + 7 \cdot 4^9 + 7 \cdot 4^{10}) \\ \\ &= 7 - 7 \cdot 4^{10} . \end{aligned}$$

From this we easily get $S = (7 - 7 \cdot 4^{10})/(1 - 4) = 7(4^{10} - 1)/3$.

GENERAL CASE: Evaluate $\displaystyle\sum_{n=1}^{M} A \cdot r^{n-1}$.

Letting S denote the sum we have $S - rS = A - Ar^n$ and therefore

$$S = A \, \frac{1 - r^n}{1 - r} .$$

Infinite series

No discussion of finite series would be complete without a mention of infinite series. There is a whole theory of convergence of infinite series that they teach in Math 104. Here we'll stick to what's practical. It should be obvious that $1 + 2 + 4 + \cdots$ does NOT converge, while $1/2+1/4+1/8+\cdots$ DOES converge, and in fact converges to 1. There are eleven theorems and tests in the book about when series converge. From a practical point of view, all you need is two things: the definition, and an example.

> *Definition:* An infinite sum $\sum_{n=1}^{\infty} a_n$ is said to converge if and only if the partial sums $S_M = \sum_{n=1}^{M} a_n$ form a convergent sequence. In other words, if $\lim_{M\to\infty} S_M$ exists and is equal to L, then $\sum_{n=1}^{\infty} a_n$ is said to equal L.

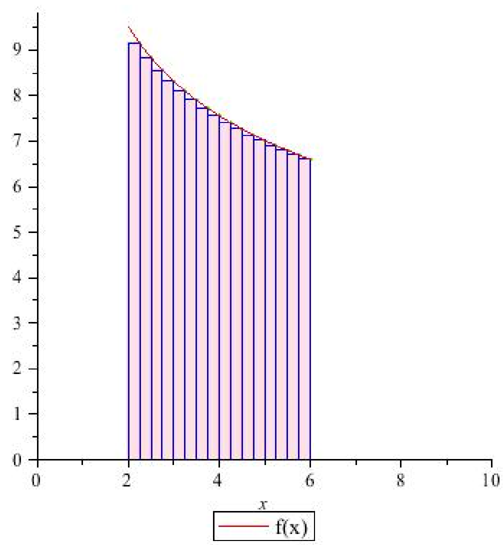
EXAMPLE: If $a_n = (1/2)^n$ then $S_M = 1 - (1/2)^M$. Clearly $\lim_{M\to\infty} S_M = 1$ so we say that $\sum_{n=1}^{\infty} (1/2)^n = 1$.

3.2 Riemann sums

In this unit we recap how areas lead to integrals and then, by the Fundamental Theorem of Calculus, to anti-derivatives.

Areas under graphs

Thankfully, Sections 5.1–5.3 do a nice job in explaining areas of regions under graphs as limits of areas of regions composed of rectangles. I will just point out the highlights. This figure shows a classical rectangular approximation to the region under a graph $y = f(x)$ between the x values of 2 and 6. The rectangular approximation is composed of 16 rectangles of equal width, all of which have their base on the x-axis and their top edge intersecting the graph $y = f(x)$. The rectangular approximation is clearly very near to the actual region, therefore the area of the region will be well approximated by the area of the rectangular approximation. This is easy to compute: just sum the width times height. The sum that gives this area is known as a Riemann sum.



Because the height is not constant over the little interval, there is no one correct height. You could certainly cover the targeted area with your rectangles by always choosing the highest point in each interval. That is called the **upper Riemann sum** (see page 300). If you go only as high as the least value of f in the interval, that is the **lower Riemann sum**, and these rectangles together will surely lie inside your targeted area. If one always chooses the top-left corner of the rectangle to lie on the graph then this is called the **left-Riemann sum**; if one always chooses the top-right corner of the rectangle to lie on the graph, this is called **right-Riemann sum**.

If f is increasing over the whole interval $[a, b]$ then a left-Riemann sum will also be a lower Riemann sum and a right-Riemann sum will be an upper Riemann sum; if f is decreasing, this correspondence is reversed. The example in the figure is of a right-Riemann sum, which is also a lower Riemann sum, with $a = 2$, $b = 4$, and a partition of the x-axis into 16 equal strips.

The definite integral is defined as such a limit. Specifically,

$$\int_a^b f(x)\,dx$$

is defined as the limit of the Riemann sums as the width of the rectangles goes to zero. So far we have not invoked the Fundamental Theorem of Calculus, so we are not connecting this with any kind of anti-derivative. We just have a definition of $\int_a^b f(x)\,dx$.

Interpretations other than area

Most people who compute integrals are not particularly interested in areas of regions. Integrals are interesting because the same math that computes the area of a region computes many other things as well. In general, it represents a total. If $f(t)$ is a quantity of something being delivered over time, such as water flow in gallons per minute, then $\int_a^b f(t)\,dt$ is the total amount delivered between time a and time b. If $f(t)$ is an acceleration then $\int_a^b f(t)\,dt$ is the total change in velocity from time a to time b.

The units of $\int_a^b f(x)\,dx$ are the units of f times the units of x. You can see this because the rectangles that make up the Riemann sum have units of height (units of f) times width (units of x). For example, suppose the x-axis is time (say hours) and the y-axis is number of people working at the given time; then the area is interpreted as person-hours of work (formerly known as man-hours). Thus $\displaystyle\int_a^b f(x)\,dx$ represents the total person-hours worked from time a to time b. If the y-axis represents a rate of change as the x-axis quantity changes, then the area represents total change. For example if x is time and y is velocity (rate of change of position with respect to time) then $\displaystyle\int_a^b f(x)\,dx$ is the total change in position from time a to time b. The units still come out right because velocity $\times$ time $=$ distance.

Fundamental Theorem of Calculus

The principles allowing us to evaluate integrals are these.

> (1) If the partition P is sufficiently fine, the upper Riemann sum U and the lower Riemann sum L will be very close. In fact the limit as P becomes finer of U and the limit as P becomes finer of L both exist and are equal to a number I which is, by definition, the value of the integral.
>
> (2) Amazingly, you can evaluate I exactly if you can find an anti-derivative F for the function f. The value of I will then be $F(b) - F(a)$.

The last part of this definition/theorem is a version of the Fundamental Theorem of Calculus. I suppose you already know it, but it's still very cool. Let's concentrate though on the other part. It says that we can use integrals to estimate sums or bound them and vice versa. In the next section we will discuss using U and L to get bounds. For now, we'll just say: if P is reasonably fine then any Riemann sum (U, L, or something in between) is pretty close to I.

A note on signed area: Area is defined to be a positive quantity. However, integrals compute *signed* area. Thus $\int_a^b f(x)\,dx$ computes the area between $x = a$ and $x = b$ below the graph of f but above the x-axis, with area below the x-axis counting as negative. Be careful about this, especially if there are both positive and negative pieces of the area.

Anti-derivatives

The FTC says areas are computed by anti-derivatives. Students from previous terms identified confusion as to exactly what an anti-derivative is. The **indefinite integral** has the notation $\int f(x)\,dx$ and represents any function whose derivative is the function $f(x)$. Let's say $F(x)$ is such a function.

The confusion lies when considering $F(x)$ as both a function of x and an integral. What integral has derivative equal to $f(x)$? Answer: $\int_a^x f(t)\,dt$. Note several things. (1) x appears as the upper limit of the integral. (2) we changed the name of the variable of integration to t. (3) the lower limit of integration can be any constant. It is (2) that is the most confusing: an integral from a constant to a variable is a function of that variable! If you're wondering why we changed x to t inside, it's to avoid confusion. The variable of integration is a **bound variable** also known as a **dummy variable**. It has no value, rather it is summed over. The value of the definite integral $\int_b^x f(t)\,dt$ depends on the values of b and x and the function f, but not on the value of t; there is no value of t. Please compare to pages 329–330 and Theorem 4 in the textbook.

3.3 Bounding and estimating integrals and sums

Both integrals and sums represent areas: an integral is the area under a curve and a sum is an area under a bunch of rectangles. You know one area is bigger than another when the first region completely covers the second region. Based on this, you can bound an integral by a sum or vice versa. To find a sum that is an upper bound for an integral, represent the integral as an area and find a sum whose area representation covers that of the integral. This is just the same as finding in upper Riemann sum. Similarly you can find a sum to give a lower bound for an integral, namely a lower Riemann sum. Going the other way, if you have a sum you can find an integral whose area completely covers that of the sum, which will give you an upper bound for the sum. Similarly, an integral whose area is completely contained in the rectangles for the sum will be a lower bound for the sum. We will practice this both ways: first, given an integral, bound it above and below by sums; secondly, given a sum, bound it above and below by integrals. At the very end of this section, we'll see how to get a good estimate for an integral that is neither an upper nor a lower bound (the trapezoidal estimate).

Estimating integrals using sums

The upper Riemann sum U is always an upper bound and the lower sum L is always a lower bound. When the function is monotone (either increasing or decreasing) then these are left- or right-Riemann sums and can therefore be computed routinely (though it may be tedious).

EXAMPLE: Find ten-term sums that are upper and lower bounds for $\displaystyle\int_1^2 \frac{1}{1+x^3}\,dx$.

The function $1/(1+x^3)$ is decreasing so the the left-Riemann sum (evaluate f at the left endpoint of each interval) is an always an upper sum and the right-Riemann sum is always a lower sum. These sums are easy to represent.

$$U = \sum_{j=0}^{9} f\left(1+\frac{j}{10}\right)\cdot\left(\frac{1}{10}\right) \tag{3.1}$$

$$L = \sum_{j=1}^{10} f\left(1+\frac{j}{10}\right)\cdot\left(\frac{1}{10}\right)$$

You can evaluate these by hand as 0.27430... and 0.2354... respectively [2].

Estimating sums using integrals

It is more interesting going the other way. Given a sum, how do we bound it by an integral? It's not hard to write an integral to which the sum is approximately equal, but to ensure that the integral lies above or below the sum we might have to do some fiddling. We use the fact that if a sum S is an upper Riemann sum for an integral I then I is a lower bound for S.

EXAMPLE: Find upper and lower bounds for the sum S_n defined by $\displaystyle\sum_{k=1}^{n} \frac{1}{k}$.

The lower bound is easy: if we put a rectangle of height $1/k$ above the interval $[k, k+1]$, for each k from 1 to n, then the union of rectangles is the upper Riemann sum for $\displaystyle\int_1^{n+1} \frac{1}{x}\,dx$. Thus $S \geq \ln(n+1)$ and we have found a lower bound for S. For an upper bound, one trick that works is to use fit all terms but the first of S underneath the graph of $1/x$ from 1 to n and then add the extra 1 from the first term. Thus $S \leq 1 + \int_1^n (1/x)\,dx = 1 + \ln(n)$. To summarize,

$$\ln(n+1) \leq S \leq 1 + \ln(n)\,.$$

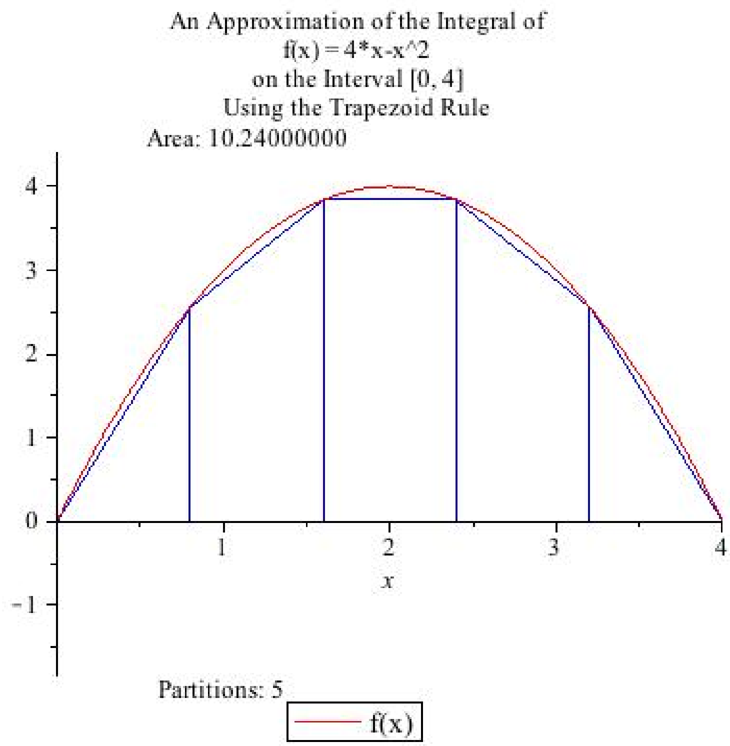
For $n = 50$ this comes out approximately as $3.93 \leq S \leq 4.92$.

Trapezoidal approximation

Sometimes it can be frustrating using Riemann sums because a lot of calculation doesn't get you all that good an approximation. You can see a lot of "white space" between the function f and the horzontal lines at the top of the rectangles that make up the upper or lower Riemann sum. If instead you let the rectangle become a right trapezoid, with both its top-left and top-right corner on the graph $y = f(x)$, then you get what is known as the **trapezoidal approximation**. The figure shows a trapezoidal approximation of an integral $\int_0^4 f(x)\,dx$ with five trapezoids. Note that the first and last trapezoid are degenerate, that is, one of the vertical sides has length

[2] If you are able to evaluate the integral exactly as $\pi/(6\sqrt{3}) + \ln(3/4)/6 \approx 0.25435...$ then you probably shouldn't be in this course.

zero and the trapezoid is actually a right triangle. It is perfectly legitimate for one or more of the trapezoids to be degenerate.



Because the tops of the slices are allowed to slant, they remain much closer to the graph $y = f(x)$ than do the Riemann sums. Because the area of a right trapezoid is the average of the areas of the two rectangles whose heights are the value of f at the two endpoints, it is easy to compute the trapezoidal approximation: it is just the average of the left-Riemann sum and the right-Riemann sum corresponding to the same partition into vertical strips.

Let's check what the trapezoidal approximation gives for the integral at the beginning of this section: $\int_1^2 \frac{1}{1+x^2}$. Adding the formlae for U and L and dividing by 2 yields

$$\frac{1}{2}\frac{f(1)}{10} + \frac{1}{2}\frac{f(0)}{10}\sum_1^9 \frac{1}{10} f\left(1 + \frac{j}{10}\right) .$$

In words, sum the values of f along a regular grid of x-values, counting endpoints as half, and multiply by the spacing between consecutive points.

The trapezoidal estimate is usually much closer than the upper or lower estimate, though it has the drawback of being neither an upper nor a lower bound. However, if you know the function to be concave upward then the trapezoidal estimate is an upper bound. Similarly if $f'' < 0$ on the interval then the trapezoidal estimate is an lower bound. In the figure, f is concave downward and the trapezoidal estimate is indeed a lower bound.

Example: The function $1/(1+x^3)$ is concave upward on $[1, 2]$ (compute and see that the second derivative is a positive quantity divided by $(1 + x^3)^3$) so the trapezoidal estimate should be not only very close but an upper bound. Indeed, the trapezoidal estimate is the average of the upper and lower previously computed and is equal to 0.25485... which is indeed just slightly higher than the true value of 0.25425....