

Solving Simultaneous Equations and Matrices

The following represents a systematic investigation for the steps used to solve two simultaneous linear equations in two unknowns. The motivation for considering this relatively simple problem is to illustrate how matrix notation and algebra can be developed and used to consider problems such as the rotation of an object. Examples of how 2D vectors are transformed by some elementary matrices illustrate the link between matrices and vectors.

Consider a system of two simultaneous linear equations:

$$ax + by = e \quad \cdots \quad (1)$$

$$cx + dy = f \quad \cdots \quad (2)$$

Multiply Equation (1) by d and Equation (2) by b :

$$adx + bdy = ed \quad \cdots \quad (3)$$

$$bcx + bdy = bf \quad \cdots \quad (4)$$

Subtract Equation (4) from Equation (3)

$$adx - bcx = ed - bf \quad \cdots \quad (5)$$

Making x the subject of the equation, assuming $ad - bc \neq 0$:

$$x = \frac{ed - bf}{ad - bc} \quad \cdots \quad (6)$$

Similarly, multiply Equation (1) by c and Equation (2) by a :

$$acx + cby = ec \quad \cdots \quad (7)$$

$$acx + ady = af \quad \cdots \quad (8)$$

Subtract Equation (7) from Equation (8)

$$ady - bcy = af - ce \quad \cdots \quad (9)$$

Making y the subject of the equation, assuming $ad - bc \neq 0$:

$$y = \frac{af - ce}{ad - bc} \quad \cdots \quad (10)$$

Equations (6) and (10) provide a solution to the simultaneous Equations (1) and (2). Introducing matrix notation for the simultaneous Equations (1) and (2) these solutions (6) and (10) form a pattern as follows.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

Define the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det(A) = ad - bc$. Then introduce two matrices formed from A by first replacing the coefficient to x in Equations (1) and (2) by the right-hand side values, then forming the second matrix by replacing the coefficient of y by the same right-hand side values yields

$$\mathbf{B} = \begin{bmatrix} e & b \\ f & d \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} a & e \\ c & f \end{bmatrix}$$

Thus $\det(\mathbf{B}) = ed - bf$ and $\det(\mathbf{C}) = af - ec$, therefore the solutions (6) and (10) can be written as follows, provided $\det(\mathbf{A}) \neq 0$.

$$x = \frac{\det(\mathbf{B})}{\det(\mathbf{A})}$$

and

$$y = \frac{\det(\mathbf{C})}{\det(\mathbf{A})}$$

As an alternative the Equations (6) and (10) can be expressed in matrix notation as follows

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{ed - bf}{ad - bc} \\ \frac{af - ec}{ad - bc} \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} ed - bf \\ -ec + af \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}$$

Given the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the matrix $\mathbf{A}^{-1} = \begin{bmatrix} \frac{d}{\det(\mathbf{A})} & \frac{-b}{\det(\mathbf{A})} \\ \frac{-c}{\det(\mathbf{A})} & \frac{a}{\det(\mathbf{A})} \end{bmatrix}$ is known as the inverse of \mathbf{A} with

the property that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

The solution to Equations (1) and (2) can therefore be expressed as follows.

Given a pair of simultaneous equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

form the matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \Rightarrow \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

calculate the inverse matrix

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

then express the solution $\begin{bmatrix} x \\ y \end{bmatrix}$ using

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} e \\ f \end{bmatrix} \quad \dots \quad (11)$$

Equation (11) shows that the solution is obtained by matrix multiplication of the right-hand side of the Equations (1) and (2) by a matrix entirely created from the coefficients of the x and y terms in these equations. Geometrically speaking, the coefficients a , b , c and d define a pair of straight lines in a 2D plane in terms of the slope for these lines. The right-hand side values for a given set of coefficients determine the intercept values for these lines and specifying the values for e and f define two lines from the infinite set of parallel lines characterised by the coefficients a , b , c and d . Equation (11) states that it is sufficient to consider the directions for these lines in order to obtain a general method for determining the point at which specific lines intersect.

For example, when given a pair of simultaneous equations

$$\begin{aligned} 2x + 3y &= 9 \\ -x + y &= -2 \end{aligned}$$

the point of intersection for all equations of the form

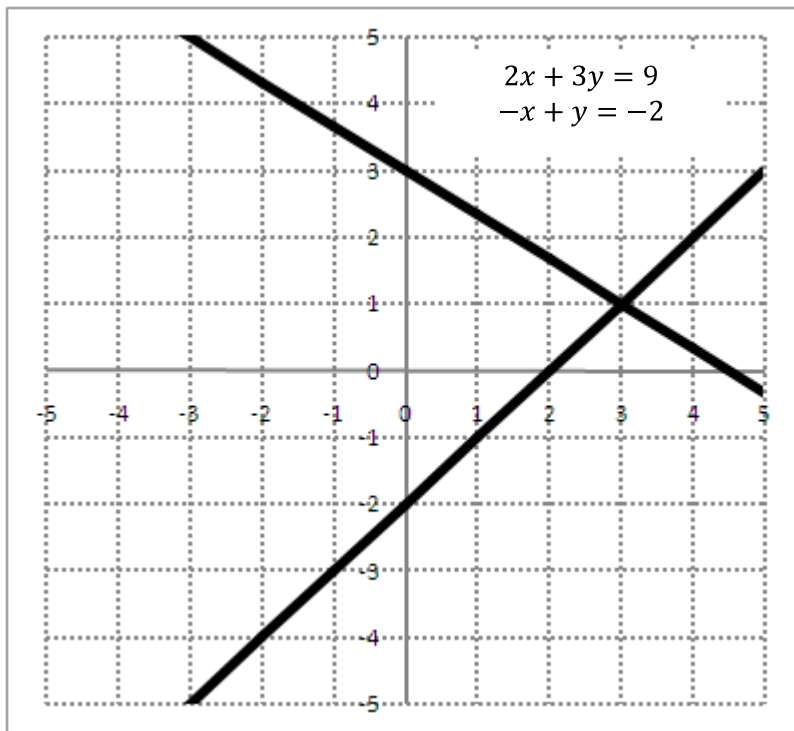
$$\begin{aligned} 2x + 3y &= e \\ -x + y &= f \end{aligned}$$

can be prepared by considering the matrix $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$.

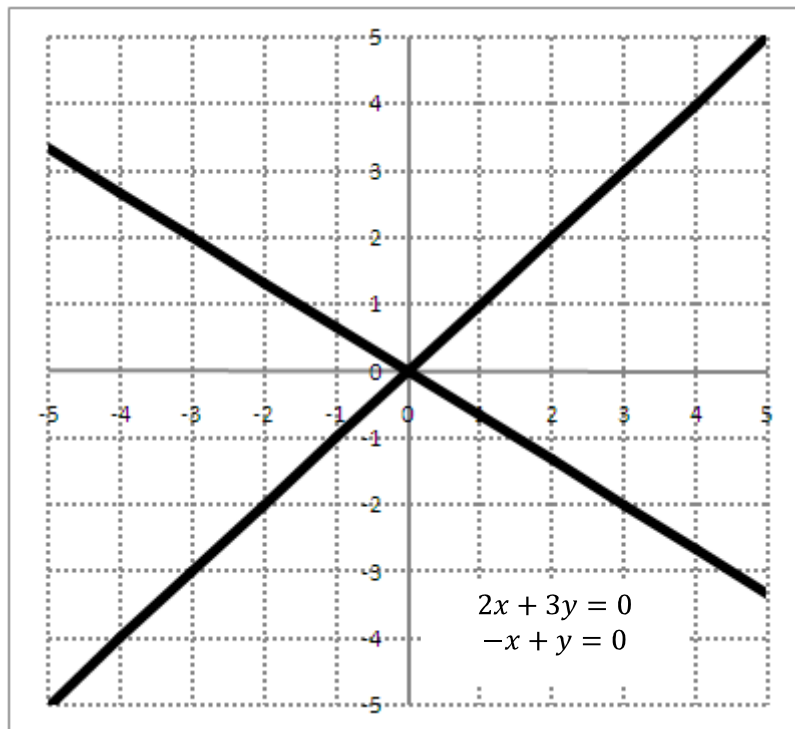
If $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ then $\det(\mathbf{A}) = (2)(1) - (-1)(3) = 5$. The inverse matrix is therefore

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ -2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} (1)(9) + (-3)(-2) \\ (1)(9) + (2)(-2) \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 15 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



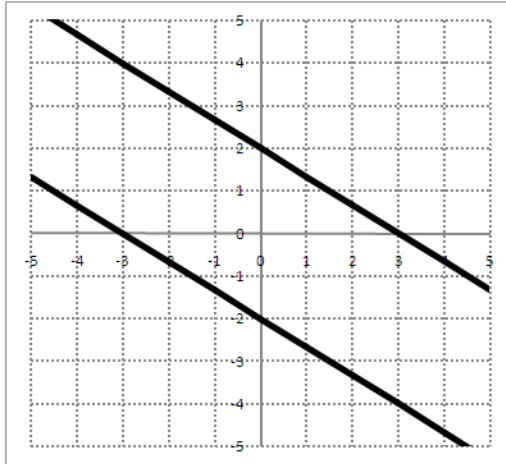
With the exception of the last step, each step in the solution can be performed without reference to the values from the right-hand side $\begin{bmatrix} 9 \\ -2 \end{bmatrix}$. The solution is therefore obtained by considering the homogenous equations corresponding to a pair of lines passing through the origin.



The method of solution fails whenever $\det(A) = 0$. Since $\det(A) = ad - bc$ the failure occurs if

$$\frac{b}{a} = \frac{d}{c}$$

which geometrically means the two lines defined by the simultaneous equations (1) and (2) have the same slope and are therefore parallel. Parallel lines are either identical or the lines never intersect one another, therefore no single point can satisfy both equations.



$$\begin{aligned} 2x + 3y &= 6 \\ 4x + 6y &= -12 \end{aligned}$$

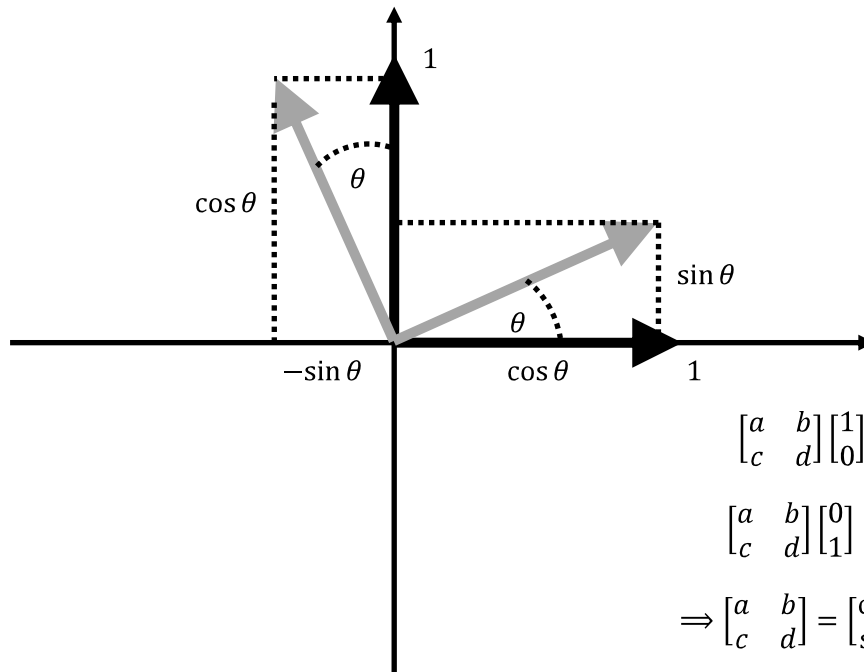
$$\therefore A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

$$\det(A) = (2)(6) - (4)(3) = 0$$

A matrix A such that $\det(A) = 0$ is said to be *singular*.

Linear Transformations defined by Matrices

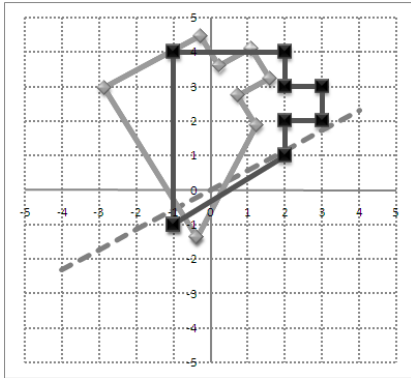
Rotation by θ in an anticlockwise direction about the origin



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



Rotation about the origin by an angle of $\frac{\pi}{6}$ is achieved by transforming a point $\begin{bmatrix} x \\ y \end{bmatrix}$ by matrix multiplication by

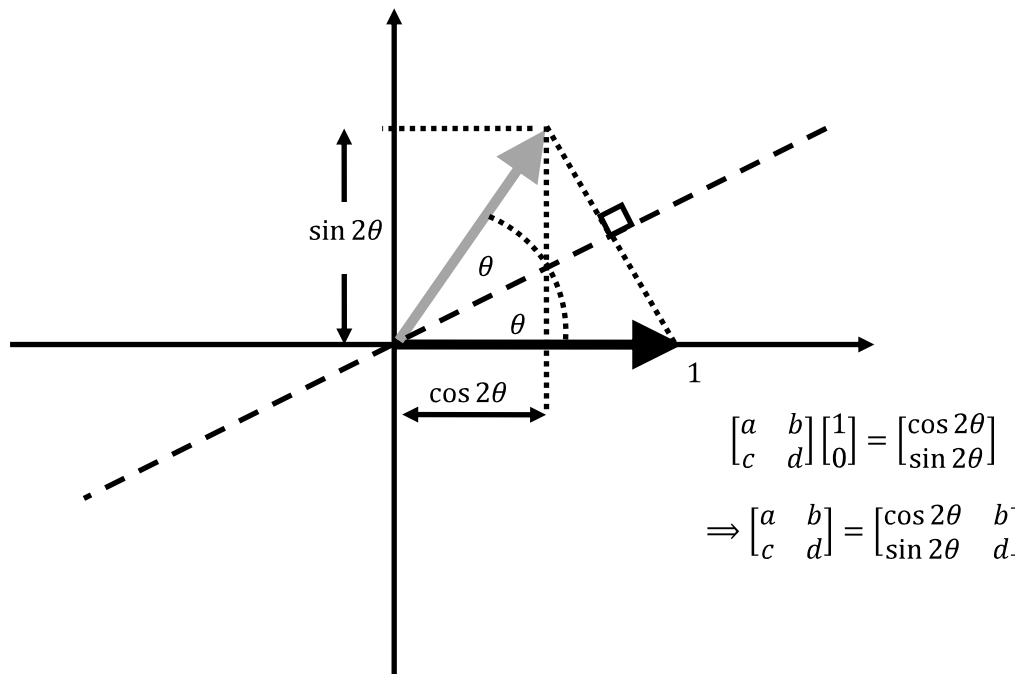
$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \frac{\pi}{6} & -\sin \frac{\pi}{6} \\ \sin \frac{\pi}{6} & \cos \frac{\pi}{6} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The shape with vertices $\begin{bmatrix} -1 & -1 & 2 & 2 & 3 & 3 & 2 & 2 \\ -1 & 4 & 4 & 3 & 3 & 2 & 2 & 1 \end{bmatrix}$ is transformed by the rotation matrix through matrix multiplication as follows.

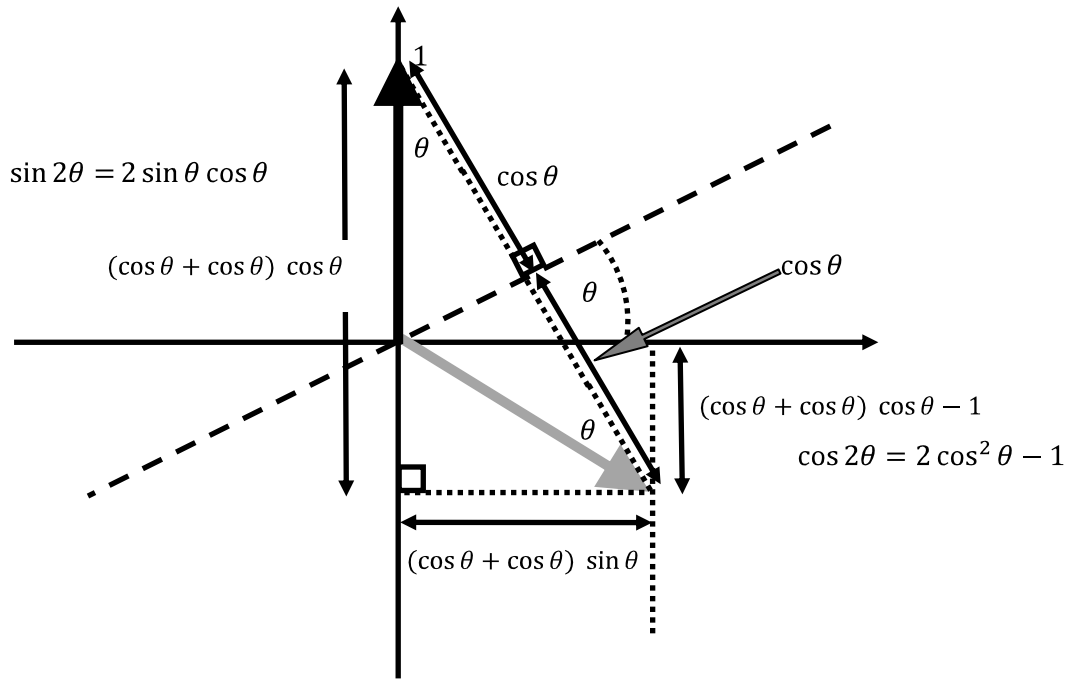
$$\begin{bmatrix} -0.366 & -2.866 & -0.268 & 0.232 & 1.098 & 1.598 & 0.732 & 1.232 \\ -1.366 & 2.964 & 4.464 & 3.598 & 4.098 & 3.232 & 2.732 & 1.866 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -1 & -1 & 2 & 2 & 3 & 3 & 2 & 2 \\ -1 & 4 & 4 & 3 & 3 & 2 & 2 & 1 \end{bmatrix}$$

Reflection about a line making an angle of θ in an anticlockwise direction with the x-axis

Consider first the result of reflecting the unit vector in the direction of the x-axis.



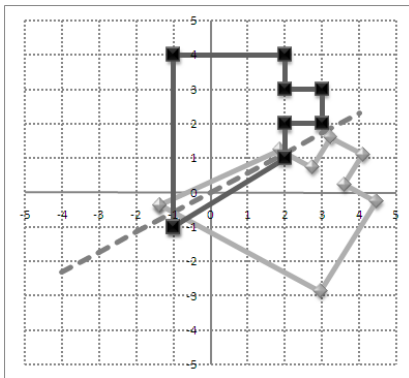
Next consider the result of reflecting the unit vector in the direction of the y-axis.



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin 2\theta \\ -\cos 2\theta \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & \sin 2\theta \\ c & -\cos 2\theta \end{bmatrix}$$

Reflection in a line through the origin making an angle θ with the x-axis is therefore

$$A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$



Reflection in a line making an angle of $\frac{\pi}{6}$ with the x-axis passing through the origin is achieved by transforming a point $\begin{bmatrix} x \\ y \end{bmatrix}$ by matrix multiplication

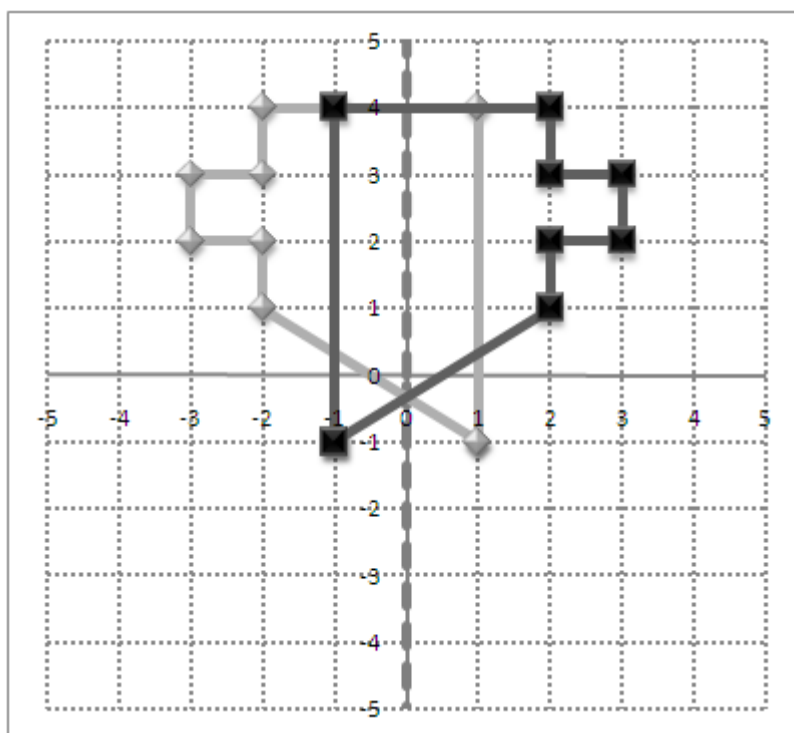
$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 2\frac{\pi}{6} & \sin 2\frac{\pi}{6} \\ \sin 2\frac{\pi}{6} & -\cos 2\frac{\pi}{6} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The shape with vertices $\begin{bmatrix} -1 & -1 & 2 & 2 & 3 & 3 & 2 & 2 \\ -1 & 4 & 4 & 3 & 3 & 2 & 2 & 1 \end{bmatrix}$ is transformed by the reflection matrix through matrix multiplication as follows.

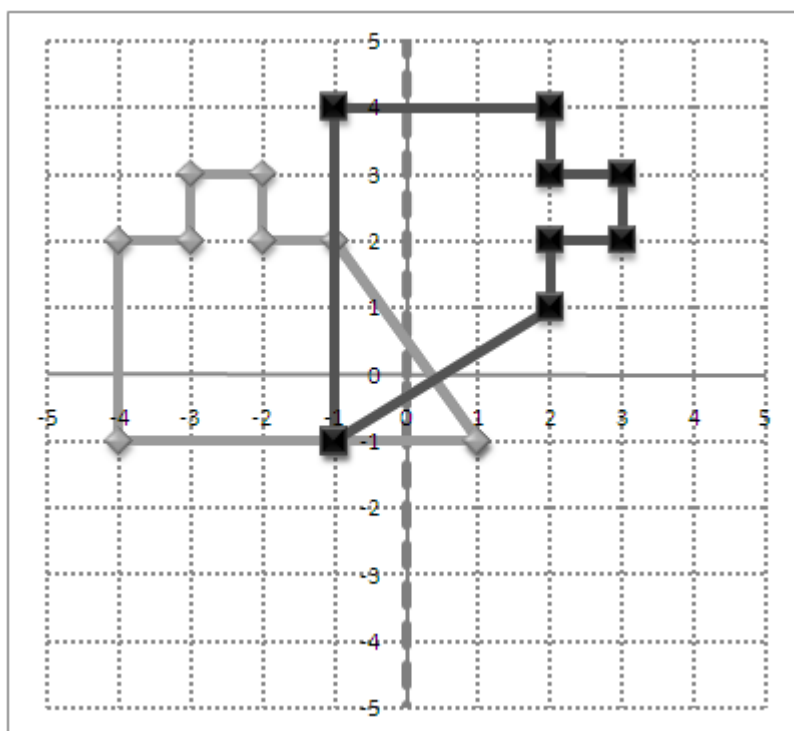
$$\begin{bmatrix} -1.366 & 2.964 & 4.464 & 3.598 & 4.098 & 3.232 & 2.732 & 1.866 \\ -0.366 & -2.866 & -0.268 & 0.232 & 1.098 & 1.598 & 0.732 & 1.232 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & -1 & 2 & 2 & 3 & 3 & 2 & 2 \\ -1 & 4 & 4 & 3 & 3 & 2 & 2 & 1 \end{bmatrix}$$

Examples of Matrix Transformations

Reflection in the y-axis $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

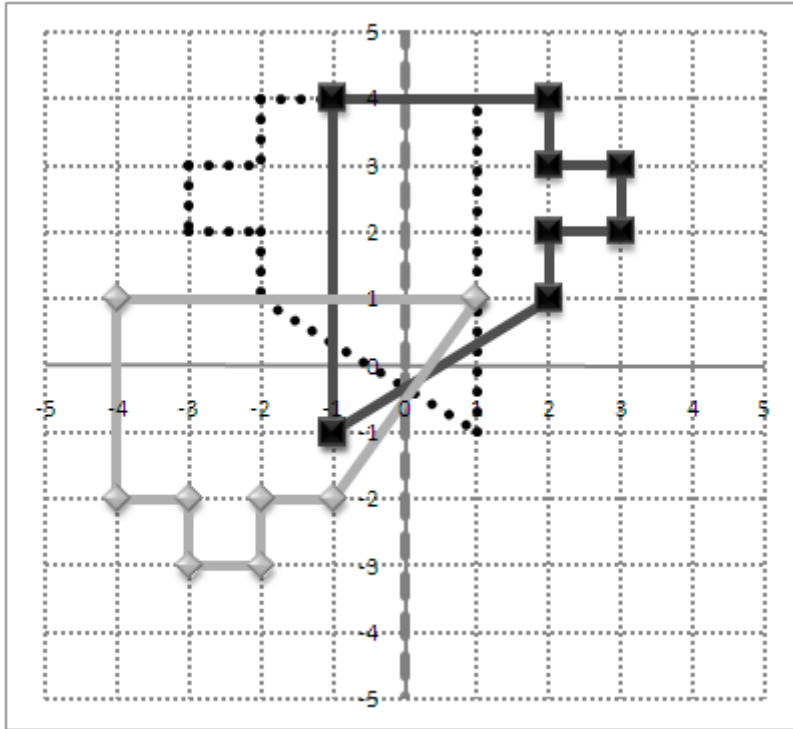


Rotation about the origin by $\frac{\pi}{2}$ radians $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



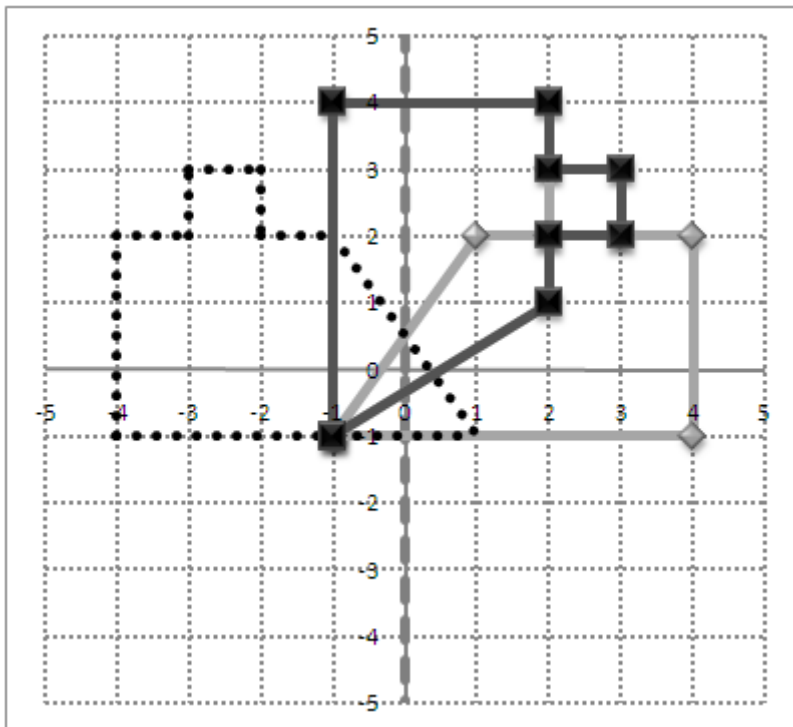
Reflection in the y-axis followed by a rotation by $\frac{\pi}{2}$ about the origin

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

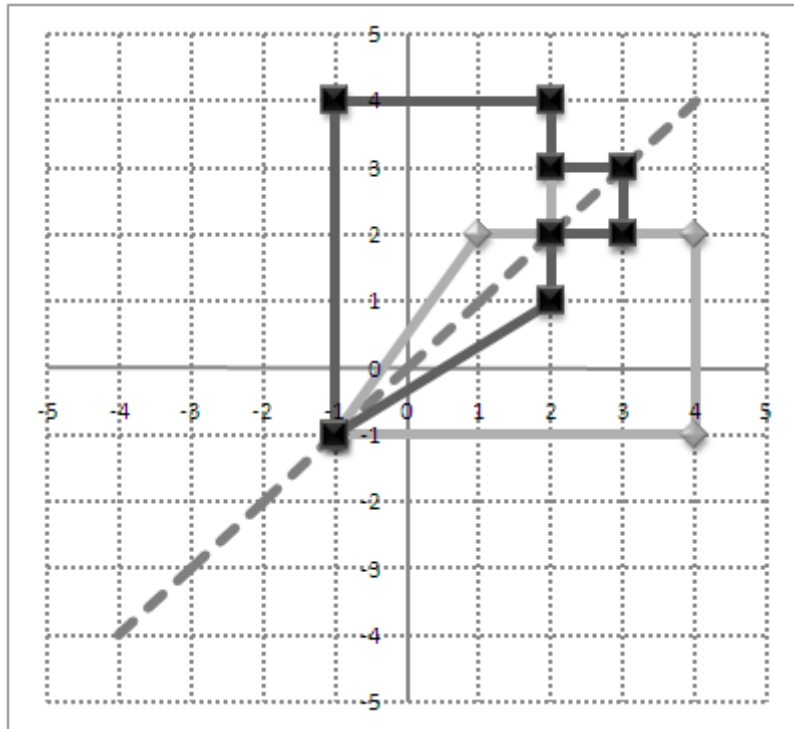


Rotation by $\frac{\pi}{2}$ about the origin followed by a Reflection in the y-axis

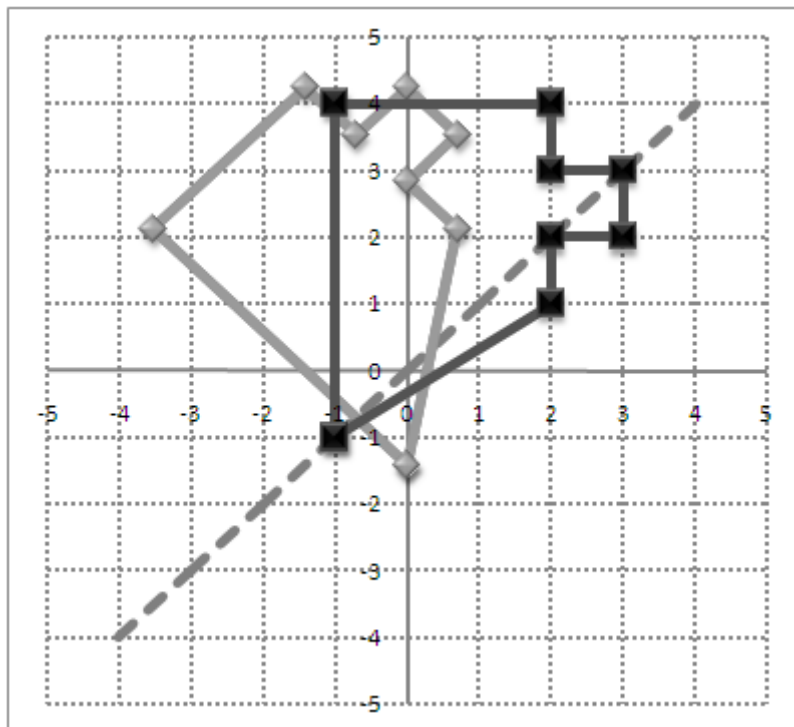
$$\mathbf{AB} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



Reflection in the line $y = x$ $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

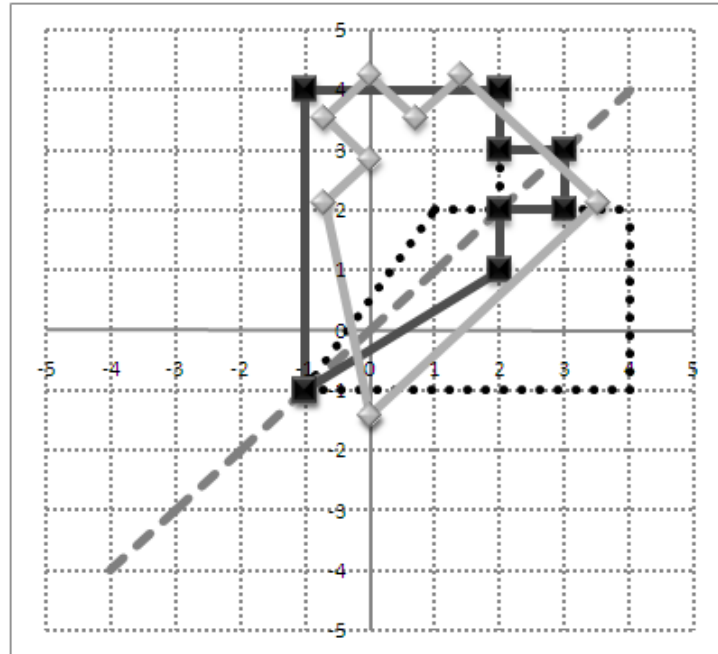


Rotation about the origin by $\frac{\pi}{4}$ radians $\mathbf{D} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$



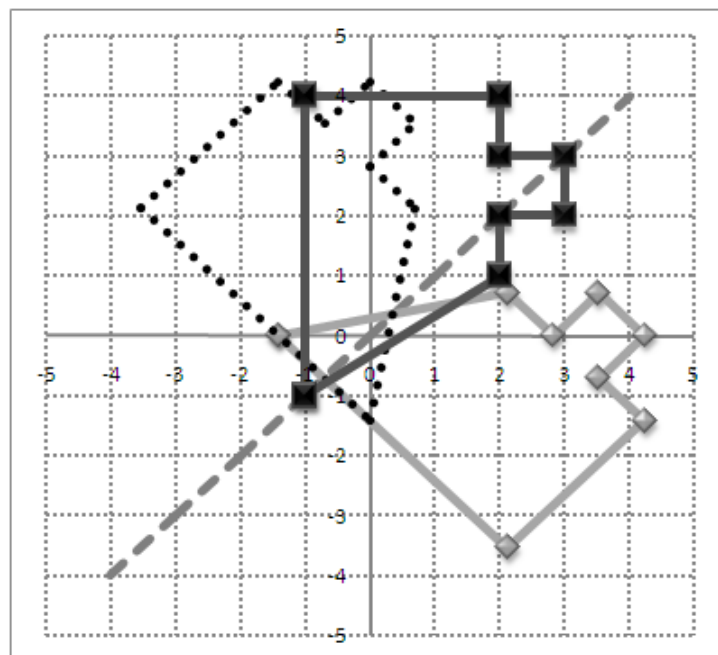
Reflection in the $y = x$ followed by a rotation by $\frac{\pi}{4}$ about the origin

$$DC = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$



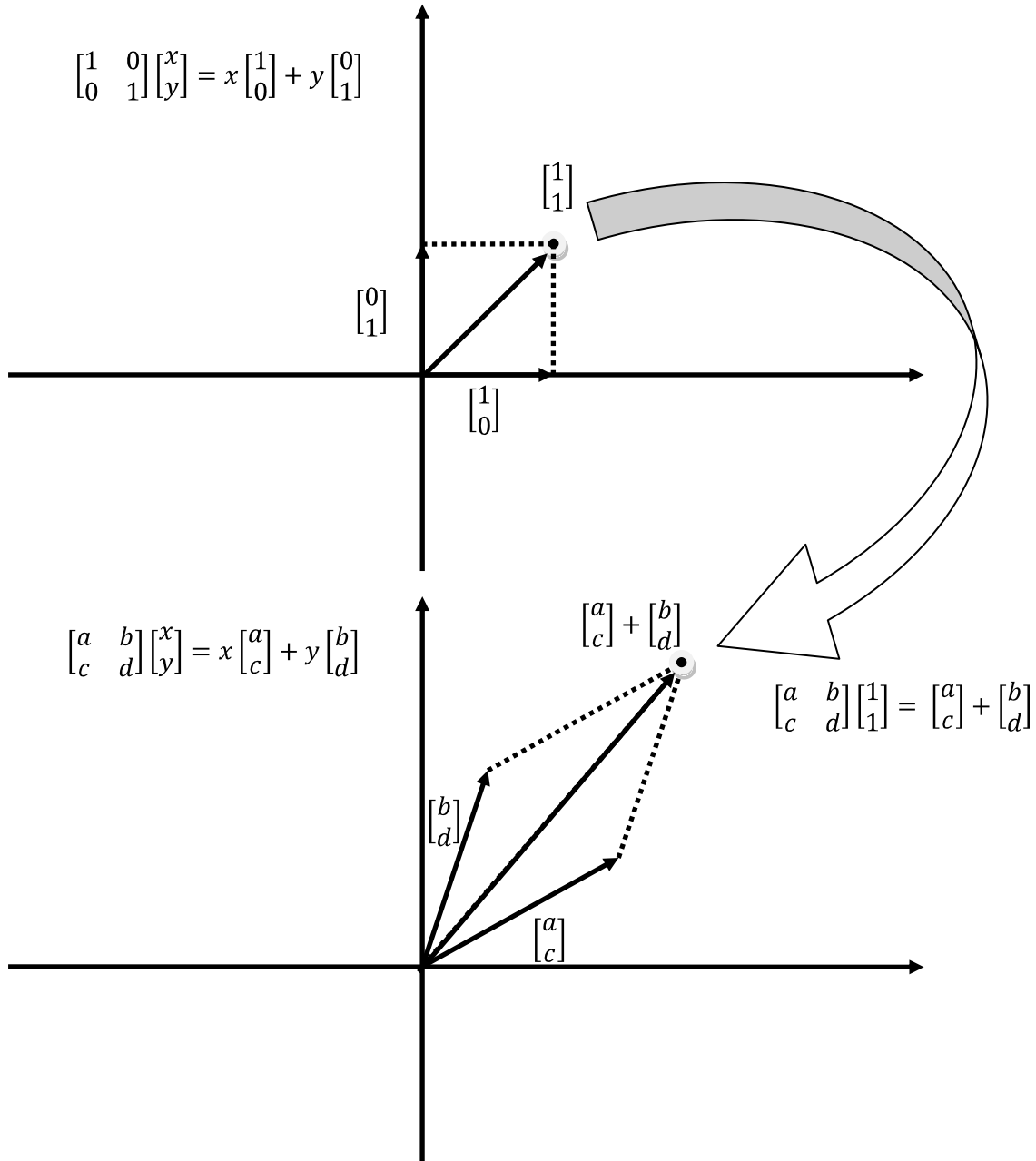
Rotation by $\frac{\pi}{4}$ about the origin followed by a reflection in the $y = x$

$$CD = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$



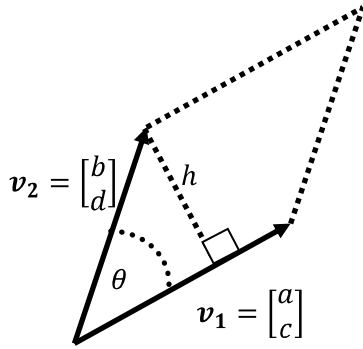
Mapping Shapes of known Area

Consider how a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ transforms an area of unit size to a new area. By considering a square defined by the unit vectors and therefore of unit area, the relationship between the initial area of a shape to the area of the transformed shape is shown to be dependent on the determinant of the matrix A . A square is transformed by A to a parallelogram as follows.



A matrix of the form $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ transforms the unit square to a parallelogram with area $|\det(A)|$.

The square defined by the unit vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ after transformation by A become the vector $v_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $v_2 = \begin{bmatrix} b \\ d \end{bmatrix}$.



The area of a parallelogram defined by \mathbf{v}_1 and \mathbf{v}_2 is given by

$$\text{area} = \text{base} \times \text{perpendicular height}$$

If the base is taken to be the vector \mathbf{v}_1 the length of the base is $|\mathbf{v}_1|$, and the perpendicular height is $h = |\mathbf{v}_2| \sin \theta$, where θ is the angle between \mathbf{v}_1 and \mathbf{v}_2

$$\text{area} = |\mathbf{v}_1| |\mathbf{v}_2| \sin \theta$$

Since the dot product between vectors gives

$$\cos \theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1| |\mathbf{v}_2|} \Rightarrow \sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1| |\mathbf{v}_2|} \right)^2}$$

$$\text{area} = |\mathbf{v}_1| |\mathbf{v}_2| \sqrt{1 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1| |\mathbf{v}_2|} \right)^2} = \sqrt{(|\mathbf{v}_1| |\mathbf{v}_2|)^2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2}$$

Now, $\mathbf{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$, therefore

$$\begin{aligned} |\mathbf{v}_1|^2 &= a^2 + c^2, |\mathbf{v}_2|^2 = b^2 + d^2 \text{ and } (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 = (ab + cd)^2 \\ \Rightarrow \text{area}^2 &= (|\mathbf{v}_1| |\mathbf{v}_2|)^2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 = (a^2 + c^2)(b^2 + d^2) - (ab + cd)^2 \\ \Rightarrow \text{area}^2 &= a^2 b^2 + a^2 d^2 + c^2 b^2 + c^2 d^2 - (a^2 b^2 + c^2 d^2 + 2abcd) \\ \Rightarrow \text{area}^2 &= a^2 d^2 + c^2 b^2 - 2abcd = (ad - cb)^2 \\ \Rightarrow \text{area} &= |ad - cb| = |\det(\mathbf{A})| \end{aligned}$$

An object with area of one unit is transformed by the matrix \mathbf{A} to an object of area $|\det(\mathbf{A})|$.

Thus, a shape with area X is transformed by the matrix \mathbf{A} to a shape with area Y such that

$$Y = |\det(\mathbf{A})|X$$

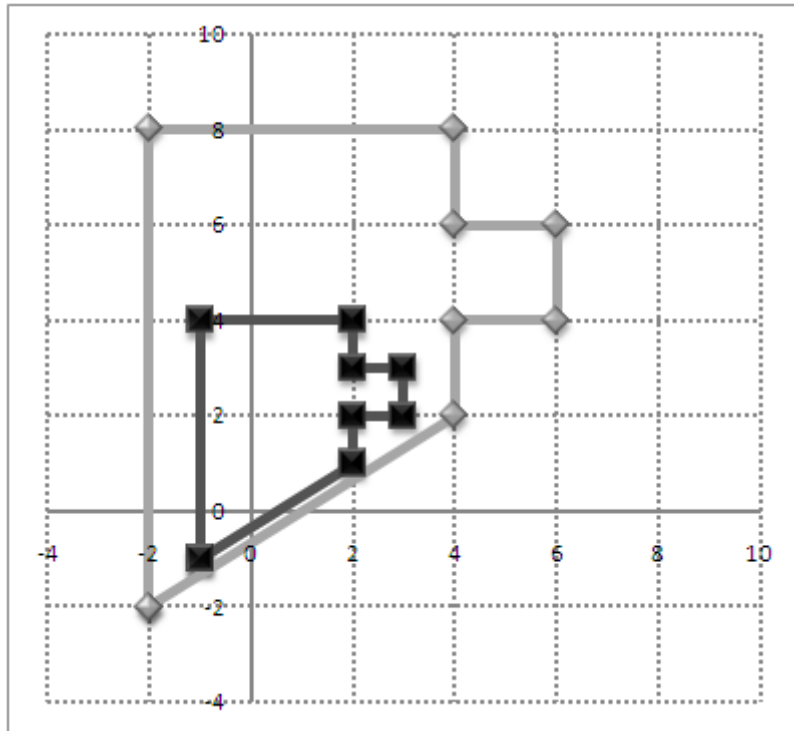
Example:

The closed shape with vertices $\begin{bmatrix} -1 & -1 & 2 & 2 & 3 & 3 & 2 & 2 \\ -1 & 4 & 4 & 3 & 3 & 2 & 2 & 1 \end{bmatrix}$ is transformed by the enlargement matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ through matrix multiplication as follows.

$$\begin{bmatrix} -2 & -2 & 4 & 4 & 6 & 6 & 4 & 4 \\ -2 & 8 & 8 & 6 & 6 & 4 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 & 2 & 2 & 3 & 3 & 2 & 2 \\ -1 & 4 & 4 & 3 & 3 & 2 & 2 & 1 \end{bmatrix}$$

Since $\det \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) = 4$, the original area of the closed shape 13 units^2 is transformed to a shape with area 52 units^2

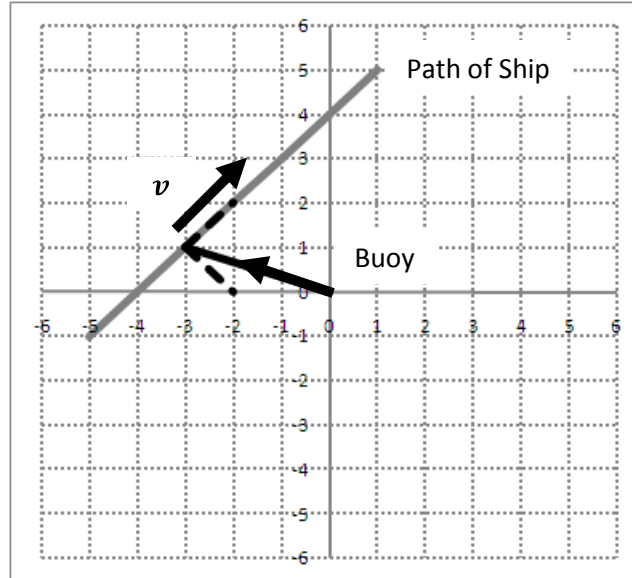
$$52 = 13 \times \left| \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right| = 13 \times 4$$



Note: since a rotation about the origin in an anti-clockwise direction by the angle θ is given by $\mathbf{A} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$, the determinant of the rotation matrix \mathbf{A} is $\det(\mathbf{A}) = \cos^2 \theta + \sin^2 \theta = 1$ therefore area is preserved by a rotation. Similarly, a reflection in a line passing through the origin is given by the matrix $\mathbf{B} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ has determinant $\det(\mathbf{B}) = -\cos^2 2\theta - \sin^2 2\theta = -1$. Again, areas do not change when a reflection transformation is performed.

Mechanics and Transformations

Consider the motion of a ship travelling in a straight line relative to a buoy marking dangerous rocks beneath the water. From the location of the buoy, the path of the ship is specified by a velocity vector and a position vector relative to the buoy.



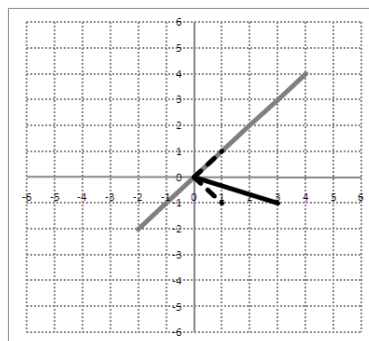
The vector equation of the line representing the path of the ship is given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

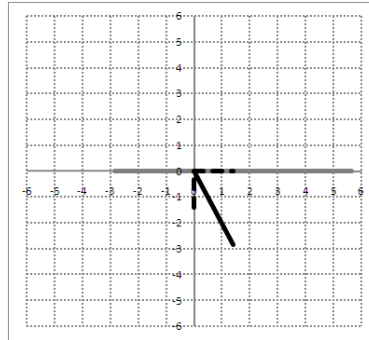
If the problem is to determine how close the ship gets to the rocks marked by the buoy, the solution is to determine the length of a line passing through the position of the buoy which is perpendicular to the velocity vectors and the point of intersection with the path of the ship.

An alternative perspective is to consider the position of the buoy relative to an observer on the ship facing at right-angles to the direction of motion of the ship looking on the side where the buoy can be seen. A sequence of transformation to the vectors involved reduces the problem to calculating the coordinates for the buoy relative to the observer on the ship. The transforms are as follows:

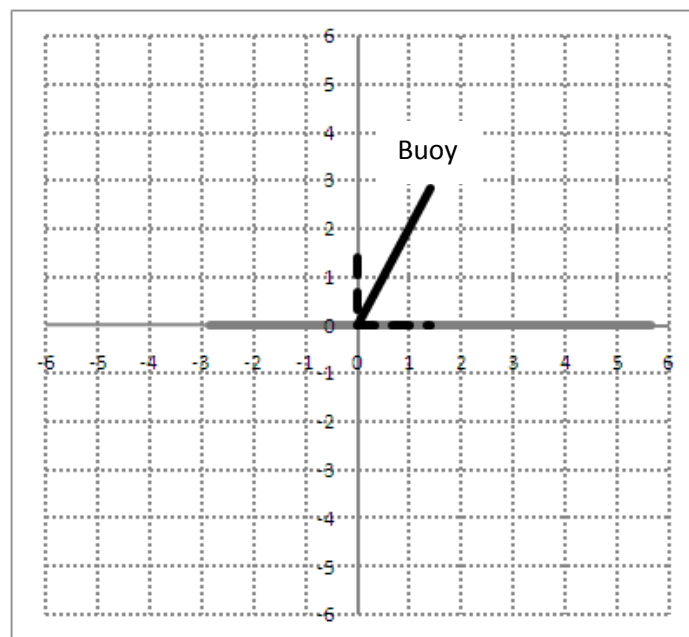
1. Translate the vectors to position the observer on the ship at the origin.



2. Rotate the vectors about the origin so that the velocity vector is in the direction of the x-axis.



3. Reflect the vectors in the x-axis.



The y-coordinate for the buoy is now the shortest distance at which the ship will pass the buoy.

The translation transformation is achieved by subtracting the position vectors \mathbf{r}_0 from all vectors involved in the calculation. Note that a translation is different from a rotation or a reflection since a translation is not a linear transformation, while both a rotation and a reflection are linear transformations.

To apply a rotation and a reflection to 2D vectors, two 2x2 matrices can be used to transform the vectors concerned. A rotation about the origin by $-\frac{\pi}{4}$ radians followed by a reflection in the x-axis

are achieved by multiplying vectors by the rotation matrix $\mathbf{A} = \begin{bmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{bmatrix} =$

$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, then multiply the result of the rotation by the reflection matrix

$$\mathbf{B} = \begin{bmatrix} \cos 2(0) & \sin 2(0) \\ \sin 2(0) & -\cos 2(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since the position of the buoy as originally described is at the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the position of the buoy relative to the observed on the ship is given by

1. Translate by subtracting $\mathbf{r}_0 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ from the original position vector for the buoy, therefore

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

2. Next, a rotation about the origin by $-\frac{\pi}{4}$ radians is achieved using matrix multiplication,

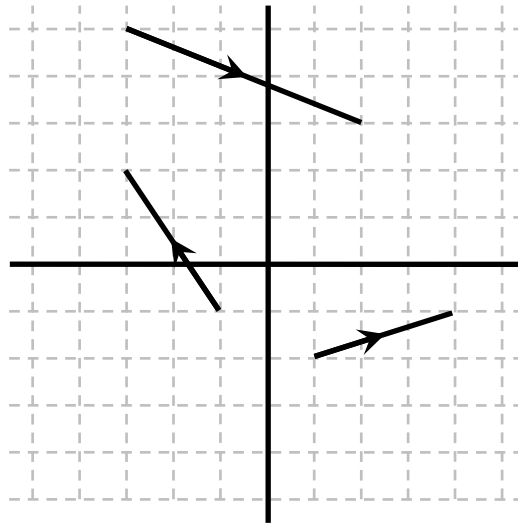
$$\mathbf{A} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -2\sqrt{2} \end{bmatrix}.$$

3. Finally a reflection about the x-axis $\mathbf{B} \begin{bmatrix} \sqrt{2} \\ -2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ -2\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$

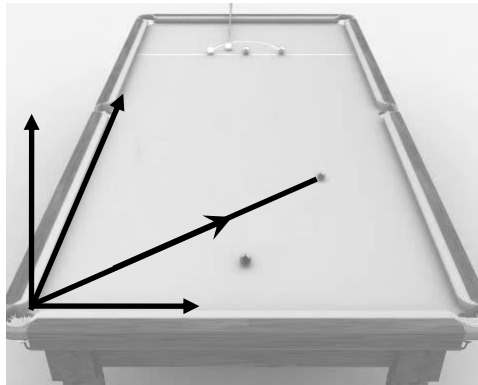
The position of the buoy relative to an observer on the ship at time $t = 0$ is therefore $\begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$. The equation of motion for the ship has been reduced to a 1D motion along the x-axis. The y-coordinate for the ship is therefore unaffected by the ship's motion and only the x-coordinate changes with time. The shortest distance between the ship and the buoy is simply the component of position for the buoy perpendicular to the direction of motion of the ship and is therefore $2\sqrt{2}$ units.

Basics of Vectors

Vectors are a means of conveying size and direction. Graphically vectors are represented by lines with an arrow, where the length of the line indicates the size of the vector, and the orientation of the line and direction of the arrow indicate the direction for the vector. In two dimensions, vectors can be drawn as lines in the xy-plane.



A table full of snooker balls could be defined in terms of a set of 2D vectors, where the position for each ball on the snooker table is defined relative to one chosen pocket by a set of lines with length equal to the distance of each ball from the pocket and the direction defined by the line connecting the ball to the specified pocket.



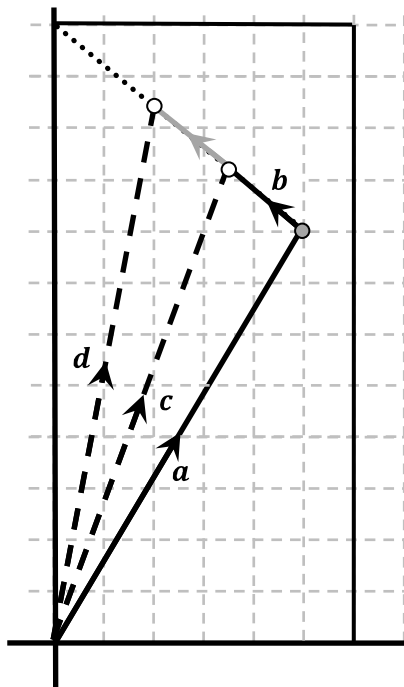
It is usual to introduce vectors as entities belonging to a 2D world such as the surface of a snooker table, but it is worth observing how this 2D model relates to other dimensions. Even for the snooker table example, the 2D world over which the snooker balls move is a 2D plane parallel to the floor on which the snooker table stands, both planes represent slices within a 3D world. Similarly, a snooker ball moving in a straight line within the plane of the table surface is moving in 1D.

Vectors are also closely linked to a mathematical understanding of dimension and the nature of the vector description changes with context. While a ball travels in a straight line a 1D model is sufficient

and such problems are solved without explicit reference to vectors. If the snooker ball strikes another ball and changes direction, a 2D model is required. If the snooker ball after the collision enters a pocket, the ball can move in a vertical direction perpendicular to the table surface and therefore a 3D model becomes essential. Vectors provide the means of moving in a systematic way between these scenarios.

Geometric Perspective of Vectors

Vectors have magnitude and direction. By implication, a set of 2D vectors must all be defined with respect to some underlying direction. For example the bottom edge of the snooker table offers a natural direction. Once the unit of size is stated and the agreed direction for a 2D problem is established all vector quantities can be defined. For a snooker table the unit for size is metres and the direction is chosen relative to an edge. The motion of balls on the table can then be analysed using vectors.

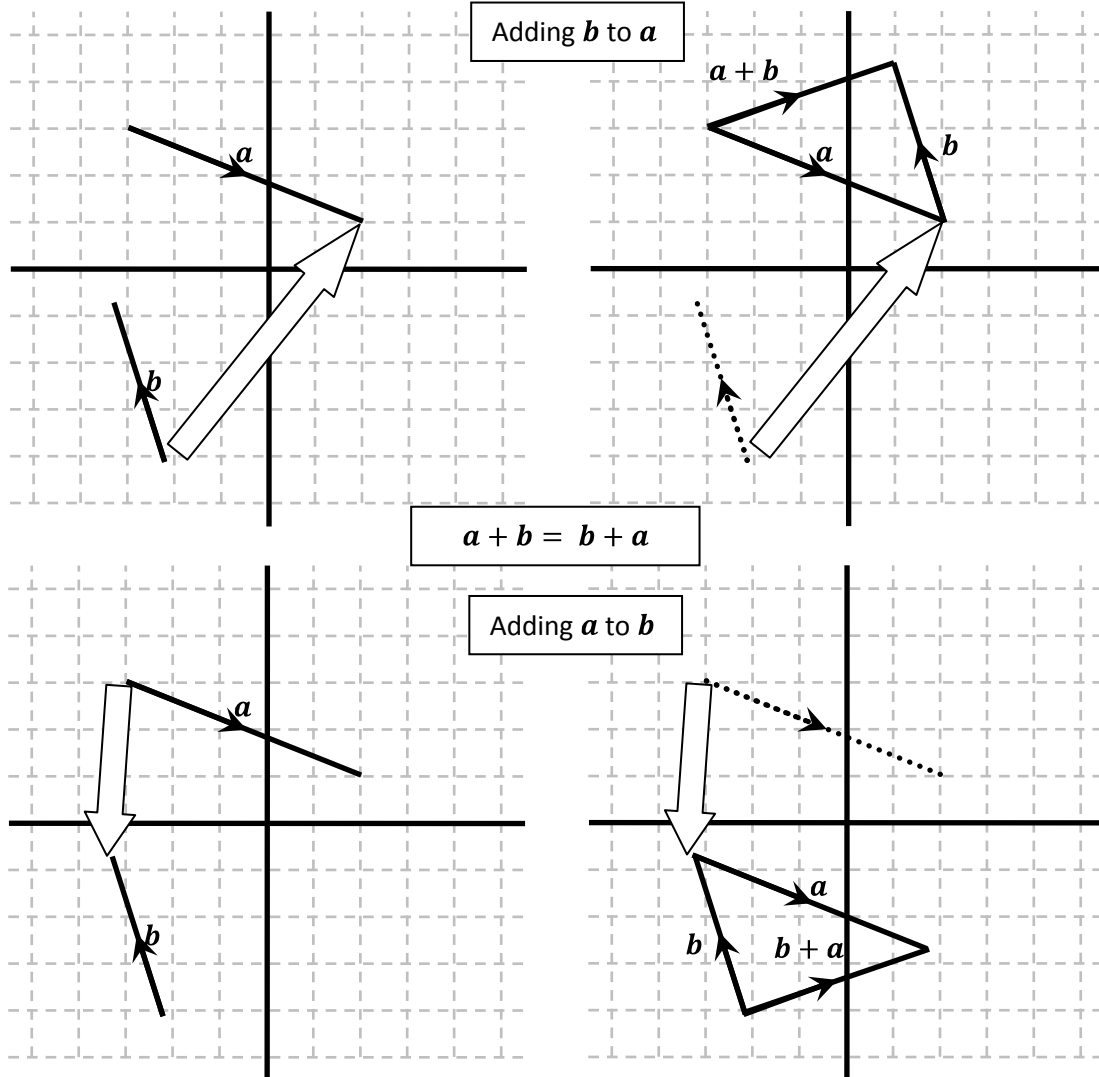


A snooker ball after being struck by the cue may be assigned a vector ***a*** representing the displacement from the reference table pocket and a second vector ***b*** indicating both the direction of motion for the ball and the distance per second moved along the direction of ***b***. Assuming constant speed for the snooker ball, together these two vectors ***a*** and ***b*** provide a means of describing the position for the snooker ball one second after the ball is at the position defined by ***a***, namely, the vector $c = a + b$.

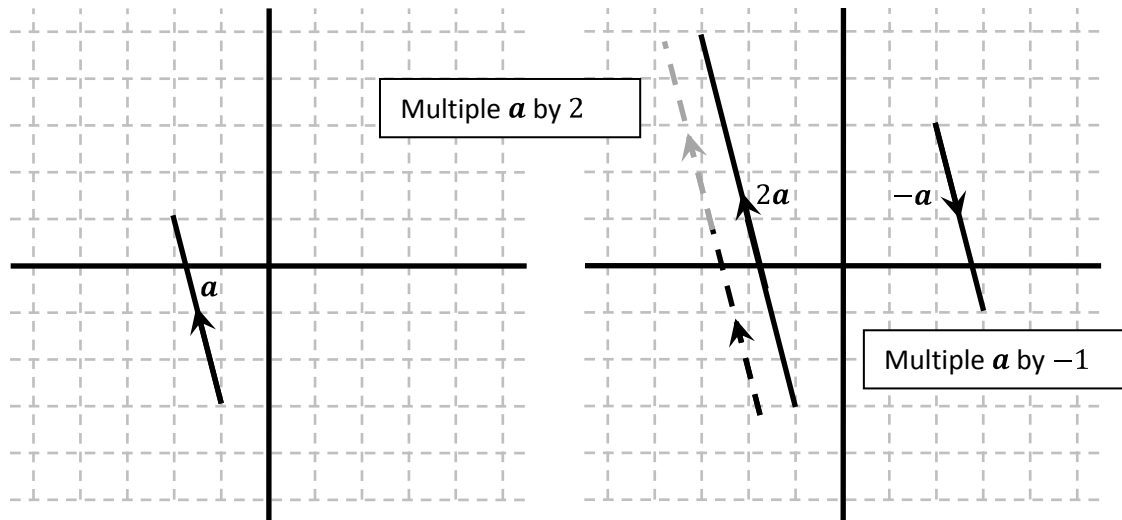
After two seconds, the snooker ball would move to $d = a + 2b$.

For 2D vectors, these vector statements can be represented using a pencil and paper to draw lines of length and direction corresponding to the vectors.

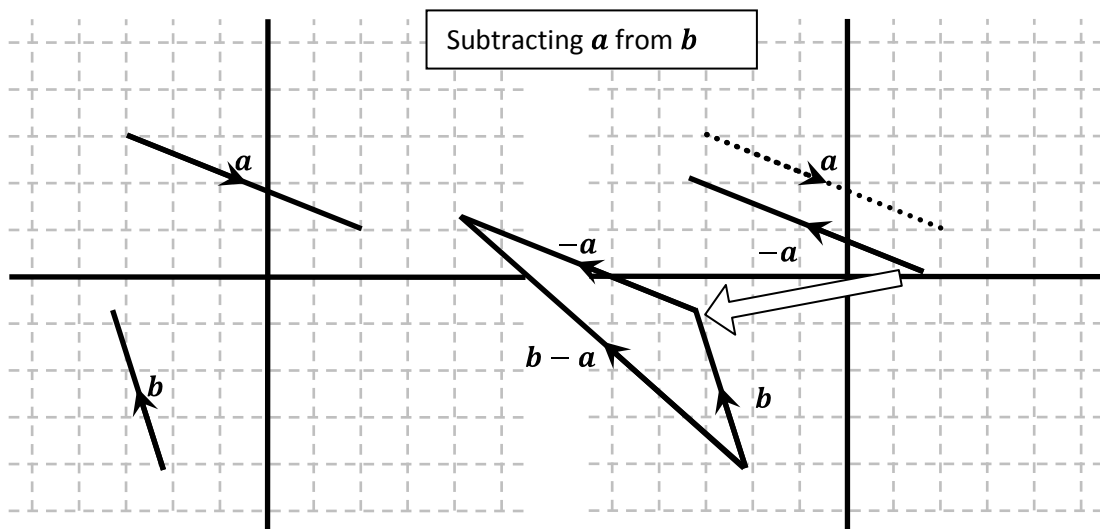
The snooker ball example illustrates vector addition and multiplication of a vector by a scale factor or, in vector terminology, multiplication of a vector by a scalar. Adding ***a*** and ***b*** in a geometric sense results in a new vector ***c***. Scaling ***b*** by a factor of 2, then adding the new vector to ***a*** results in yet another vector ***d***. The lines representing ***c*** and ***d*** are literally obtained by the steps one would take when drawing the lines using a ruler, a protractor, a compass and a sheet of paper. First the line representing ***a*** would be drawn, the ruler would then be moved to the end of ***a*** pointed to by the arrow for the vector, then the ruler would be aligned in the direction of ***b*** before measuring using the ruler the magnitude or size of ***b*** as a distance from the end of ***a***. The beginning point for ***b*** (or the tail of the arrow) must be placed at the end of ***a*** (or the nose of the arrow). These drafting steps are the definition of how to add two 2D vectors.



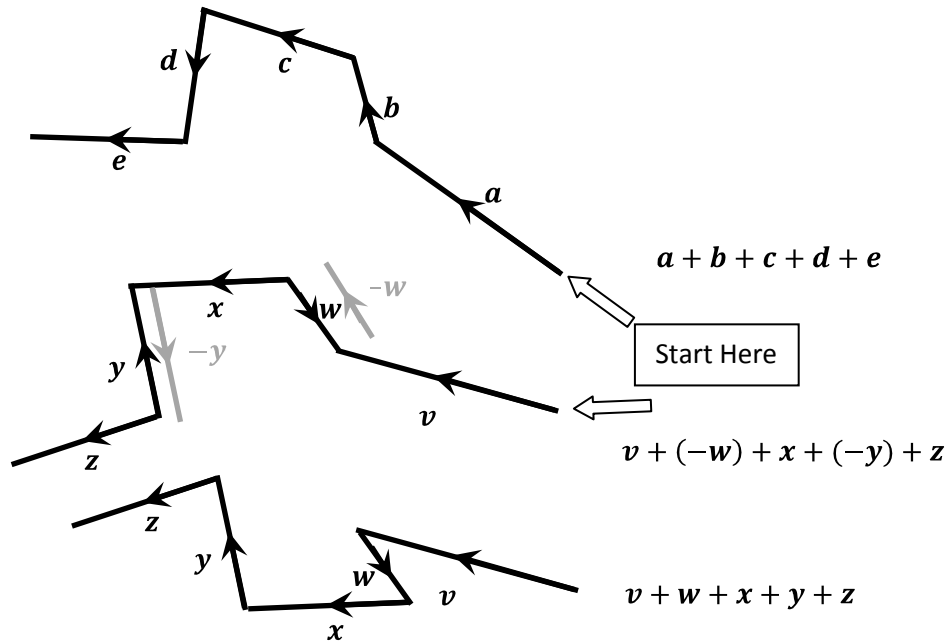
A geometric line drawn on a piece of paper defines an orientation. A line representing a vector is also specified using an arrow. The arrow however is vector notation for defining the positive direction for the line representing the vector. If a vector is multiplied by minus one, the arrow is reversed, meaning the positive direction has been switched.



Subtracting a from b as vectors is performed by adding $-a$ to b . The vector $-a$ is achieved graphically by reversing the arrow on the vector representing a . The expression $b - a$ is then constructed by first drawing the line representing b , moving the ruler to the end of b , before drawing the line in the direction of the vector $-a$.

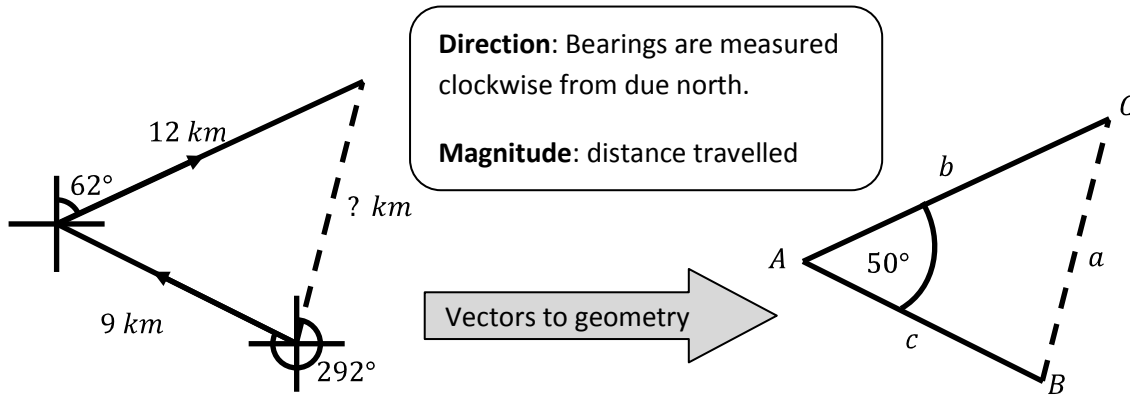


When two or more vectors are added together, a diagram for the vector sum should show all the vectors aligned such that if a pencil were to trace from the beginning of the lines drawn to represent the vector sum, the pencil would always encounter the arrows all facing in the same direction.



In 2D, constructing the triangle corresponding to the lengths and angles specified by the vectors \mathbf{a} and \mathbf{b} will permit a graphical determination of the vectors such as $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} + 2\mathbf{b}$. In mechanics, the ability to work with right-angle triangles and apply the cosine and sine rules when determining lengths and angles from general triangles is essential.

Example: Consider a ship sailing from port on a bearing of 292° for 9 km before changing direction to a bearing of 62° . What is the distance of the ship from port after the ship sails a further 12 km following changing course?



Two sides and the included angle, therefore apply the cosine rule:

$a^2 = b^2 + c^2 - 2bc \cos A$ where $b = 12, c = 9, \angle BAC = 50^\circ$. Therefore

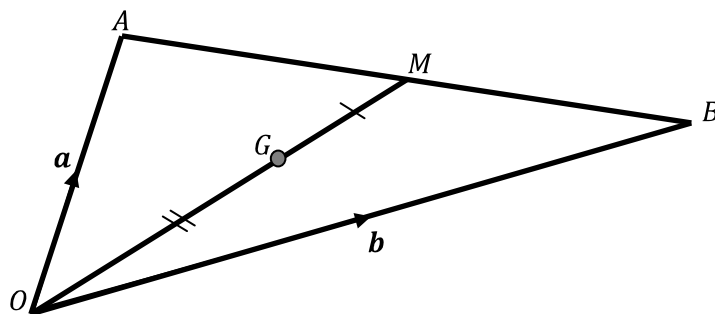
$$a = \sqrt{12^2 + 9^2 - 2 \times 12 \times 9 \times \cos 50} = 9.28 \text{ km (to 3 sf)}.$$

If the bearing for the ship were also calculated using the sine rule:

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \sin B = \frac{\sin 50 \times 12}{9.28} \Rightarrow \angle ABC = 82.1^\circ$$

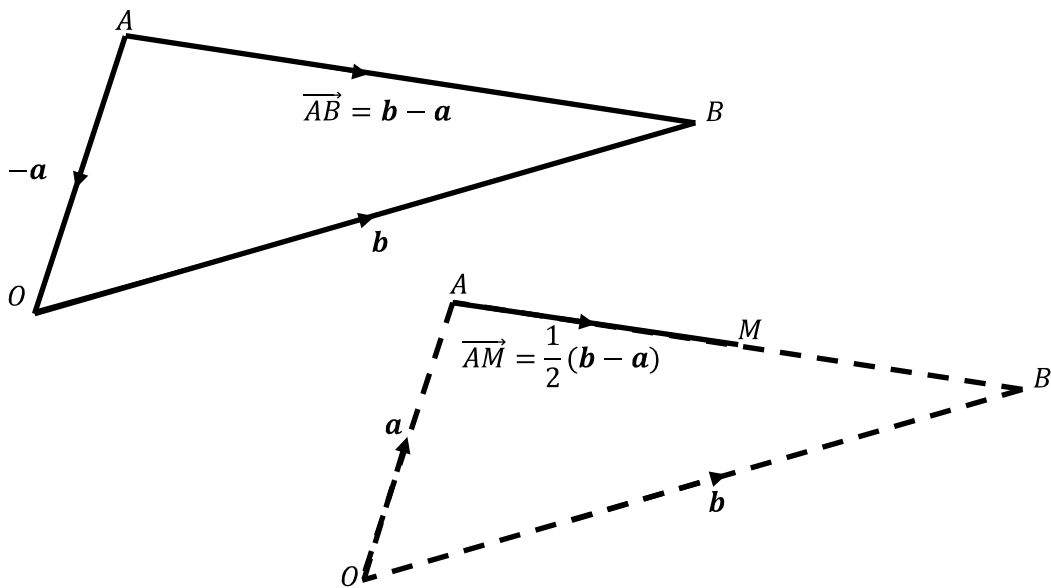
Both the magnitude and direction for the ship as a bearing, namely, $82.1 - (90 - 22) = 14.1^\circ$, are established using the geometry of a triangle congruent to the displacement vectors for the ship movements after leaving port.

Example: Express the position for the particle at location G in terms of the vectors \mathbf{a} and \mathbf{b} , where G is located $\frac{2}{3}$ of the length of OM and M is the mid-point of the line AB .

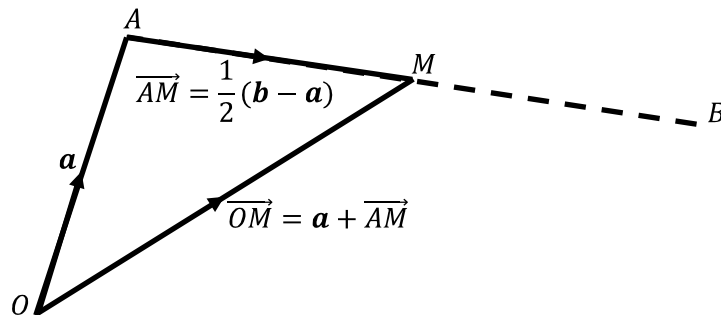


The position vector \vec{OG} and \vec{OM} are two vectors which are parallel but of different magnitude. The only difference between these two vectors is the length of the lines which would be used to draw the vectors on a piece of paper. As a consequence, a scale factor can be applied to \vec{OM} to produce the same vector as \vec{OG} . The question states the ratio of the line lengths $\frac{OG}{OM} = \frac{2}{3}$ therefore \vec{OG} and

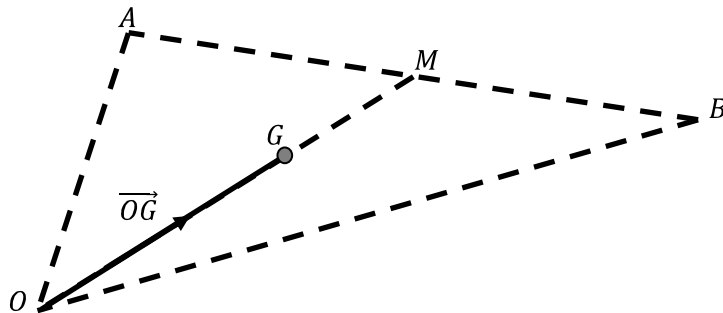
$\frac{2}{3}\overrightarrow{OM}$ must be identical vectors or $\overrightarrow{OG} = \frac{2}{3}\overrightarrow{OM}$. The problem is therefore to express \overrightarrow{OM} in terms of \mathbf{a} and \mathbf{b} .



The vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and since M is the mid-point of the line AB , the vector $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$.

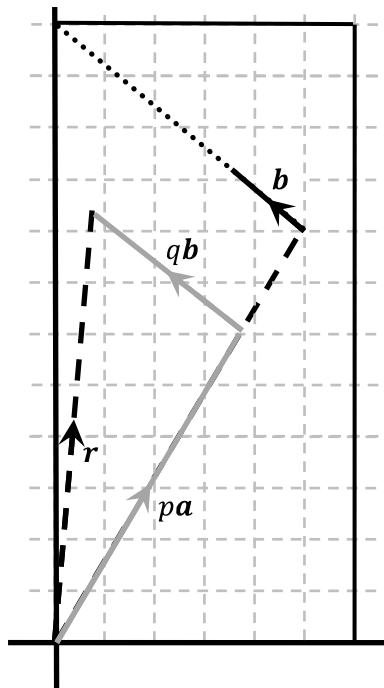


The vector $\overrightarrow{OM} = \mathbf{a} + \overrightarrow{AM} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{b} + \mathbf{a})$.



Since OG is $\frac{2}{3}$ of the length of OM , $\overrightarrow{OG} = \frac{2}{3}\overrightarrow{OM} = \frac{1}{3}(\mathbf{b} + \mathbf{a})$.

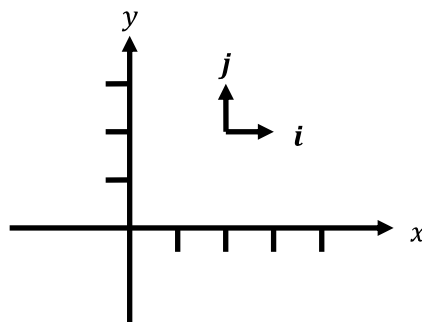
Moving Vectors from Geometry to Algebra



From the previous example, it is easy to see that any position on a line joining the initial position for the snooker ball and the pocket at which the ball is aimed can be defined using a scale factor for adjusting the length of \mathbf{b} . Further, sets of parallel lines of positions for balls can be described using \mathbf{a} and \mathbf{b} . If \mathbf{a} is scaled using a factor less than one, the position for the ball simply moves closer to the reference pocket. If the ball still moves from this new position in the direction of \mathbf{b} , a different line of positions on the table is defined. In fact, if both \mathbf{a} and \mathbf{b} are scaled appropriately, any point on the snooker table can be defined. Assuming the two vectors \mathbf{a} and \mathbf{b} are not parallel in direction, any vector \mathbf{r} representing the position of a ball on the snooker table can be written as the sum of these two vectors scaled by some pair of scale factors p and q .

$$\mathbf{r} = p\mathbf{a} + q\mathbf{b}$$

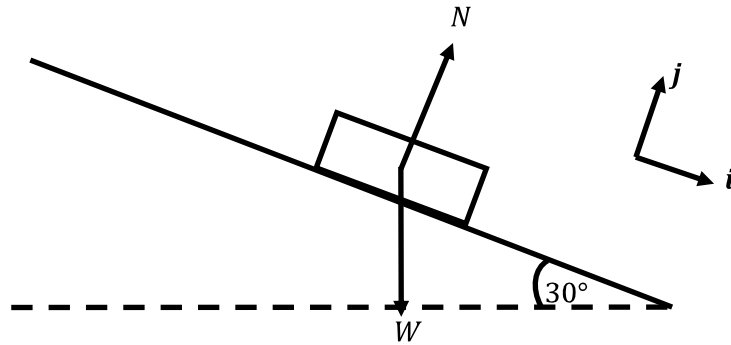
The vectors \mathbf{a} and \mathbf{b} , as drawn, provide the basis for constructing all vectors representing positions for balls on the snooker table. A key property for these two vectors \mathbf{a} and \mathbf{b} is the vectors are neither depicted as parallel nor anti-parallel. That is, the sine of the angle between the lines representing these vectors is non-zero.



If we decided on two such vectors \mathbf{a} and \mathbf{b} for a given snooker table, the position of a snooker ball could be specified by stating the values for p and q . The convention for working in 2D is to indeed define two vectors, denoted by \mathbf{i} and \mathbf{j} which are used in exactly the same context as the vectors \mathbf{a} and \mathbf{b} to allow any point in a 2D plane to be specified by a pair of numbers equivalent to the scalar values p and q already discussed.

The vectors \mathbf{i} and \mathbf{j} are chosen to be unit vectors, meaning vectors with magnitude equal to unity or in other words 1 unit in length. Further, these two vectors are defined to be perpendicular to each other such that if the vector \mathbf{i} is rotated through 90° in an anti-clockwise direction, the resulting vector is the \mathbf{j} unit vector.

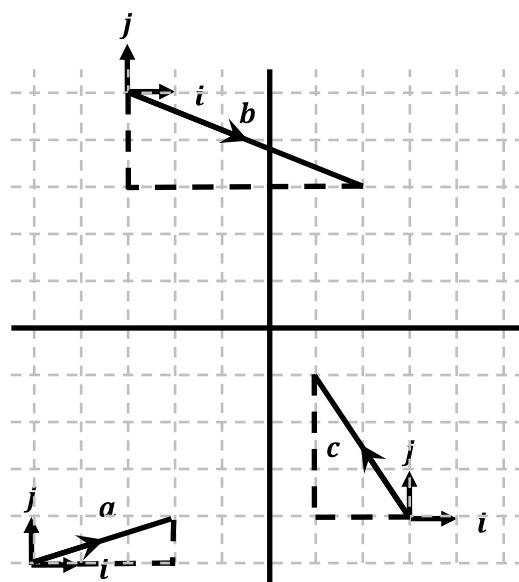
The meaning for these unit vectors depends on the context for the problem under analysis. Using the example of the snooker table, the unit vectors would be positioned parallel to the edges of the table. Alternatively, if a car were driving up a road and the motion of the car is modelled as a particle on an inclined plane, the \mathbf{i} unit vector might be chosen to be parallel with the inclined plane forcing the \mathbf{j} unit vector to be perpendicular to the direction of motion.



Once defined, these unit vectors allow both magnitude and direction for other vector to be specified in the form

$$\mathbf{a} = x\mathbf{i} + y\mathbf{j} \text{ or as in column vector notation } \mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Examples of vectors depicted geometrically are now represented using these unit vectors as follows.



$$\mathbf{a} = 3\mathbf{i} + \mathbf{j}$$

$$\mathbf{b} = 5\mathbf{i} - 2\mathbf{j}$$

$$\mathbf{c} = -2\mathbf{i} + 3\mathbf{j}$$

Or

$$\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

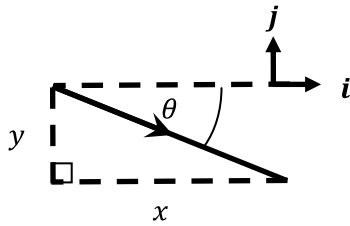
$$\mathbf{b} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$\mathbf{c} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

The vector $\mathbf{b} = 5\mathbf{i} - 2\mathbf{j}$ above scales the unit vectors by factor of 5 and -2 before summing the two resulting vectors. These scale factors used for each of these unit vectors are referred to as component values.

Given a vector specified using the \mathbf{i} and \mathbf{j} unit vectors, the magnitude or length of the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j}$, is calculated using Pythagoras' Theorem where the length of the vector is the hypotenuse of a right-angle triangle. The direction for a vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j}$ is also calculated from the geometry of a right-angle triangle.

$$\mathbf{a} = x\mathbf{i} + y\mathbf{j}$$



Magnitude or length of a vector

$$|\mathbf{a}| = \sqrt{x^2 + y^2}$$

Direction relative to the \mathbf{i} unit vector is given by the angle θ where

$$\tan \theta = \frac{y}{x}$$

$$\mathbf{a} = 3\mathbf{i} + \mathbf{j}$$

$$|\mathbf{a}| = \sqrt{x^2 + y^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$$

$$\tan \theta = \frac{1}{3} \Rightarrow \theta = 18.4^\circ$$

$$\mathbf{b} = 5\mathbf{i} - 2\mathbf{j}$$

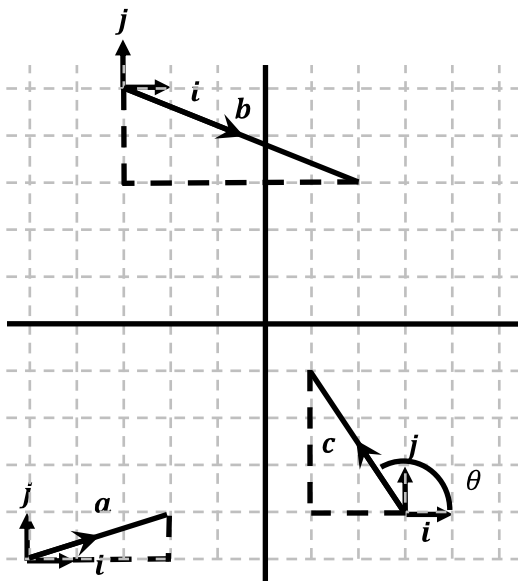
$$|\mathbf{b}| = \sqrt{x^2 + y^2} = \sqrt{5^2 + (-2)^2} = \sqrt{29}$$

$$\tan \theta = \frac{-2}{5} \Rightarrow \theta = -21.8^\circ$$

$$\mathbf{c} = -2\mathbf{i} + 3\mathbf{j}$$

$$|\mathbf{c}| = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$$

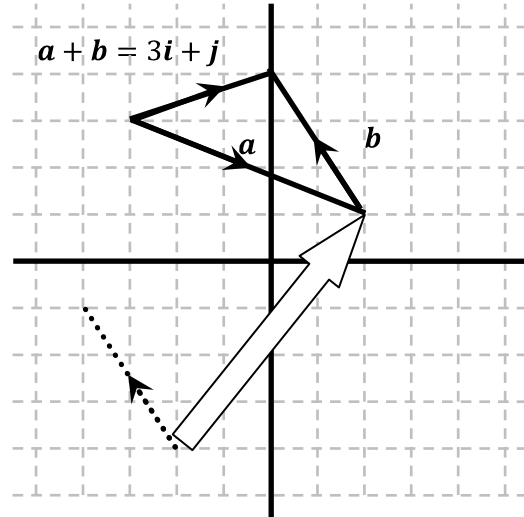
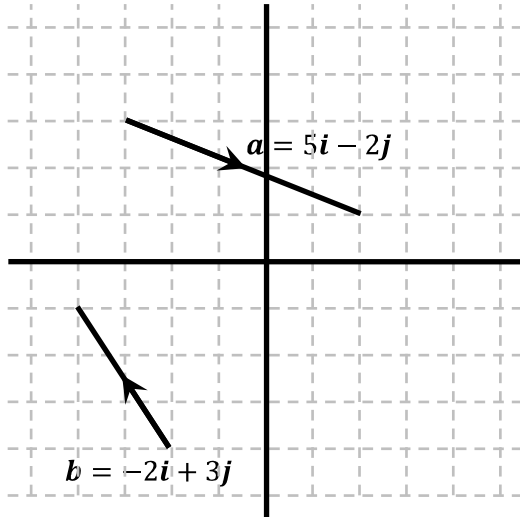
$$\tan \theta = \frac{3}{-2} \Rightarrow \theta = 123.7^\circ$$



While vectors in the form $\mathbf{a} = x\mathbf{i} + y\mathbf{j}$ can be manipulated in many ways, for the mechanics subjects treated in this text, only addition and scaling of vectors are important. The geometric interpretation of these two operations discussed above is embodied in the \mathbf{i} and \mathbf{j} description for vectors.

Given two vectors $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$ and $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j}$ then $\mathbf{a} + \mathbf{b} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j}$

The sum of two vectors is therefore performed by adding the components independently, a result entirely consistent with the geometry operations for summing two vectors.



$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} = 5\mathbf{i} - 2\mathbf{j},$$

$$\mathbf{b} = b_x \mathbf{i} + b_y \mathbf{j} = -2\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{a} + \mathbf{b} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} = (5 + (-2))\mathbf{i} + ((-2) + 3)\mathbf{j} = 3\mathbf{i} + \mathbf{j}$$

Multiplying a vector $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j}$ by a scalar k is also achieved by component wise multiplication:

$$k\mathbf{a} = k(a_x \mathbf{i} + a_y \mathbf{j}) = ka_x \mathbf{i} + ka_y \mathbf{j}$$

This definition is again consistent with the geometric interpretation of adjusting the length of the line representing the vector. The magnitude of the vector after multiplication by a scalar $k \geq 0$ is

$$|k\mathbf{a}| = |ka_x \mathbf{i} + ka_y \mathbf{j}| = \sqrt{(ka_x)^2 + (ka_y)^2} = \sqrt{k^2 (a_x^2 + a_y^2)} = k \sqrt{a_x^2 + a_y^2} = k|\mathbf{a}|$$

These algebraic steps simply show that adjusting the length of a vector $|\mathbf{a}|$ by a scalar is the same as multiplying the components of the vector by the scalar and then calculating the length for the scaled vector from these new components.

Two vectors $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j}$ and $\mathbf{b} = b_x \mathbf{i} + b_y \mathbf{j}$ are equal if and only if the components are all equal, that is

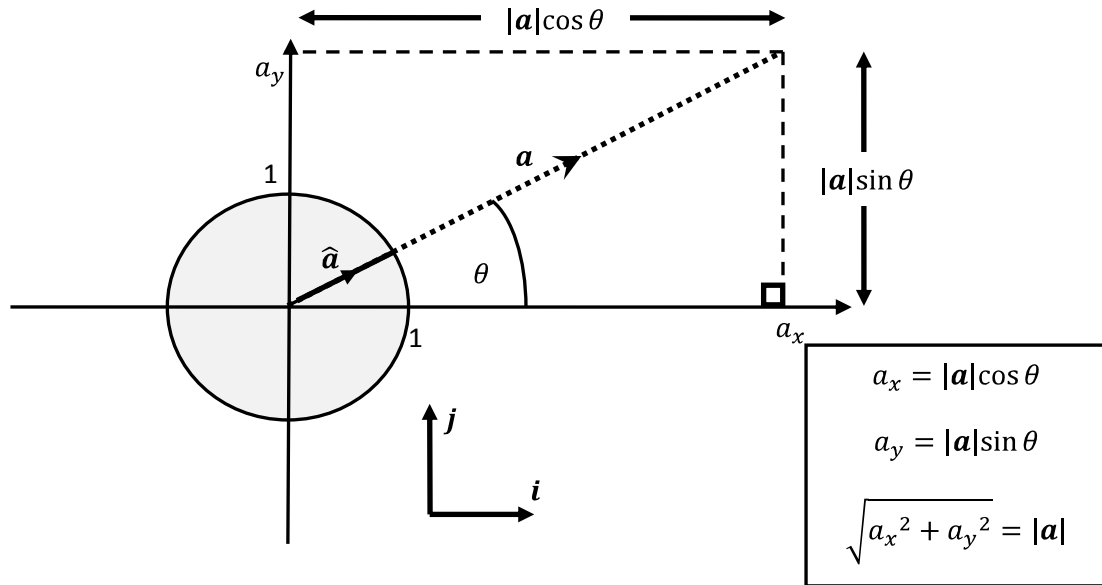
$$\mathbf{a} = \mathbf{b} \text{ if and only if } a_x = b_x \text{ and } a_y = b_y$$

As stated earlier, a unit vector is a vector with a length of unity or 1 unit in length. Since $|\mathbf{a}| =$

$\sqrt{a_x^2 + a_y^2}$ is the length of the vector \mathbf{a} , provided the length is greater than zero a unit vector can be calculated for all such vectors, namely, $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|} \mathbf{a}$. The importance of these unit vectors calculated from an arbitrary vector is that if the unit vector for any two or more vectors are equal, then the vectors are parallel and therefore represent the same direction.

The initial statement regarding the nature of vectors was that a vector quantity has magnitude and direction. The \mathbf{i} and \mathbf{j} notation for describing vectors fits entirely with this statement. When a vector is expressed using the angle between the direction of the vector and the direction of \mathbf{i} unit vector together with the magnitude of the vector $|\mathbf{a}|$:

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} = |\mathbf{a}|(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) = \text{"size"} \times \text{"direction"}$$



Examples of Vectors in Mechanics

Example: Three forces acting on a particle are given by the vectors $(-3\mathbf{i} + 4\mathbf{j})\text{N}$, $(21\mathbf{i} - 7\mathbf{j})\text{N}$ and $(-18\mathbf{i} + 3\mathbf{j})\text{N}$. Calculate the resultant force acting on the particle.

Solution:

Force has magnitude and direction, hence force is a vector quantity. The resultant force acting on a particle is the vector sum of all the forces in play.

Resultant force \mathbf{F}

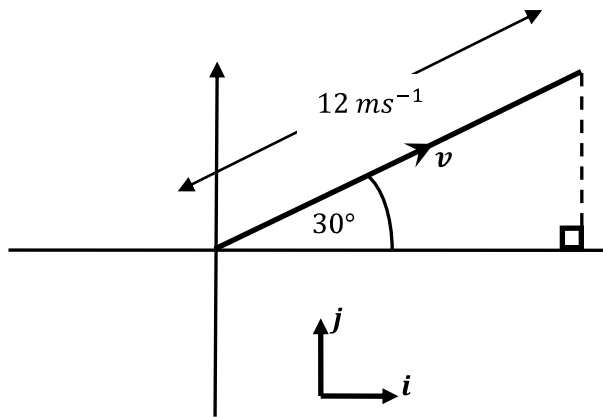
$$\mathbf{F} = (-3\mathbf{i} + 4\mathbf{j}) + (21\mathbf{i} - 7\mathbf{j}) + (-18\mathbf{i} + 3\mathbf{j}) = (-3 + 21 - 18)\mathbf{i} + (4 - 7 + 3)\mathbf{j}$$

Therefore $\mathbf{F} = 0\mathbf{i} + 0\mathbf{j}$ also known as the zero vector or the null vector. The vector sum for these forces shows the particle is in equilibrium.

Example: A ball is thrown at an angle of 30° to the horizontal with a speed of 12 ms^{-1} . Express the velocity for the ball using unit vectors \mathbf{i} and \mathbf{j} where the \mathbf{i} unit vector is parallel to the horizontal.

Solution:

Velocity is expressed in terms of a magnitude referred to a speed and a direction specified using the angle from the horizontal at which the ball is thrown. The velocity vector $\mathbf{v} = (v_x \mathbf{i} + v_y \mathbf{j}) \text{ ms}^{-1}$ is calculated from the right-angle triangle:



$$|v| = 12$$

$$v_x = 12 \cos 30^\circ$$

$$v_y = 12 \sin 30^\circ$$

Therefore,

$$v = \left(12 \frac{\sqrt{3}}{2} i + 12 \frac{1}{2} j \right) \text{ms}^{-1}$$

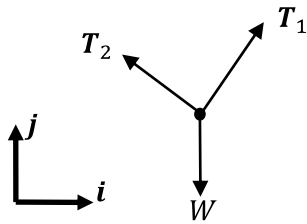
$$v = (6\sqrt{3}i + 6j) \text{ms}^{-1}$$

Example: A small bird feeder is held in equilibrium by the tension exerted by two light inextensible strings. The strings exert tension of $T_1 \text{ N}$ and $T_2 \text{ N}$ on the bird feeder. Given $T_1 = (5i + 6j) \text{ N}$, calculate the magnitude for T_2 and the angle the direction of T_2 makes with the vertical.

Solution:

The magnitude of T_1 is given by

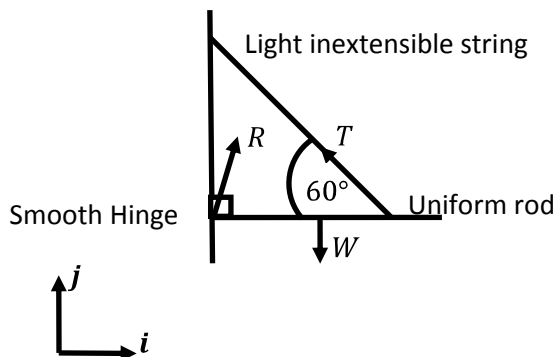
$$|T_1| = \sqrt{5^2 + 6^2} = \sqrt{25 + 36} = \sqrt{61} = 7.8 \text{ N}$$



The angle between T_1 and the direction of i is

$$\theta = \tan^{-1} \left(\frac{6}{5} \right) = 50.2^\circ$$

The tension T_1 acts in a direction which makes the angle 50.2° to the horizontal, therefore the angle with the vertical is $90 - 50.2 = 39.8^\circ$.



Example: A uniform rod is hinged to a vertical wall and supported in a horizontal position by a light inextensible string. The magnitudes for two of the forces acting on the rod are $T = 12 \text{ N}$, $W = 30 \text{ N}$ while the reaction force at the hinge is $R = (6i + 6(5 - \sqrt{3})j) \text{ N}$.

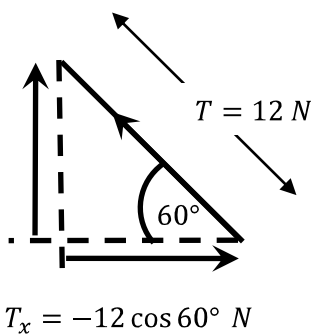
Express all the forces acting on the rod as vectors using the unit vectors i and j . Calculate the resultant force acting on the rod.

Solution:

The reaction force on the hinge is expressed in the required vector notation. The problem is to specify the weight as a vector \mathbf{W} and the tension vector \mathbf{T} acting on the rod by the string. The unit vectors \mathbf{i} and \mathbf{j} are specified by the diagram with the \mathbf{i} unit vector parallel to the rod.

Since the weight acts vertically downwards and is of magnitude 30 N , the weight vector is $\mathbf{W} = 0\mathbf{i} - 30\mathbf{j}$.

The tension force is of magnitude 12 N acting at an angle of 60° to the horizontal in the opposite direction to the \mathbf{i} unit vector.



$$T_y = 12 \sin 60^\circ \text{ N}$$

$$T_x = -12 \cos 60^\circ \text{ N}$$

$$\mathbf{T} = T_x \mathbf{i} + T_y \mathbf{j}$$

$$\Rightarrow \mathbf{T} = -12 \cos 60^\circ \mathbf{i} + 12 \sin 60^\circ \mathbf{j} \text{ N}$$

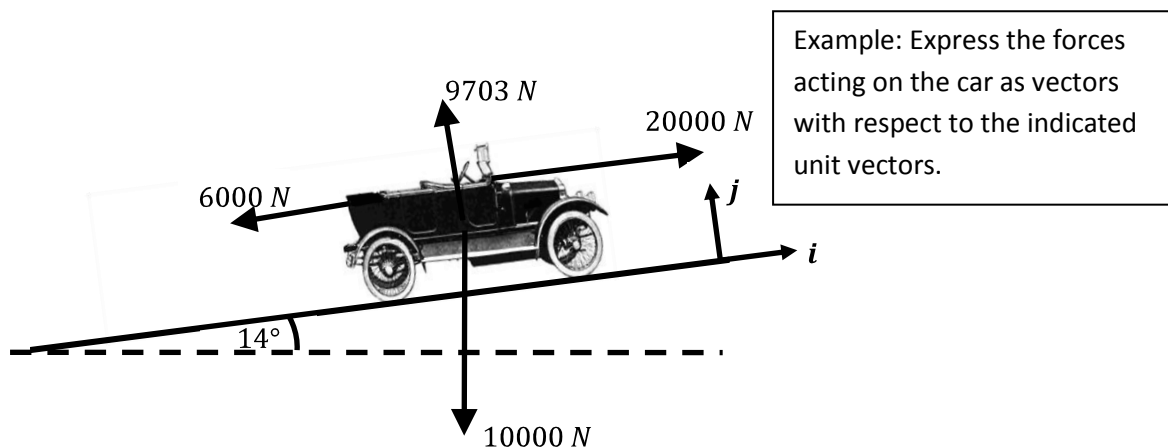
$$\Rightarrow \mathbf{T} = -6 \mathbf{i} + 6\sqrt{3} \mathbf{j} \text{ N}$$

The resultant force \mathbf{F} acting on the rod is the vector sum of all the forces acting on the rod, namely,

$$\mathbf{F} = \mathbf{R} + \mathbf{W} + \mathbf{T}$$

Writing these vectors as column vectors,

$$\mathbf{F} = \begin{pmatrix} 6 \\ 6(5 - \sqrt{3}) \end{pmatrix} + \begin{pmatrix} 0 \\ -30 \end{pmatrix} + \begin{pmatrix} -6 \\ 6\sqrt{3} \end{pmatrix} = \begin{pmatrix} 6 + 0 - 6 \\ 30 - 6\sqrt{3} - 30 + 6\sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Solution:

The driving force \mathbf{D} for the car has magnitude 20000 N . The direction for the driving force is parallel to the road and since the unit vectors are chosen to be parallel and perpendicular to the road, the vector representing the driving force is

$$\mathbf{D} = (20000 \mathbf{i} + 0 \mathbf{j}) \text{ N}$$

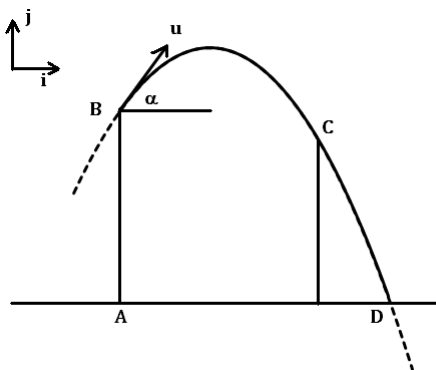
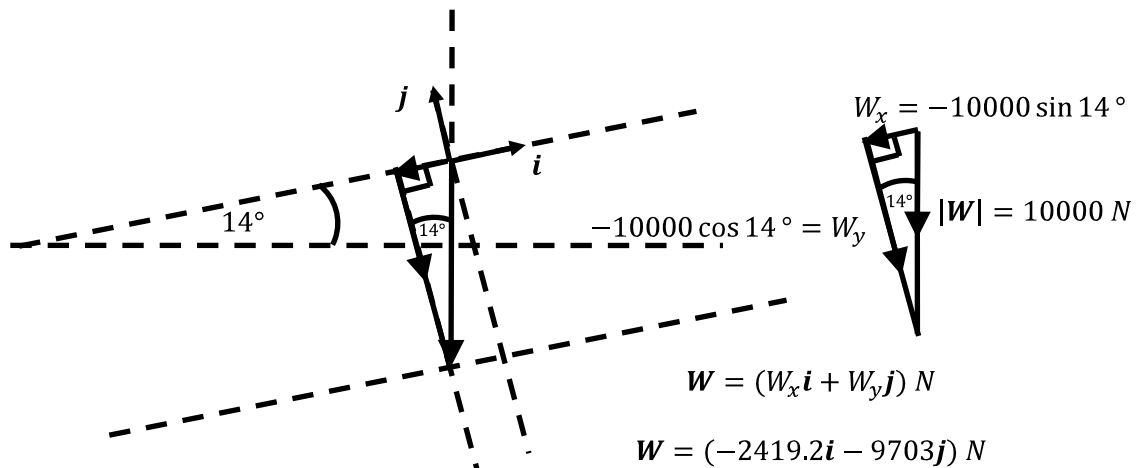
Similarly, the resistance force \mathbf{R} to the motion of the car has magnitude 6000 N acting parallel to the road but in the opposite direction to the motion of the car and also the \mathbf{i} unit vector, therefore the vector representation for the resistance force is

$$\mathbf{R} = (-6000 \mathbf{i} + 0 \mathbf{j}) \text{ N}$$

The normal reaction force of the road on the car \mathbf{N} is in the direction of the \mathbf{j} unit vector and has magnitude 9703 N , therefore the normal reaction force is expressed as follows

$$\mathbf{N} = (0 \mathbf{i} + 9703 \mathbf{j}) \text{ N}$$

Since the weight of the car acts vertically downwards with magnitude 10000 N , the weight vector \mathbf{W} acts both perpendicular and parallel to the direction of the road. The components for the weight vector are calculated using right-angled triangles where the hypotenuse is of length equal to the magnitude of the weight and the angle is specified by the angle of the road.



Example: A particle is projected from a point B with speed 29.4 ms^{-1} in a direction 60° to the horizontal. The velocity of the particle after a time t seconds is given by

$$\mathbf{v}(t) = \begin{pmatrix} 14.7 \\ 14.7\sqrt{3} \end{pmatrix} + \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} t$$

- Calculate the speed of the particle after 0.5 seconds.
- Calculate the speed of the particle at the point when the velocity makes an angle of 30° to the horizontal.

Solution:

Part a) The velocity vector for the particle is given as a function of time t . To calculate the speed of the particle after 0.5 seconds, the magnitude for the velocity vector is calculated by substituting $t = 0.5$ into the equation:

$$\mathbf{v}(t) = \begin{pmatrix} 14.7 \\ 14.7\sqrt{3} \end{pmatrix} + \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} t$$

$$\mathbf{v}(0.5) = \begin{pmatrix} 14.7 \\ 14.7\sqrt{3} \end{pmatrix} + \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} (0.5) = \begin{pmatrix} 14.7 \\ 14.7\sqrt{3} - 9.8(0.5) \end{pmatrix} = \begin{pmatrix} 14.7 \\ 20.6 \end{pmatrix}$$

The speed of the particle is therefore $|\mathbf{v}| = \sqrt{14.7^2 + 20.6^2} = 25.3 \text{ ms}^{-1}$

Part b): The angle of the velocity vector to the horizontal is specified as 30° . Given the angle between a vector and the positive direction of the \mathbf{i} unit vector, a unit vector for the velocity vector can be written down:

$$\hat{\mathbf{v}} = \cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j} = \frac{\sqrt{3}}{2} \mathbf{i} + \frac{1}{2} \mathbf{j} \quad \dots \quad (1)$$

From the above equation for velocity in terms of time, the same unit vector is given by

$$\hat{\mathbf{v}}(t) = \frac{1}{|\mathbf{v}(t)|} (14.7 \mathbf{i} + (14.7\sqrt{3} - 9.8t) \mathbf{j}) \quad \dots \quad (2)$$

Taking the ratio of the components from the expressions (1) and (2),

$$\sqrt{3} = \frac{14.7}{(14.7\sqrt{3} - 9.8t)}$$

$$\Rightarrow (14.7\sqrt{3} - 9.8t)\sqrt{3} = 14.7 \Rightarrow t = \frac{14.7(3) - 14.7}{9.8\sqrt{3}} = \frac{29.4}{9.8\sqrt{3}} = \sqrt{3} \text{ s}$$

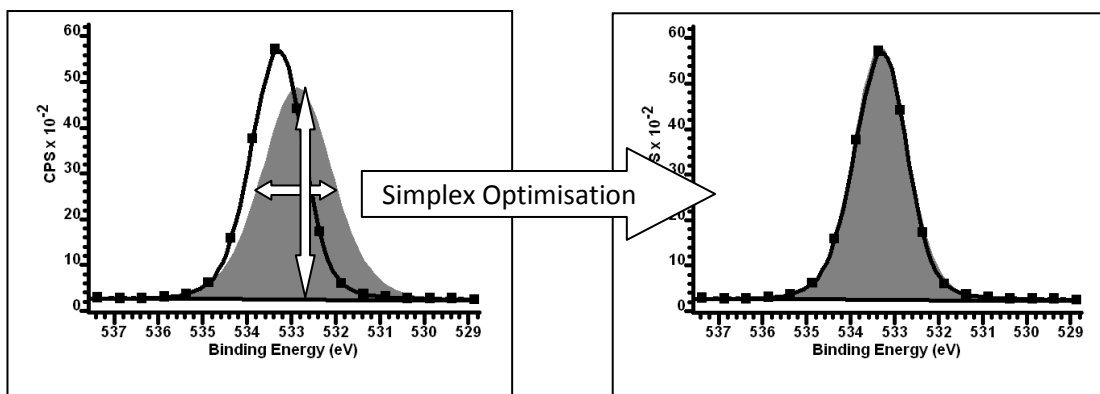
The velocity vector with direction 30° to the horizontal is therefore

$$\mathbf{v}(\sqrt{3}) = 14.7\mathbf{i} + (14.7\sqrt{3} - 9.8\sqrt{3})\mathbf{j} = 14.7\mathbf{i} + 4.9\sqrt{3}\mathbf{j}$$

The speed of the particle at this point in the trajectory is

$$|\mathbf{v}(\sqrt{3})| = \sqrt{14.7^2 + (4.9\sqrt{3})^2} = 17.0 \text{ ms}^{-1} \text{ (3 sf)}$$

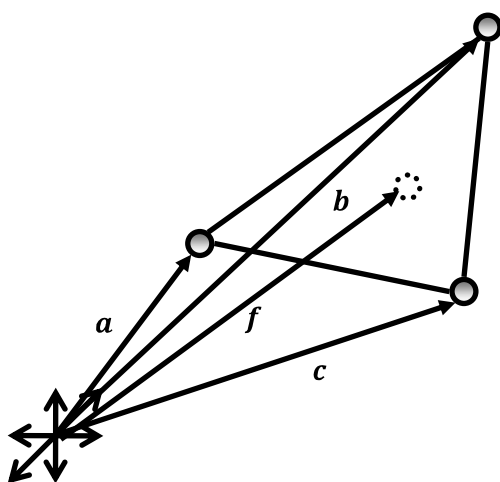
Application of Vectors to Surface Analysis



A peak defined in terms of peak-maximum position P , the full width at half maximum F and peak height H using an approximation of the form $G(x) = He^{-4\ln 2 \left(\frac{(x-P)}{F}\right)^2}$ is fitted to a set of experimental intensities by adjusting the three parameters P , F and H to minimise a function of these three variables of the form

$$R(P, F, H) = \sum_{i=1}^n \left(d_i - He^{-4\ln 2 \left(\frac{(x_i-P)}{F}\right)^2} \right)^2$$

For non-linear optimisation using the simplex algorithm, these three parameters P , F and H are considered as coordinate values in a 3D vector space. A series of tests for a minimum are performed as part of the simplex algorithm via transforming a set of initial coordinates defining the vertices of a tetrahedron (the 3D case for a simplex) to new sets of coordinates with similar tetrahedral geometry.



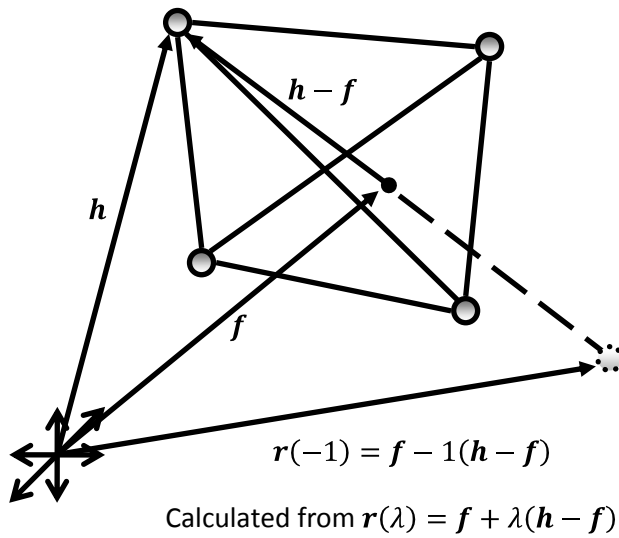
One step in the simplex method involves calculating a position vector for a point in the opposite face to the vertex with greatest function value. The opposite face in the 3D case is a triangle.

Vector equation of plane containing position vectors \mathbf{a} , \mathbf{b} and \mathbf{c} is:

$$\mathbf{r}(\alpha, \beta) = \mathbf{a} + \alpha(\mathbf{b} - \mathbf{a}) + \beta(\mathbf{c} - \mathbf{a})$$

$$\mathbf{f} = \mathbf{r}\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$$

therefore \mathbf{f} lies in the plane too.



Optimisation by the simplex method involves transforming a simplex to a new simplex by moving the coordinates for the vertex with greatest function value (position vector \mathbf{h}) to a point along the line passing through the opposite face of the simplex (position vector \mathbf{f}) and the vertex with greatest function value. The new vertex is calculated from the vector equation of a line defined by \mathbf{h} and \mathbf{f} .

The vector equation of a line passing through the position vector \mathbf{a} and in the direction of the vector \mathbf{d} is given in terms of the scalar λ by $\mathbf{r}(\lambda) = \mathbf{a} + \lambda\mathbf{d}$. The units of λ are determined by the magnitude of \mathbf{d} .