4.1 Inequalities and Applications

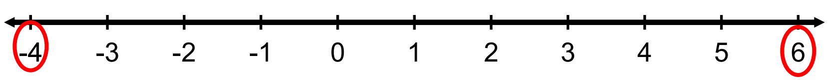
- **Solving Inequalities**
- **Interval Notation**

Solutions of Inequalities

An inequality is a number sentence containing > (is greater than), < (is less than), ≥ (is greater than or equal to), or ≤ (is less than or equal to).

Example Determine whether the given number is a solution of $x < 5$: a) –4 b) 6

Solution



a) Since –4 < 5 is true, –4 is a solution.

b) Since 6 < 5 is false, 6 is not a solution.

Graphs of Inequalities

Because solutions of inequalities like $x < 4$ are too numerous to list, it is helpful to make a drawing that represents all the solutions.

The **graph** of an inequality is such a drawing. Graphs of inequalities in one variable can be drawn on a number line by shading all the points that are solutions. Parentheses **( )** are used to indicate endpoints that are *not* solutions and brackets [ ] indicate endpoints that *are* solutions.

Interval Notation and Graphs

We will use two types of notation to write the **solution set** of an inequality: set-builder notation and interval notation.

Set-builder notation $\{x \mid x < 5\}$

The set of all x such that x is less than 5

Interval notation uses parentheses, ( ), and brackets, [ ].

Open interval: (a, b)

Closed interval: $[a, b]$

Half-open intervals: $(a, b]$ and $[a, b)$

$$(-\infty, 5)$$

Example

Solve and graph $4x - 1 \le x - 10$.

Solution

$$4x - 1 \le x - 10$$

$$4x - 1 + 1 \le x - 10 + 1 \quad \text{Adding 1 to both sides}$$

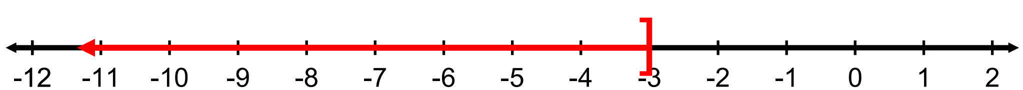
$$4x \le x - 9 \quad \text{Simplifying}$$

$$4x - x \le x - x - 9 \quad \text{Subtracting } x \text{ from both sides}$$

$$3x \le -9 \quad \text{Simplifying}$$

$$x \le -3 \quad \text{Dividing both sides by 3}$$

The solution set is $\{x | x \le -3\}$.



Example

Solve and graph:

a) $\dfrac{1}{7}x \le 4$ b) $-4y < 20$

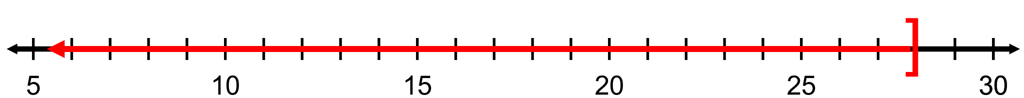
Solution

a) $\dfrac{1}{7}x \le 4$

$7 \cdot \dfrac{1}{7}x \le 7 \cdot 4$ Multiplying both sides by 7

$x \le 28$ Simplifying

The solution set is $\{x | x \le 28\}$. The graph is shown below.



b) $-4y < 20$

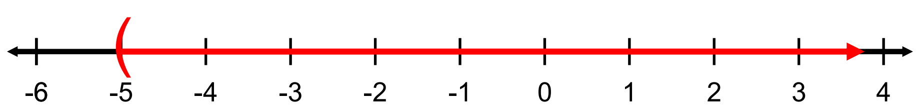
$$\frac{-4y}{-4} > \frac{20}{-4}$$

Dividing both sides by -4

At this step, we reverse the inequality, because -4 is negative.

$$y > -5$$

The solution set is $\{y|y > -5\}$. The graph is shown below.



Example Solve. $3x - 3 > x + 7$

Solution

$$3x - 3 > x + 7$$

$$3x - 3 + 3 > x + 7 + 3$$ Adding 3 to both sides

$$3x > x + 10$$ Simplifying

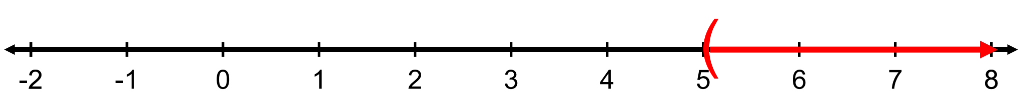
$$3x - x > x - x + 10$$ Subtracting *x* from both sides

$$2x > 10$$ Simplifying

$$\frac{2x}{2} > \frac{10}{2}$$ Dividing both sides by 2

$$x > 5$$ Simplifying

The solution set is $\{x \mid x > 5\}$.



4.2 Solving Equations and Inequalities by Graphing

- **Solving Equations Graphically: The Intersect Method**
- **Solving Equations Graphically: The Zero Method**
- **Solving Inequalities Graphically**

Solving Equations Graphically: The Intersect Method

Recall that to *solve* an equation or inequality means to find all the replacements for the variable that make the equation or inequality true.

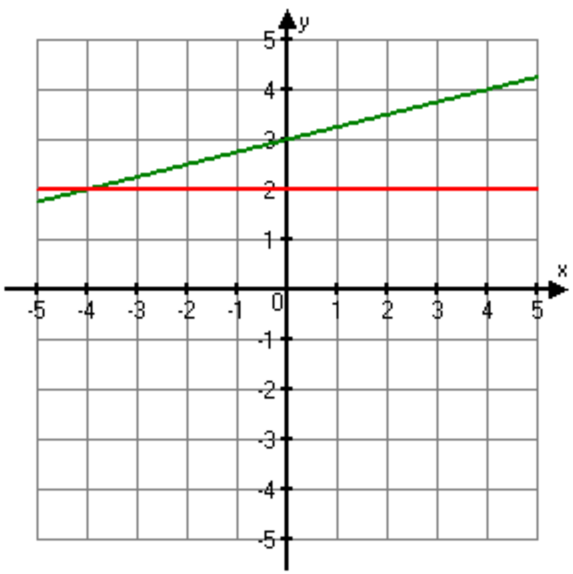
We have seen how to do this algebraically; we now use a graphical method to solve.

Example

Solve graphically: $\frac{1}{4}x+3=2.$

Solution

Graph $f(x)=\frac{1}{4}x+3$ and $g(x) = 2$ on the same set of axes.



Check:

$$\frac{1}{4}x+3=2$$

$$\frac{1}{4}(-4)+3 \;\Big|\; 2$$

$$-1+3$$

$$2=2$$

The intersection appears to be (–4, 2), the solution is apparently –4.

Example Solve: $\frac{2}{3}x+3=2x-5$

Solution Algebraic Approach

$$\frac{2}{3}x+3=2x-5$$

$$\frac{2}{3}x+3-3=2x-5-3$$

$$\frac{2}{3}x=2x-8$$

$$\frac{2}{3}x-2x=2x-8-2x$$

$$\frac{2}{3}x-\frac{6}{3}x=-8$$

$$-\frac{4}{3}x=-8$$

$$\left(-\frac{3}{4}\right)\left(-\frac{4}{3}x\right)=\left(-\frac{3}{4}\right)(-8)$$

$$x=6$$

Check:

$$\frac{2}{3}x+3=2x-5$$

$$\frac{2}{3}(6)+3 \stackrel{?}{=} 2(6)-5$$

$$4+3 \stackrel{?}{=} 12-5$$

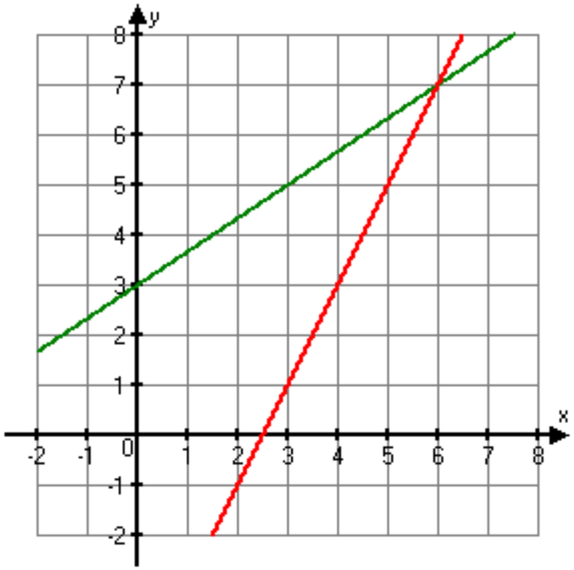
$$7=7$$

Solution Graphical Approach

We graph:

$$f(x) = \frac{2}{3}x + 3$$

$$g(x) = 2x - 5$$



It appears that the lines intersect at (6, 7).

Check $x = 6$ in the equations and it checks.

The solution is $x = 6$

CAUTION! When using a hand-drawn graph to solve an equation, it is important to use graph paper and to work as neatly as possible. Use a straightedge when drawing lines and be sure to erase any mistakes.

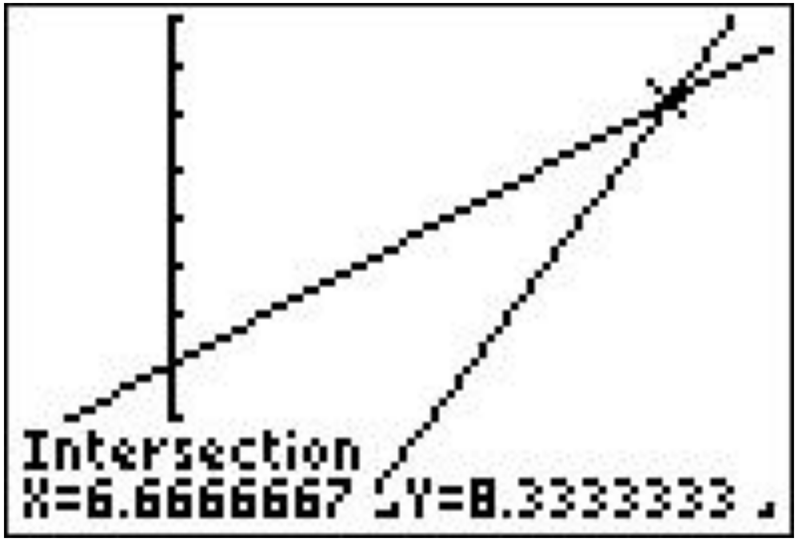
Example

Solve using a graphing calculator. $\frac{4}{5}x + 3 = 2x - 5$

Solution

We graph: $f(x) = \frac{4}{5}x + 3$

$g(x) = 2x - 5$



It appears from the screen that the solution is 6.6666.

Solving Equations Graphically: The Zero Method

It can be challenging to determine a portion of the *x*, *y*-coordinate plane that contains the point of intersection.

The Zero method make that determination easier because we are only interested in the point at which the graph crosses the *x*-axis.

Example

Solve graphically, using the Zero method: $3x - 2 = 5x - 7$

Solution

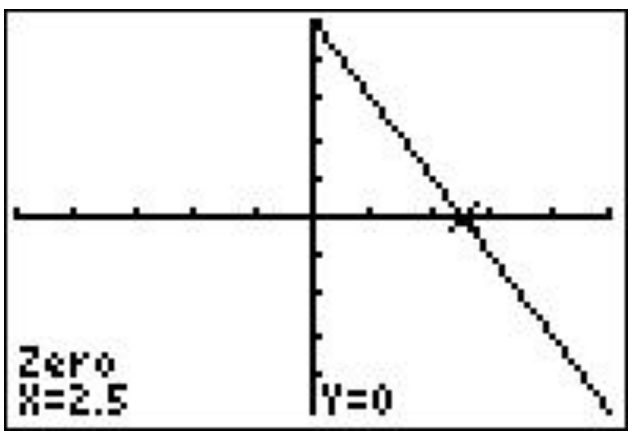
We first get 0 on one side of the equation.

$3x - 2 = 5x - 7$

$-2x - 2 = -7$ Subtracting $5x$ from both sides

$-2x + 5 = 0$ Adding 7 to both sides

Graph $f(x) = -2x + 5$, and find the x-intercept.



The x-intercept of the graph appears to be (2.5, 0). We check 2.5 in the original equation and it checks.

Solving Inequalities Graphically

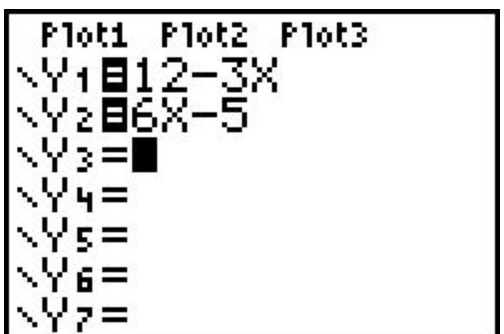
Solving Inequalities Graphically

Example

Solve graphically: $12 - 3x > 6x - 5$.

Solution

We let $y_1 = 12 - 3x$ and $y_2 = 6x - 5$,
and graph y_1 and y_2.
To the left of the point of intersection, $y_1 > y_2$.



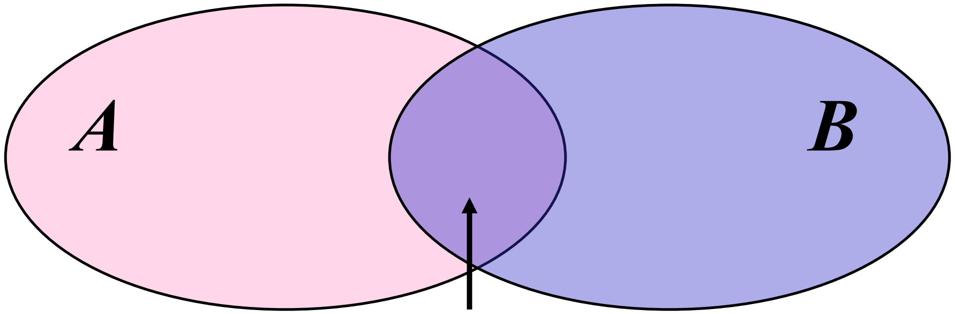
The solution set will be all x-values to the left of the point of intersection. $(-\infty, 1.889)$

4.3 Intersections, Unions, and Compound Inequalities

- Intersections of Sets and Conjunctions of Sentences
- Unions of Sets and Disjunctions of Sentences
- Interval Notation and Domains

Intersection of Sets and Conjunctions of Sentences

The **intersection** of two sets A and B is the set of all elements that are common to both A and B. We denote the intersection of sets A and B as $A \cap B$.



$A \cap B.$

Example Find the intersection:

$$\{a,b,c,d,e,f,g\}\cap\{a,e,i,o,u\}.$$

Solution

The letters a and e are common to both sets, so the intersection is $\{a, e\}$.

When two or more sentences are joined by the word *and* to make a compound sentence, the new sentence is called a **conjunction** of the sentences.

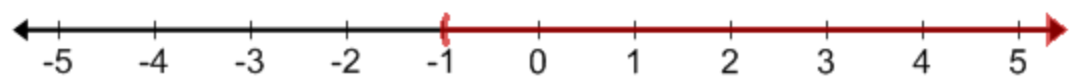
The following is a conjunction of inequalities:

$$-1 < x \quad \textit{and} \quad x < 3.$$

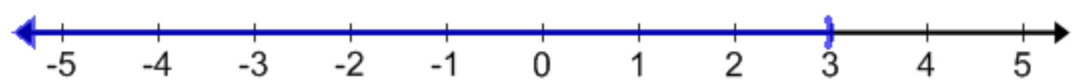
A number is a solution of a conjunction if it is a solution of *both* of the separate parts. For example, 0 is a solution because it is a solution of $-1 < x$ as well as $x < 3$.

Note that the solution set of a conjunction is the intersection of the solution sets of the individual sentences.

$\{x \mid -1 < x\}$

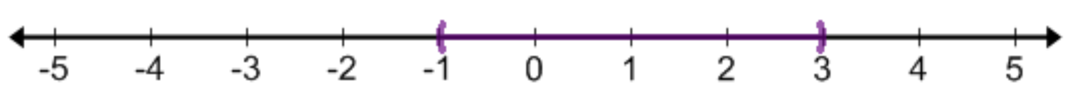


$\{x \mid x < 3\}$



$\{x \mid -1 < x\} \cap \{x \mid x < 3\}$

$\{x \mid -1 < x \text{ and } x < 3\}$



Example Solve and graph:

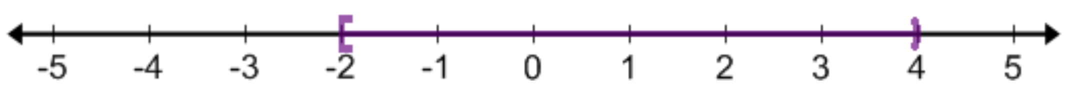
$$2x+1\geq -3 \quad and \quad 3x<12.$$

Solution

$$2x+1\geq -3 \quad and \quad 3x<12$$

$$2x\geq -4 \quad and \quad 3x<12$$

$$x\geq -2 \quad and \quad x<4.$$



Note that for $a < b$,

$a < x$ *and* $x < b$ can be abbreviated $a < x < b$;

and, equivalently,

$b > x$ *and* $x > a$ can be abbreviated $b > x > a$.

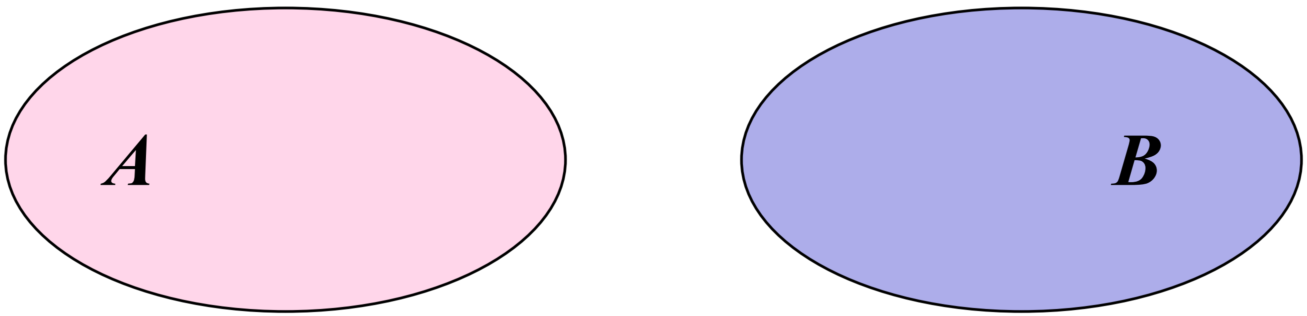
So $3 < 2x + 1 < 7$ can be solved as

$3 < 2x + 1$ *and* $2x + 1 < 7$

Mathematical Use of the Word "and"

The word "and" corresponds to "intersection" and to the symbol "$\cap$". Any solution of a conjunction must make each part of the conjunction true.

Sometimes there is no way to solve both parts of a conjunction at once.



$$A \cap B = \varnothing$$

In this situation, A and B are said to be *disjoint*.

Example Solve and graph:

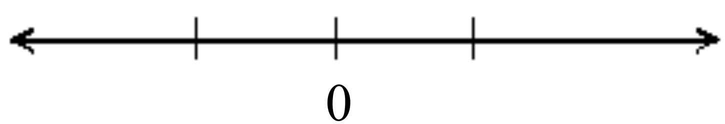
$5 + x > 10 \;\; and \;\; x - 4 < -3.$

Solution

$$5 + x > 10 \;\; and \;\; x - 4 < -3$$

$$x > 5 \;\; and \;\; x < 1.$$

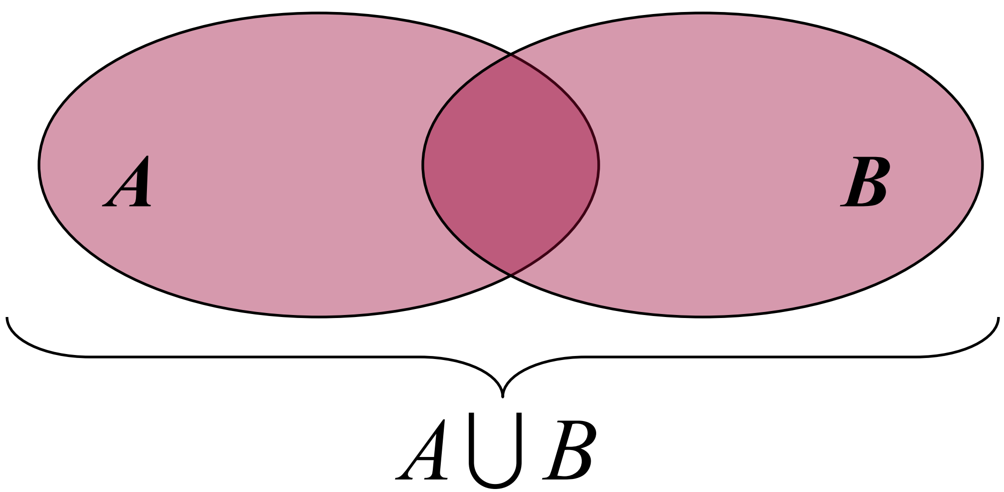
Since no number is greater than 5 and less than 1, the solution set is the empty set $\varnothing$.



Unions of Sets and Disjunctions of Sentences

The **union** of two sets A and B is the collection of elements belonging to A and/or B. We denote the union of sets A and B by

$$A \cup B.$$



Example Find the union:

$$\{a,b,c,d,e,\}\cup\{a,e,i,o,u\}.$$

Solution

The letters in either or both sets are a, b, c, d, e, i, o and u, so the union is $\{a,b,c,d,e,i,o,u\}.$

When two or more sentences are joined by the word *or* to make a compound sentence, the new sentence is called a **disjunction** of the sentences.

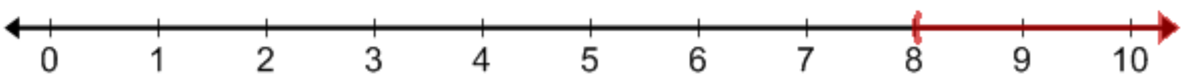
Example

$x < 2 \;\; or \;\; x > 8$

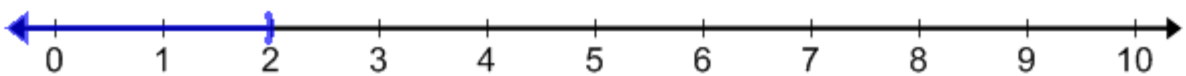
A number is a solution of a disjunction if it is a solution of at least one of the separate parts. For example, $x = 12$ is a solution since $12 > 8$.

Note that the solution set of a disjunction is the union of the solution sets of the individual sentences.

$\{x \mid x > 8\}$

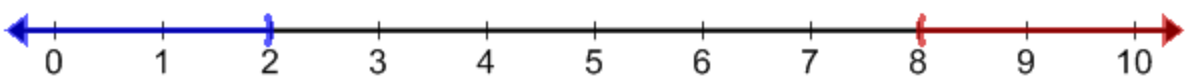


$\{x \mid x < 2\}$



$\{x \mid x < 2\} \cup \{x \mid x > 8\}$

$\{x \mid x < 2 \text{ or } x > 8\}$



Mathematical Use of the Word "or"

The word "or" corresponds to "union" and to the symbol "$\cup$". For a number to be a solution of a disjunction, it must be in *at least one* of the solution sets of the individual sentences.

Example Solve and graph:

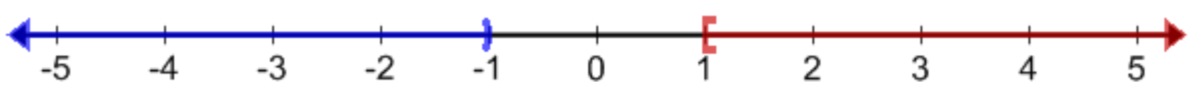
$$2x + 1 \geq 3 \quad or \quad 3x < -3.$$

Solution

$$2x + 1 \geq 3 \quad or \quad 3x < -3$$

$$2x \geq 2 \quad or \quad 3x < -3$$

$$x \geq 1 \quad or \quad x < -1.$$



Interval Notation and Domains

We saw earlier that if $f(x)=\dfrac{2x+1}{x-7}$, then the domain of $f(x)=\{x\,|\,x \text{ is a real number } \textit{and } x\neq 7\}$.

We can now represent such a set using interval notation:

$$\{x\,|\,x \text{ is a real number } \textit{and } x\neq 7\}=(-\infty,7)\cup(7,\infty).$$