

34. What is the smallest integer that is greater than $\frac{348}{37}$?
 35. What is the smallest even integer that is greater than $12 + \sqrt{5}$?
 36. What is the largest prime number that is less than $\frac{23}{0.23}$?

Exercises 37–50 Complex Number Arithmetic Perform the indicated operations. Express the result as a complex number in standard form.

37. $(5 + 2i) + (3 - 6i)$ 38. $(3 - i) + (-1 + 5i)$
 39. $(6 - i) - (3 - 4i)$ 40. $8 - (3 + 5i) + 2i$
 41. $(2 + i)(3 - i)$ 42. $(-1 + i)(2 + 3i)$
 43. $(1 + 3i)(1 - 3i)$ 44. $(7 - 2i)(7 + 2i)$
 45. $\frac{1 + 3i}{i}$ 46. $\frac{1 + i}{1 - i}$
 47. $(1 + \sqrt{3}i)^2$ 48. $\frac{2i}{(1 - i)(2 - i)}$
 49. (a) i^2 (b) i^6 (c) i^{12} (d) i^{18}
 50. (a) i^5 (b) i^9 (c) i^{15} (d) i^{21}

Exercises 51–55 If z is $1 - i$ and w is $-2 + i$, express in standard form.

51. $z + 3w$ 52. $zw - 4$ 53. $\bar{z} \cdot \bar{w}$
 54. $|z + w|$ 55. $\frac{z - \bar{w}}{w}$

Exercises 56–59 Complex Plane For the given z and w , show in the complex plane:

- (a) z (b) w (c) \bar{z} (d) $z + w$ (e) $z \cdot w$

56. $z = 2 - 2i$; $w = 3 + 4i$
 57. $z = -3 + 2i$; $w = -2 - i$
 58. $z = -1 + 2i$; $w = 3i$
 59. $z = 5 - i$; $w = -1 + i$
 60. (a) If $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$; find z^2, z^3, z^4, z^5, z^6 .
 (b) Evaluate each of $|z|, |z^2|, |z^3|, \dots, |z^6|$.
 61. From Exercise 60 draw a diagram showing z, z^2, z^3, z^4, z^5 , and z^6 in the complex plane. Note the distance from each of these points to the origin. On what circle do these points lie?

Exercises 62–63 In Exercises 60–61, replace z with $z = \frac{1}{\sqrt{2}}(1 + i)$.

1.4 RECTANGULAR COORDINATES, TECHNOLOGY, AND GRAPHS

Creative people live in two worlds. One is the ordinary world which they share with others and in which they are not in any special way set apart from their fellow men. The other is private and it is in this world that the creative acts take place. It is a world with its own passions, elations and despairs, and it is here that, if one is as great as Einstein, one may even hear the voice of God.

Mark Kac

Rectangular Coordinates

Few intellectual discoveries have had more far-reaching consequences than coordinating the plane by René Descartes nearly 400 years ago. We speak of **Cartesian** or rectangular coordinates in his honor.

A rectangular coordinate system uses two perpendicular number lines in the plane, which we call coordinate axes. The more common orientation is a horizontal **x-axis** and a vertical **y-axis**, but other variable names and orientations are sometimes useful.

Each point P in the plane is identified by an ordered pair of real numbers (c, d) , called the **coordinates** of P , where c and d are numbers on the respective axes as shown in Figure 10. Conversely, every pair of real numbers names a unique point on the plane.

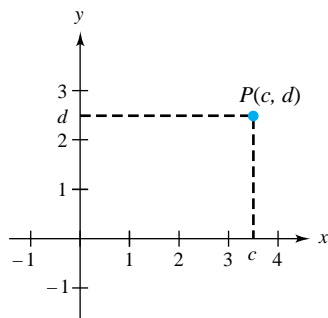


FIGURE 10

A rectangular system of coordinates provides a one-to-one correspondence between the set of ordered pairs of real numbers and the points in the plane.

I had lots of exams at school. At sixteen I took [a nationwide exam] in mathematics, physics, and chemistry, and was told that if I passed chemistry I could then drop it and do just pure math, applied math, and physics. So I did. . . I now realize that I quite enjoyed organic chemistry because that ties in somewhat with graph theory.

Robin Wilson

The axes divide the plane into four quadrants labeled I, II, III, IV, as shown in Figure 11. In the figure, points A and B are in Quadrant I, C is in II, D is in III, and E is in IV. Point F is on the x -axis while G is on the y -axis; points on the coordinate axes are not in any quadrant.

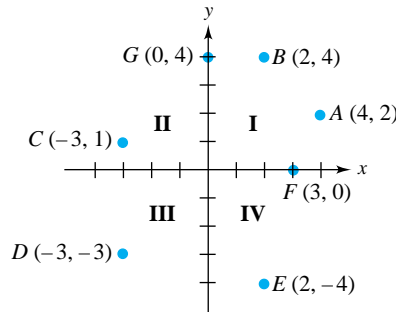


FIGURE 11

The distance $d(r, s)$ between points r and s on a number line is expressed in terms of absolute value.

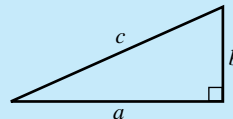
$$d(r, s) = |r - s|$$

We extend the idea of distance to the coordinate plane by means of the familiar Pythagorean Theorem.

Pythagorean theorem

Suppose a and b are the lengths of the legs of a right triangle and c is the hypotenuse. Then

$$a^2 + b^2 = c^2.$$



Conversely, suppose $a^2 + b^2 = c^2$. Then the triangle must be a right triangle with hypotenuse c .

Distance Between Points in a Plane

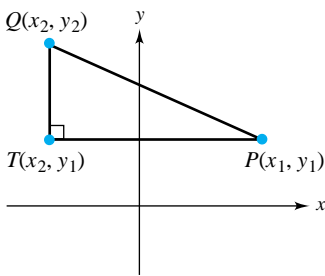


FIGURE 12

Suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are any two points in the plane. The distance between P and Q , denoted $d(P, Q)$ or $|\overline{PQ}|$, is defined to be the length of the line segment between P and Q . (\overline{PQ} denotes the line segment from P to Q .) Figure 12 shows right triangle PTQ whose legs are given in terms of absolute values.

$$|\overline{PT}| = |x_1 - x_2| \quad \text{and} \quad |\overline{TQ}| = |y_1 - y_2|$$

Applying the Pythagorean theorem gives

$$|\overline{PQ}|^2 = |\overline{PT}|^2 + |\overline{TQ}|^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2.$$

Since $|x_1 - x_2|^2 = (x_1 - x_2)^2$ and $|y_1 - y_2|^2 = (y_1 - y_2)^2$, we have the following.

Distance formula

Suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are any two points in the plane. If $d(P, Q)$ denotes the distance between P and Q , then

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

We can also write $d(P, Q)$ as $|\overline{PQ}|$.

Midpoint of a Line Segment

Suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are any two points in the plane. To get the midpoint M of the line segment \overline{PQ} , we take the average of the two x -values and the average of the two y -values. M is the point

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

It is easy to show that $d(P, M) = d(Q, M)$ and that $d(P, Q) = 2 \cdot d(P, M)$.

► **EXAMPLE 1 Midpoint** Given points $A(-4, -1)$ and $B(2, 3)$, find the coordinates of the midpoint M of the segment \overline{AB} and locate all three points on a diagram. In which quadrant is A ? B ? M ? Find $d(A, B)$ and $d(A, M)$.

Solution

The coordinates of M are $\left(\frac{-4+2}{2}, \frac{-1+3}{2}\right)$, or $(-1, 1)$. Point A is in Quadrant III, M is in Quadrant II, and B is in Quadrant I, as shown in Figure 13.

$$d(A, B) = |\overline{AB}| = \sqrt{[2 - (-4)]^2 + [3 - (-1)]^2} = \sqrt{52} = 2\sqrt{13}.$$

$$d(A, M) = |\overline{AM}| = \sqrt{[-1 - (-4)]^2 + [1 - (-1)]^2} = \sqrt{13}. \quad \blacktriangleleft$$

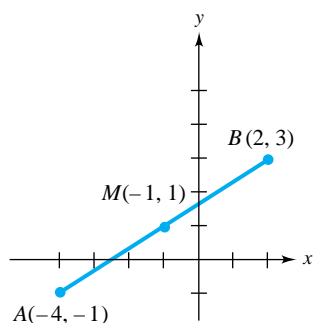


FIGURE 13

Graphs

A coordinate plane allows us to make an algebraic relation visible in the form of a graph. We can then apply visual and geometric tools to reveal analytic properties.

Definition: graph of an equation in two variables

The graph of an equation in variables x and y is the set of points whose coordinates (x, y) satisfy the equation.

Technically, there is a difference between a graph as a set of points and a sketch or picture of a set. Here we use *graph* to refer to the set or to any representation of the set, most often a pencil sketch, a figure in the book, or a display on a graphing calculator or computer. Without the aid of technology, graphing can be a very tedious process, but graphing is one of the things a graphing calculator does best. To make the best use of a graphing calculator, we need to understand a little about how such a calculator (or computer) works.

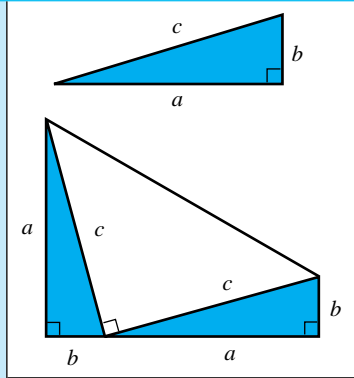
How a Graphing Calculator Represents a Graph

A graphing calculator screen is a rectangular array of *picture elements* or *pixels*. After we have entered an equation (usually of the form $y = \dots$), we choose the **window** through which we will view the graph. This is done by setting an x -range

HISTORICAL NOTE**A PROOF OF THE PYTHAGOREAN THEOREM**

How many United States presidents have made an original contribution to mathematics? There is at least one. In 1876 while a member of Congress, four years before he became president, James A. Garfield discovered an original proof for the Pythagorean theorem, one of dozens of proofs given after Euclid's (ca. 300 B.C.).

President Garfield's proof uses two facts. First, the area of a right triangle is half the product of the legs (base \times altitude). Second, the area of a trapezoid equals its base times its average height. Given any right



triangle, two copies and an isosceles right triangle can be put together to form a trapezoid, as shown in the figure. The sum of the areas of the triangles is $2\frac{ab}{2} + \frac{c^2}{2}$; the area of

the trapezoid is $(a + b)\left(\frac{a + b}{2}\right)$.

Equating these expressions and multiplying by 2 gives

$$2ab + c^2 = a^2 + 2ab + b^2$$

or

$$c^2 = a^2 + b^2.$$

and a y -range. The calculator divides the x -range into as many pieces as there are columns of pixels (the number differs on each type of calculator, but current graphing calculators have from 94 to 130 pixel columns; see inside front cover) and computes a y -value for every column. The pixel in each column nearest the computed y -value is turned on, to make a graph we can see. In **connected mode**, the calculator turns on as many pixels in a column as needed to “connect the dots,” in **dot mode**, we see at most one lighted pixel in each column.

It is essential to understand that a graph produced by any calculator or computer is obtained by computing one value for each pixel column; the calculator only *samples* a graph. Whatever happens (if anything) between pixels does not show on the screen. No matter what window we use, we see at most about a hundred points of the graph from the specified x -range.

Decimal and “Friendly” Windows

When we **TRACE** on a graph, coordinates are displayed on the screen. The x -coordinate of the n th pixel is given by $X_{\min} + (n - 1)(X_{\max} - X_{\min})/k$, where k is the number of pixel columns. Although usually we don't care about making the x -coordinates “nice” numbers, there are times when it is convenient. Most calculators either have a default window or a **ZOOM** or **RANGE** option (labeled something like **ZDECM** or **INIT** that sets a window in which x -coordinates are tenths (as 1.1, 1.2, . . .). For obvious reasons we refer to such a window as a **Decimal Window**, although there are other windows with similarly “nice” coordinates. For convenient reference, we give the settings for several calculators.

TECHNOLOGY TIP ◆ *Decimal windows*

<i>Calculator</i>	<i>Set</i>	<i>#cols</i>	<i>x-range</i>	<i>y-range</i>
<i>TI-81</i>	<i>Range Values</i>	<i>95</i>	$[-4.8, 4.7]$	$[-3.2, 3.1]$
<i>TI-82</i>	ZOOM 4	<i>94</i>	$[-4.7, 4.7]$	$[-3.1, 3.1]$
<i>TI-85</i>	ZOOM MORE ZDECM	<i>126</i>	$[-6.3, 6.3]$	$[-3.1, 3.1]$
<i>Casio</i> ⁷⁷⁰⁰ ₉₇₀₀	Range INIT	<i>94</i>	$[-4.7, 4.7]$	$[-3.1, 3.1]$
	Range INIT	<i>126</i>	$[-6.3, 6.3]$	$[-3.7, 3.7]$
<i>HP-38</i>	■PLOT ■CLEAR	<i>130</i>	$[-6.5, 6.5]$	$[-3.1, 3.2]$
<i>HP-48</i>	PLOT NXT Reset Plot	<i>130</i>	$[-6.5, 6.5]$	$[-3.1, 3.2]$

From the Decimal Window as outlined above, there are obvious adjustments that keep nice x -pixel coordinates. Without being more specific, we sometimes call such a window “Friendly.” For example, if we divide all range values by 2, or multiply each by 2, the result is a friendly window. We can shift a window right, left, up or down, by adding the same quantity to both ends of a range. We could reasonably call such a friendly window a “shifted decimal window.” When we suggest a window for a calculator graph in this book, we use the notation $[a, b] \times [c, d]$, where the interval $[a, b]$ is the x -range and $[c, d]$ is the y -range. More often, we encourage you to choose your window as you wish. Experiment to find the picture that is most helpful for your purposes at the moment.

Equal Scale Windows

When we sketch a graph by hand, we usually use the same scale for the two coordinate axes. The unit of distance is the same horizontally and vertically. That relationship does not hold with calculator graphs unless we take special care to set what we call an **Equal Scale Window**. The ranges for an equal scale window depend on the pixel dimensions of your calculator, and can only be approximate. If your calculator screen is 94 by 62, then the ratio of the x -range to the y -range must be approximately $\frac{94}{62}$, or about $\frac{3}{2}$; if your default screen is 130 by 63, then you need a ratio of about $\frac{2}{1}$. Experiment with your calculator. If you have a window that distorts the picture you want to see, you may be able to “square-up” the display by using a command from the ZOOM menu such as ZSQR.

We begin our graphing with two simple and important graphs, both of which we revisit in later chapters: lines and circles.

Lines: Graphs of Linear Equations

Linear equations and lines

A **linear equation** in x and y is an equation equivalent to

$$ax + by + c = 0, \quad (1)$$

where a , b , and c are real numbers and at least one of a and b is nonzero. The graph of any linear equation is a **line**. We will often identify the line with its equation, so that we will speak of “the line $ax + by + c = 0$.”

We find points on a line by substituting a value for x or y , and solving the equation for the corresponding value of the other variable. Since a line is determined by any

two points, to sketch the graph of a linear equation, it is sufficient to locate any pair of points on the line. A point where a graph crosses a coordinate axis is called an **intercept point**. Intercept points are often convenient for drawing a line. If the line contains the origin (that is, if $c = 0$), then of course the x - and y -intercept points coincide, and we must find another point to enable us to draw the line.

Intercept points of a line

To find the **y -intercept point**,

substitute 0 for x in Equation (1) and solve for y .

To find the **x -intercept point**,

substitute 0 for y in Equation (1) and solve for x .

► **EXAMPLE 2 Graphing a line** Given the line $2x - 3y = 6$, find the intercept points and draw a graph (a) by hand, and (b) on the graphing calculator.

Solution

Substituting 0 for x , we have $-3y = 6$, or $y = -2$, so the y -intercept point is $A(0, -2)$. Similarly, if $y = 0$, then $x = 3$, giving the x -intercept point $B(3, 0)$.

(a) Plotting the intercept points A and B and drawing the line containing them gives the graph in Figure 14a.

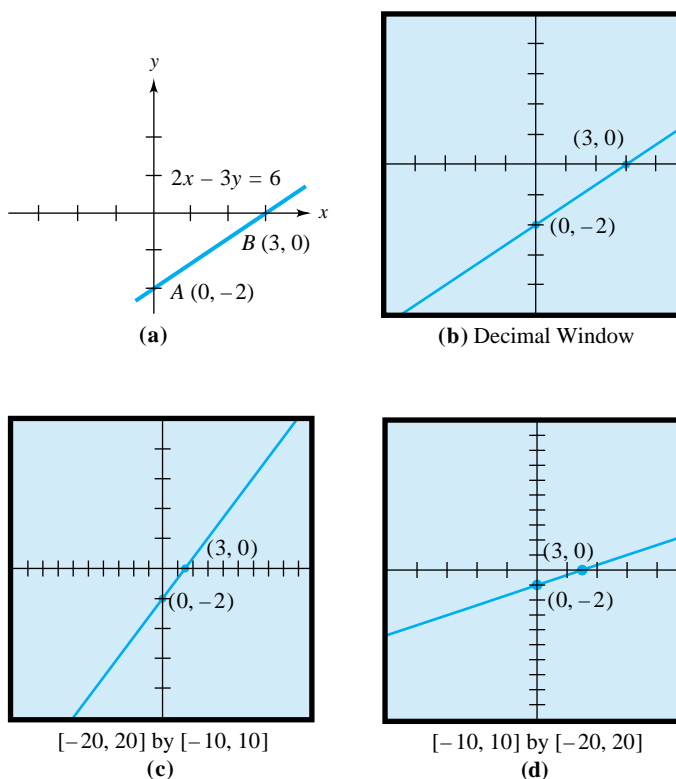


FIGURE 14

(b) To enter an equation for graphing, we must first solve for y : $y = \frac{2x-6}{3}$, so we enter $Y = (2X - 6)/3$, and we then must select a window. In a decimal window (Figure 14b) the graph looks much like the line we drew by hand. Other windows can greatly affect the appearance of the line. For example, in the $[-20, 20] \times [-10, 10]$ window, we get something like Figure 14c, and in the $[-10, 10] \times [-20, 20]$ window, we see a much less steep line (Figure 14d). We need to keep in mind that the line remains the same; by changing the window we alter the portion of the line that appears on our (non-square) screen. ◀

TECHNOLOGY TIP Proper use of parentheses

Since equations are entered on a single line in a calculator, it is hard to overemphasize the importance of the proper use of parentheses. The equation in Example 2 must be entered as $Y = (2X - 6)/3$. See what happens if you forget parentheses by graphing $Y = 2X - 6/3$ on the same screen. How would you enter $y = \frac{2}{3}x - 2$? Check by graphing.

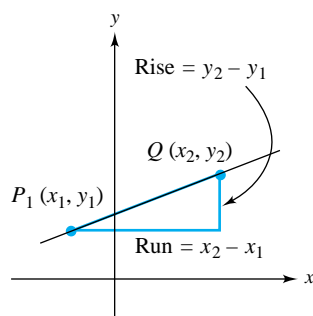


FIGURE 15
Slope: $m = \frac{y_2 - y_1}{x_2 - x_1}$

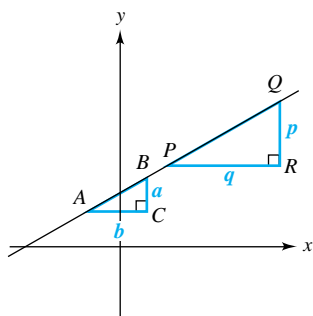


FIGURE 16
Slope: $m = \frac{a}{b} = \frac{p}{q}$

Slope of a Nonvertical Line

The intuitive idea we have of the direction of a line may not be apparent in a calculator graph, as we can see from the figures for Example 2. We make this idea more precise with the concept of **slope**, which measures how steeply a line rises or drops as we move along the line to the right. See Figure 15.

Definition: slope of a nonvertical line

If L is a nonvertical line and $P(x_1, y_1)$ and $Q(x_2, y_2)$ are any two points on L , then

$\Delta y = y_2 - y_1$ is called the **rise** from P to Q ,

$\Delta x = x_2 - x_1$ is called the **run** from P to Q ,

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \text{ is called the } \mathbf{slope} \text{ of } L. \quad (2)$$

The slope of a line is independent of the two points we choose on the line, as Figure 16 shows. The slope from Equation (2) is the ratio of two sides of a triangle. Triangles ABC and PQR are similar, so ratios of corresponding sides are the same.

The slope of L is either $\frac{a}{b}$ or $\frac{p}{q}$.

The slope of a line is very handy in drawing a graph. Express the slope m as a fraction (with denominator 1 if needed). The denominator is the run and the numerator is the corresponding rise. From any point on the line, move to the right for the run and up or down for the rise, to get the coordinates of another point on the line.

► **EXAMPLE 3 Using slopes to draw lines** Line L_1 passes through the origin, and the y -intercept point of line L_2 is $B(0, 1)$. Their slopes are given respectively by $m_1 = 2$ and $m_2 = -\frac{1}{2}$. Draw both lines on the same set of coordinate axes.

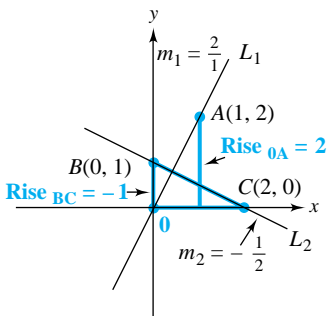


FIGURE 17

Solution

Express m_1 as a fraction, $m_1 = \frac{2}{1}$. To locate another point on L_1 , from $O(0, 0)$ plot a run of 1 unit to the right and a rise of 2, giving point $A(1, 2)$. Similarly, since $m_2 = -\frac{1}{2}$, a run of 2 corresponds to a rise of -1 (so the line drops as it moves rightward). From the y -intercept point, 2 units right and 1 down gives point $C(2, 0)$. Line L_1 is determined by O and A ; L_2 contains B and C . Both lines are shown in Figure 17. ◀

Horizontal and Vertical Lines

A horizontal line has slope zero because the rise from one point to any other is zero; $\Delta y = 0$. Since every point on a horizontal line has the same y -coordinate, **every horizontal line has an equation of the form $y = c$** for some constant c . On the other hand, it is impossible to define the slope of a vertical line because the run between any two points is zero; $\Delta x = 0$, and we cannot divide by 0. Every point on a vertical line has the same x -coordinate, so **every vertical line has an equation of the form $x = c$** .

Circles

A circle is defined as the set of points that are a fixed distance, called the *radius*, from a fixed point, called the *center*. If the center is point $C(h, k)$, then $P(x, y)$ is on the circle with radius r precisely when the distance $d(P, C)$ equals r (see Figure 18). Using the distance formula,

$$d(P, C) = \sqrt{(x - h)^2 + (y - k)^2} = r.$$

Since the radius r is a positive number, we may square both sides to get the standard form for an equation of a circle.

Standard form for equation of a circle

Suppose h, k , and r are given real numbers ($r > 0$). Point (x, y) lies on a circle of radius r and center (h, k) if and only if (x, y) satisfies

$$(x - h)^2 + (y - k)^2 = r^2 \tag{3}$$

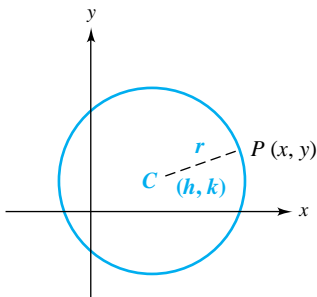


FIGURE 18
Circle with center $C(h, k)$ and radius r

▶ **EXAMPLE 4** Equation and graph of a circle

- (a) Write an equation for the circle with center $C(2, -3)$ and radius 3. Sketch the graph.
- (b) Determine which of the points $O(0, 0)$, $A(2, 0)$, and $B(4, -1)$ are inside the circle.

Solution

- (a) Given the coordinates of the center, $h = 2$ and $k = -3$, replace h by 2, k by -3 , and r by 3 in Equation 3 to get

$$(x - 2)^2 + [y - (-3)]^2 = 3^2 \quad \text{or} \quad (x - 2)^2 + (y + 3)^2 = 9.$$

The graph, with points O , A , and B , is shown in Figure 19.

- (b) From the graph, it appears that O is outside the circle, but it is not as clear whether A and B are inside or outside. Using the distance formula,

$$d(A, C) = \sqrt{(2 - 2)^2 + (0 + 3)^2} = \sqrt{9} = 3,$$

$$d(B, C) = \sqrt{(4 - 2)^2 + (-1 + 3)^2} = \sqrt{8}.$$

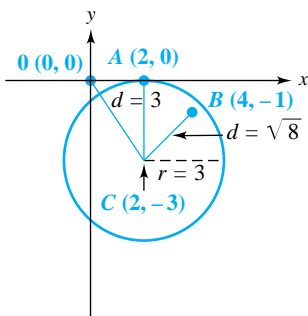


FIGURE 19

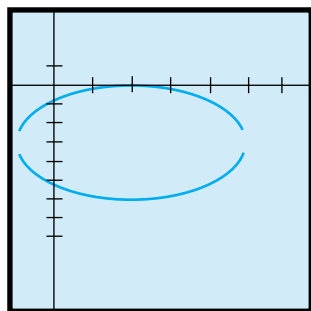
From the definition of a circle, A is on the circle and B is inside because $\sqrt{8} < 3$. ◀

Calculator Graphs of Circles

While there are some calculators that have special adaptations for drawing circles, most graphing calculators do not graph circles as easily as lines. At this point we are limited to entering equations in the form $y = \dots$. For the circle in Example 4, $(x - 2)^2 + (y + 3)^2 = 9$, we would have to solve the equation for y by subtracting $(x - 2)^2$ from both sides, taking square roots (there are two), and finally subtracting 3:

$$(y + 3)^2 = 9 - (x - 2)^2, \quad y + 3 = \pm \sqrt{9 - (x - 2)^2},$$

$$y = -3 \pm \sqrt{9 - (x - 2)^2}.$$



$[-1, 7]$ by $[-6, 2]$

FIGURE 20

Thus we must enter *two* equations, one for each sign, $Y1 = -3 + \sqrt{9 - (X - 2)^2}$ and $Y2 = -3 - \sqrt{9 - (X - 2)^2}$ (watch the parentheses), choose a window, and graph. In the standard decimal window we see only the upper portion of the circle, so we need a larger window. Looking carefully at Figure 19, we see that our window must include at least the interval $[-1, 5]$ in the x -direction and $[-6, 0]$ in the y -direction. If we try, say, $[-1, 7] \times [-6, 2]$, the result doesn't look much like the circle in Figure 19. See Figure 20. There are two problems: We don't have an equal scale window, so the circle is "squashed," and when we trace along the curve, we see that there is no x -pixel coordinate for -1 or for 5 . Thus the two pieces of the circle don't meet at the ends. Both obstacles may be overcome with some work (in this instance by simply shifting all the x - and y -values the same amount from the decimal window), but we suggest that you learn to interpret what the calculator shows. You should recognize that what appears on your display screen need not look like what you would draw yourself.

There is another way to draw circles on a graphing calculator that avoids gaps. In Chapter 6 we discuss the use of trigonometric functions to graph circles in parametric mode.

The standard form for an equation of a circle identifies the center and the radius. If we expand the squared terms on the left side of Equation (3) and collect the constants, we get a general form for an equation of a circle.

General form for the equation of a circle

For any real numbers A , B , and C , the graph of the equation

$$x^2 + y^2 + Ax + By + C = 0 \quad (4)$$

is either a circle, a point, or no points.

We can show that Equation (4) is equivalent to Equation (3) by completing the square on $x^2 + Ax$ and on $y^2 + By$, as illustrated in Example 5.

► **EXAMPLE 5 Finding center and radius** Find the center and radius. Draw a graph.

(a) $x^2 + y^2 = 4x$ (b) $2x^2 + 2y^2 - 4x + 6y + \frac{1}{2} = 0$

Strategy: Write in the form of Equation (4) and then complete the squares. Remember to add the same quantity to both sides of the equation.

Solution

(a) Follow the strategy.

$$(x^2 - 4x) + y^2 = 0$$

$$(x^2 - 4x + 4) + y^2 = 4 \quad \text{or} \quad (x - 2)^2 + (y - 0)^2 = 2^2$$

The center is (2, 0) and the radius is 2.

(b) Divide by 2 and collect x - and y -terms. Complete the squares.

$$x^2 + y^2 - 2x + 3y + \frac{1}{4} = 0$$

$$(x^2 - 2x) + (y^2 + 3y) = -\frac{1}{4}$$

$$(x^2 - 2x + 1) + \left(y^2 + 3y + \frac{9}{4}\right) = -\frac{1}{4} + 1 + \frac{9}{4}$$

$$(x - 1)^2 + \left(y + \frac{3}{2}\right)^2 = 3.$$

The center is $(1, -\frac{3}{2})$ and the radius is $\sqrt{3}$. Both circles are shown in Figure 21.

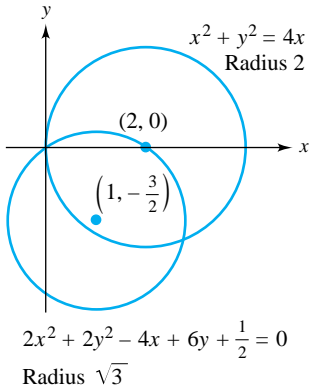


FIGURE 21

► **EXAMPLE 6 Finding extreme points** Find the coordinates of the highest and the lowest points on the graph of $x^2 + y^2 - 4x + 6y - 3 = 0$.

Solution

Looking at the equation, our first observation is that the graph appears to be a circle. From the graph we should be able to locate the high and low points. Find the center and radius by completing the squares on $x^2 - 4x$ and $y^2 + 6y$.

$$(x - 2)^2 + (y + 3)^2 = 16.$$

The center of the circle is at (2, -3) and the radius is 4. The graph is shown in Figure 22. The graph shows the highest point 4 units above the center, at (2, 1), and the lowest point 4 units below the center, at (2, -7).

► **EXAMPLE 7 Intercept points** Find the distance between the x -intercept points on the graph of $x^2 + y^2 - 4x + 6y - 3 = 0$.

Solution

This is the equation from Example 6, with the graph in Figure 22. Find the distance between points A and B . To find the x -intercept points, replace y by 0 and get

$$x^2 - 4x - 3 = 0.$$

Using the quadratic formula, the roots are $2 \pm \sqrt{7}$.

Here we have two different values of x . The graph makes it clear that the x -coordinate of A is negative, namely $2 - \sqrt{7}$ (≈ -0.65), and the x -coordinate of B is $2 + \sqrt{7}$ (≈ 4.65), so A and B are the points $(2 - \sqrt{7}, 0)$ and $(2 + \sqrt{7}, 0)$, respectively. Since A and B lie on the same horizontal line, the distance between them is given by the difference in their x -coordinates:

$$d(A, B) = |(2 + \sqrt{7}) - (2 - \sqrt{7})| = 2\sqrt{7} \approx 5.29.$$

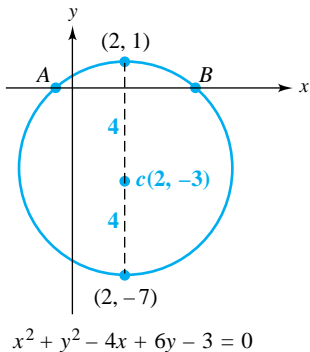


FIGURE 22

► **EXAMPLE 8 Point and imaginary circles** Describe the graphs of

(a) $x^2 + y^2 - 2x + 4y + 5 = 0$. (b) $x^2 + y^2 - 2x + 4y + 6 = 0$.

Solution

(a) Completing the squares on $x^2 - 2x$ and $y^2 + 4y$ gives

$$(x - 1)^2 + (y + 2)^2 = 0.$$

This equation has the appearance of the equation of a circle with center at $(1, -2)$ and radius 0! We want the set of points whose coordinates satisfy the equation. The sum of two squares can be 0 only if both terms are 0. The graph is a single point, $(1, -2)$. This is sometimes called a *point circle*.

(b) Completing the squares on the x and y terms, we get

$$(x - 1)^2 + (y + 2)^2 = -1.$$

This is even worse; there are no points whose coordinates satisfy the given equation. No graph is associated with the equation, but, by analogy with the point circle of part (a), the equation in part (b) is sometimes called an *imaginary circle*. ◀

EXERCISES 1.4

Check Your Understanding

Exercises 1–7 True or False. Given reasons.

- The graph of $x^2 + y^2 + 2x + 1 = 0$ is a single point.
- The point $(1, -2)$ is the center of the circle whose equation is $x^2 + y^2 + 2x - 4y = 0$.
- The graph of $x^2 + y^2 - 2x = 0$ is a circle with diameter 2.
- If the graphs of $5x - 2y = 4$ and $3x + y = 20$ are drawn simultaneously using the window $[-10, 10]$ by $[-5, 5]$, the display will show two lines intersecting at a point on the screen.
- When the graphs of $2x + 3y = 12$ and $x^2 + y^2 = 8$ are drawn simultaneously using the window $[-9.4, 9.6]$ by $[-6.2, 6.4]$, the display will show a line and a circle intersecting at two points.
- There is no real number c such that the point $(1, c)$ is 1 unit from $(-1, 2)$. (Hint: Think geometrically.)
- There are two numbers c for which the point $(1, c)$ is 4 units from $(-1, 2)$. (Hint: Think geometrically.)

Exercises 8–10 Fill in the blank so that the resulting statement is true.

- The graph of $x^2 + y^2 - 2x + 4y - 5 = 0$ is a circle having center in Quadrant _____.
- If (a, b) is any point in the second quadrant, then (b, a) is in Quadrant _____.
- The graph of $x + y - 1 = 0$ does not contain any points in Quadrant _____.

Develop Mastery

Exercises 1–5 **Applying Distance Formula** (a) Draw a diagram showing points A and B and find the distance between them. (b) Find the coordinates of the midpoint M of the line segment \overline{AB} . (c) Verify that $d(A, M) = \frac{1}{2}d(A, B)$.

- $A(1, 3)$, $B(-2, 4)$
- $A(-2, 3)$, $B(4, -1)$
- $A(-\frac{1}{2}, 2)$, $B(1, -\frac{1}{3})$
- $A(1, -2)$, $B(-\frac{1}{3}, -\frac{1}{3})$
- $A(2\sqrt{2}, -3\sqrt{2})$, $B(-2\sqrt{2}, \sqrt{2})$

Exercises 6–11 **Special Triangles** Determine whether the three points are vertices of a right triangle, an equilateral triangle, an isosceles triangle, or none of these. (Hint: For a right triangle, use the Pythagorean theorem.)

- $A(-1, 2)$, $B(4, -2)$, $C(8, 3)$
- $A(4, -2)$, $B(-4, 2)$, $C(7, 4)$
- $A(0, 0)$, $B(2\sqrt{3}, 2)$, $C(0, 4)$
- $A(0, 0)$, $B(4, 2)$, $C(0, 4)$
- $A(-1, -1)$, $B(4, 1)$, $C(1, 4)$
- $A(-2, -3)$, $B(4, 6)$, $C(-6, -\frac{1}{3})$

Exercises 12–21 **Equation of a Circle** Write an equation for the circle that satisfies the given conditions. First draw a diagram showing the circle. Give the result in expanded form.

- Center $(0, 0)$; radius 3
- Center $(1, 1)$; radius $\sqrt{3}$
- Center $(2, -1)$; radius $\sqrt{5}$

15. Center $(-1, 5)$; diameter 1
 16. Center $(-2, -1)$; tangent to the x -axis
 17. Center $(-2, -1)$; tangent to the y -axis
 18. The segment from $A(-3, 4)$ to $B(1, 1)$ is a diameter.
 19. The segment from $A(3, -2)$ to $B(5, 4)$ is a diameter.
 20. The circle is circumscribed about the triangle having vertices $A(0, 0)$, $B(8, 0)$ and $C(8, 6)$. (*Hint*: Triangle ABC is a right triangle.)
 21. The circle passes through the three points $A(2, 1)$, $B(6, 1)$ and $C(6, 4)$. (*Hint*: $\angle ABC$ is a right angle.)

Exercises 22–23 Extreme Points An equation of a circle is given. Find (a) the highest and lowest points and (b) the points furthest to the right and left. See Example 6.

22. $x^2 + y^2 + 4x - 4y - 8 = 0$
 23. $x^2 + y^2 - 6x - 2y + 1 = 0$

Exercises 24–25 An equation of a circle is given. Find (a) the x - and y -intercept points, (b) the distance between the x -intercept points, and (c) the distance between the y -intercept points. See Example 7.

24. $x^2 + y^2 + 2x - 2y - 8 = 0$
 25. $x^2 + y^2 - 6x - 2y + 1 = 0$

Exercises 26–29 Intercept Points An equation of a line is given. Find (a) the x - and y -intercept points, (b) the distance between the intercept points, (c) the slope of the line.

26. $2x - 3y = 6$ 27. $3x - 4y + 12 = 0$
 28. $4x + 3y = 6$ 29. $3x - 4y = 6$

Exercises 30–33 Using Distance Formula Find the value(s) of x or y so that the distance d between the two points is the given distance.

30. $(-2, 3)$, $(x, -1)$; $d = 6$
 31. $(3, -1)$, $(x, 4)$; $d = 8$
 32. $(-4, 2)$, $(2, y)$; $d = 5$
 33. $(3, -1)$, $(5, y)$; $d = 3$

Exercises 34–44 Graph of Equation (a) Identify the graph of the equation as a line or a circle. (b) For a line, find the coordinates of the intercept points. For a circle, find the radius and the coordinates of the center. (c) Sketch the graph.

34. $2x + 3y = 6$ 35. $x + y = 4$
 36. $x^2 + y^2 = 4$ 37. $y = 3x - 2$
 38. $2y = x^2 + y^2$ 39. $3x^2 + 3y^2 = 21$
 40. $7x + 7y = 21$ 41. $x^2 + 2x + y^2 = 0$
 42. $x^2 + y^2 = 2x + 4y$
 43. $x^2 - 4x + y^2 + 2y + 1 = 0$
 44. $2x^2 + 2y^2 + 4y = 12x + 15$

Exercises 45–55 Draw a calculator graph of each equation in Exercises 34–44. Experiment with windows of various sizes.

Exercises 56–59 Windows The graphs of the given equations are perpendicular lines. Draw the graphs. Experiment with windows of different sizes until you get lines that appear to be perpendicular. Indicate the window size you use. Use TRACE to find the point of intersection of the lines (one decimal place).

56. $2x - y = 4$, $x + 2y - 2 = 0$
 57. $3x - 2y - 6 = 0$, $2x + 3y = 12$
 58. $8x - 5y = 36$, $5x + 8y = 4$
 59. $3x - y - 7 = 0$, $x + 3y + 5 = 0$

Exercises 60–63 Viewing Windows In order to draw a graph of the equation that will show the x - and y -intercept points, which window would you use? Try each one.

60. $2x + y = 25$
 (i) $[-10, 10] \times [-10, 10]$
 (ii) $[-15, 5] \times [-15, 15]$
 (iii) $[-5, 20] \times [-5, 30]$
 61. $3x - 2y + 40 = 0$
 (i) $[-10, 10] \times [-10, 10]$
 (ii) $[-20, 5] \times [-10, 10]$
 (iii) $[-15, 5] \times [-5, 25]$
 62. $(x - 2)^2 + (y + 4)^2 = 64$
 (i) $[-10, 10] \times [-10, 10]$
 (ii) $[-4.7, 4.8] \times [-3.2, 3.6]$
 (iii) $[-13.1, 14.5] \times [-13.3, 5.6]$
 63. $x^2 + y^2 - 2x + 4y - 100 = 0$
 (i) $[-10, 10] \times [-10, 10]$
 (ii) $[-5.5, 18.3] \times [-5.2, 18.3]$
 (iii) $[-17.0, 17.3] \times [-12.5, 8.9]$

Exercises 64–65 The graph of the equation is a circle. Which of the windows gives a graph that is nearly circular?

64. $x^2 + y^2 = 16$
 (i) $[-10, 10] \times [-10, 10]$
 (ii) $[-8, 8] \times [-5, 5]$
 (iii) $[-10, 10] \times [-5, 5]$
 65. $(x - 1)^2 + (y + 4)^2 = 25$
 (i) $[-10, 10] \times [-10, 10]$
 (ii) $[-12, 12] \times [-10, 2]$
 (iii) $[-8, 10] \times [-10, 2]$

Exercises 66–69 Half Circle Graphs Draw graphs to determine the quadrant(s) in which the two half circles intersect (if any).

66. Lower half of $x^2 + y^2 = 16$ and the lower half of $x^2 + y^2 - 12x + 11 = 0$.
 67. Lower half of $x^2 + y^2 = 16$ and the lower half of $x^2 + y^2 - 12x - 28 = 0$.

68. Upper half of $x^2 + y^2 = 16$ and the upper half of $x^2 + y^2 - 12x + 20 = 0$.
69. Upper half of $x^2 + y^2 = 16$ and the upper half of $x^2 + y^2 + 12x + 20 = 0$.

Exercises 70–72 Lattice Points For these exercises, remember that point (x, y) is a lattice point if both x and y are integers.

70. Give an example of two lattice points in Quadrant I that define a line segment whose midpoint is not a lattice point.
71. Give an example of two lattice points in Quadrant I that define a line segment whose midpoint is also a lattice point.
72. Find a pair of lattice points, A and B , with A in Quadrant II and B in Quadrant IV for which the midpoint of segment \overline{AB} is (a) in Quadrant I, (b) in Quadrant II, (c) in Quadrant III, (d) not in any quadrant.

Exercises 73–75 Find all lattice points in Quadrant I on the graph of the equation.

73. $2x + y = 6$ 74. $x + 3y = 13$
75. $2x + 3y = 12$

Exercises 76–77 Pythagorean Triples If the lengths of the three sides of a right triangle are integers, then the triplet of numbers is called a Pythagorean triple. For example, $a = 3$, $b = 4$, and so $3^2 + 4^2 = 25 = 5^2$, and $[3, 4, 5]$ is a Pythagorean triple. We illustrate an interesting way to find Pythagorean triples:

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}; \quad 3^2 + 4^2 = 5^2 \text{ gives } [3, 4, 5]$$

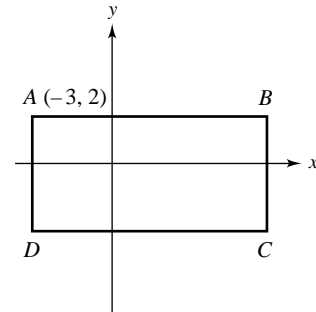
$$\frac{1}{3} + \frac{1}{5} = \frac{8}{15}; \quad 8^2 + 15^2 = 17^2 \text{ gives } [8, 15, 17]$$

$$\frac{1}{7} + \frac{1}{9} = \frac{16}{63}; \quad 16^2 + 63^2 = 65^2 \text{ gives } [16, 63, 65]$$

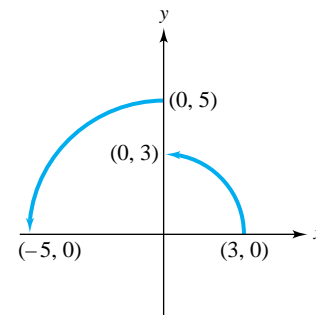
76. Follow the pattern suggested above and find three additional Pythagorean triples.
77. Prove that the pattern illustrated here always yields a Pythagorean triple. (Hint: Consider $\frac{1}{n-1} + \frac{1}{n+1}$ where n is any positive integer greater than 1.)
78. If $A(0, 3)$, $B(-1, -1)$, and $C(4, 1)$ are three vertices of a parallelogram, what are the coordinates of the fourth vertex? Draw a diagram. Is the answer unique?

Exercises 79–80 Find the point P that is equidistant from the three points A , B , and C ; that is, find P such that $|PA| = |PB| = |PC|$. (Hint: First show that the three points are vertices of a right triangle and consider the circle circumscribing $\triangle ABC$.)

79. $A(8, 3)$, $B(4, 10)$, $C(2, 6)$
80. $A(2, 3)$, $B(8, 0)$, $C(5, 9)$
81. A rectangle has sides parallel to the coordinate axes. Two of its vertices are at $(-5, -7)$ and $(4, -2)$. Find the coordinates of the other two vertices and the length of a diagonal.
82. A rectangle has sides parallel to the coordinate axes and its upper left corner at $A(-3, 2)$ as shown in the diagram (which is not drawn to scale). The length (horizontal side) is twice the width, and the perimeter is 30. Find the coordinates of the other three vertices.



83. A 90° rotation of a plane counterclockwise about the origin moves point $(3, 0)$ to $(0, 3)$ and point $(0, 5)$ to $(-5, 0)$ (see the diagram). What is the image of each point under the same rotation? Draw diagrams.
- (a) $(-4, 0)$ (b) $(0, -3)$
(c) $(3, 4)$ (d) $(-3, -4)$



84. Find the area of the region that is inside the circle $x^2 + y^2 - 2x - 3 = 0$ and outside the circle $x^3 + y^2 = 1$. (Hint: First draw a diagram.)
85. What is the area of a circle if the reciprocal of its circumference equals the length of its radius?
86. The line $x + y = 3$ divides the interior of the circle $x^2 + y^2 = 9$ into two regions. If A_1 and A_2 are the areas of the larger and smaller of the two regions, respectively, find the ratio $\frac{A_1}{A_2}$.