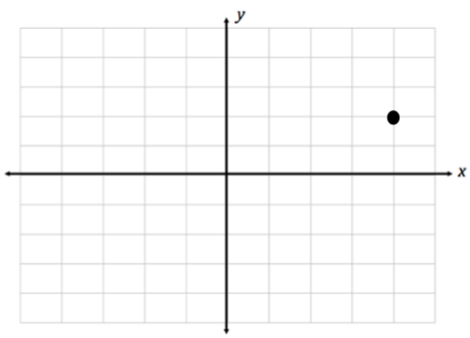
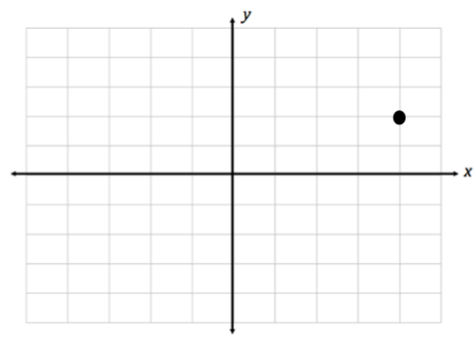
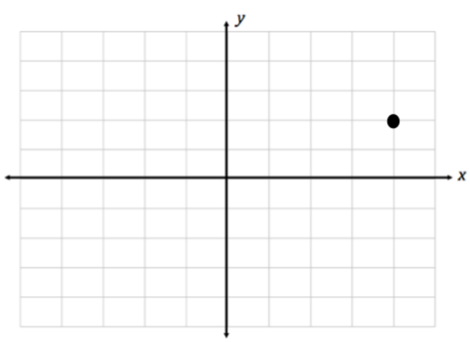
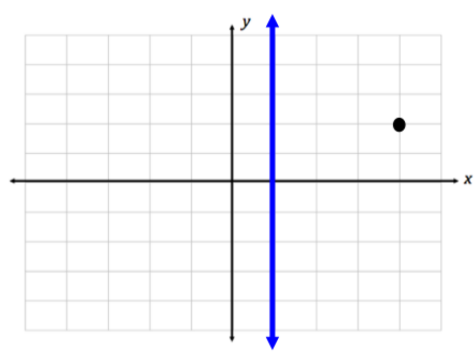
Unit 2
Transformations / Rigid Motions

Lesson 1: Reflections on the Coordinate Plane

Opening Exercise

What do you remember about reflections???

Take the point (4, 2) and reflect it as stated. Plot the new point and state its coordinates.

Reflection in the *x*-axis  Coordinates of the New Point: ___________	**Reflection in the *y*-axis**  Coordinates of the New Point: ___________
Reflection in the line *y* = *x*  Coordinates of the New Point: ___________	**Reflection in the line *x* = 1**  Coordinates of the New Point: ___________

Summary of the Rules:

$r_{x-axis}: \ (x, y) \rightarrow$

$r_{y-axis}: \ (x, y) \rightarrow$

$r_{y=x}: \ (x, y) \rightarrow$

Vocabulary

A **transformation** is a change in the position, shape, or size of a figure.

A **rigid motion** is a transformation that changes only the position of the figure (length and angle measures are preserved).

An **image** is the result of a transformation of a figure (called the pre-image). To identify the image of a point, use prime notation. The image of point A is A' (read as A prime).

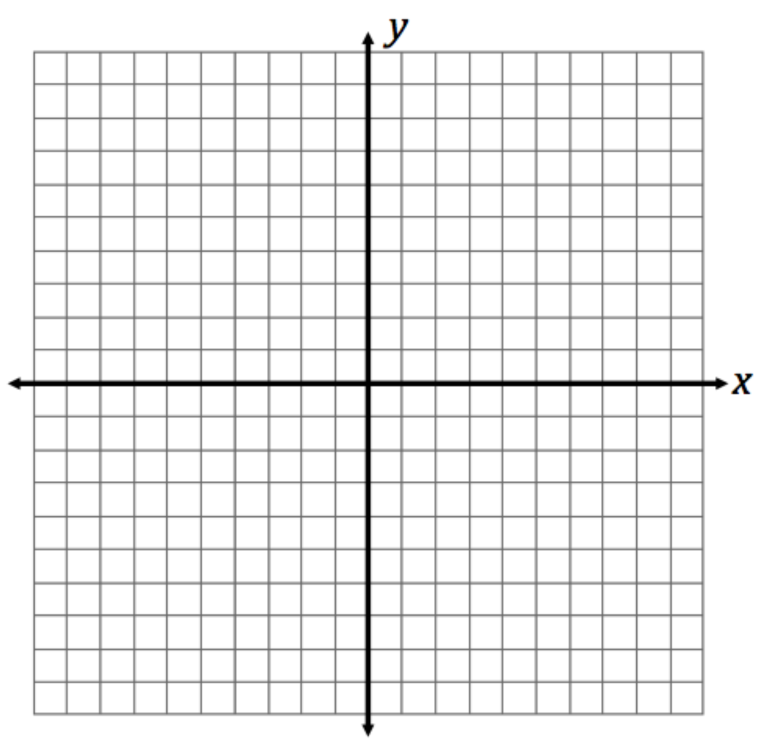
Example 1

Given ΔABC with vertices A (-5, 1), B (-1, 1) and C (-1, 7).

a. Graph ΔABC on the axes provided below.

b. On the same set of axes, graph $\Delta A'B'C'$, the image of ΔABC reflected over the x-axis.

c. On the same set of axes, graph $\Delta A''B''C''$, the image of ΔABC reflected over the y-axis.



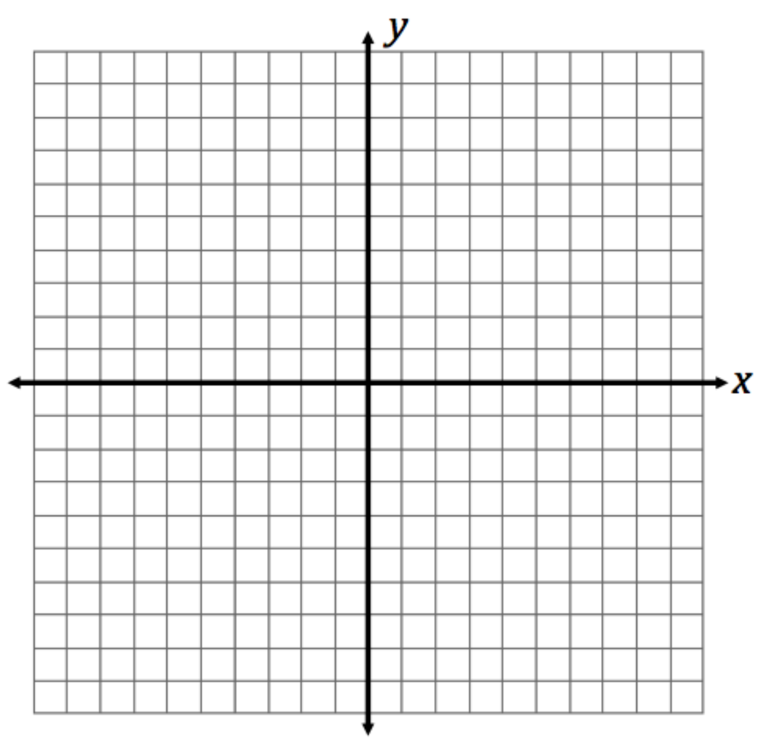
Example 2

Given ΔABC with vertices A (2, 3), B (0, 6) and C (2, 6).

a. Graph ΔABC on the axes provided below.

b. Graph and state the coordinates of $\Delta A'B'C'$, the image of ΔABC reflected over the line $y = x$.

c. Graph and state the coordinates of $\Delta A''B''C''$, the image of $\Delta A'B'C'$ reflected over the line $y = -2$.



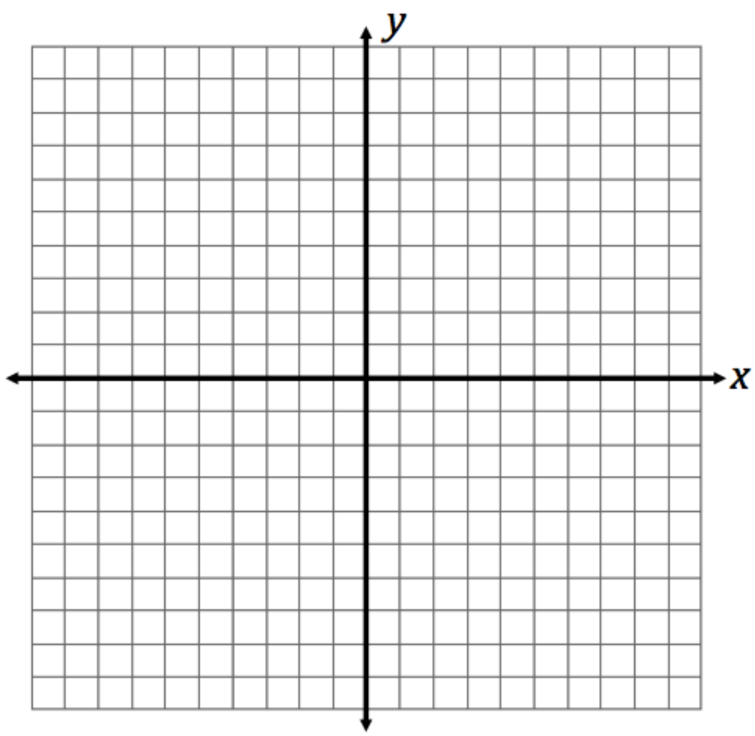
Example 3

ΔDOG has vertices *D* (-1, 1), *O* (-2, 5) and *G* (-5, 2) and $\Delta D'O'G'$ has vertices *D'* (3, -3), *O'* (7, -4) and *G'* (4, -7).

a. Graph and label ΔDOG and $\Delta D'O'G'$

b. Graph the line $y = x - 2$

c. What is the relationship between the line of reflection and the segments connecting the corresponding points? (Think back to Unit 1)



Homework

1. Using the point (-5, 3), find its image after the following reflections (the use of the grid is optional).

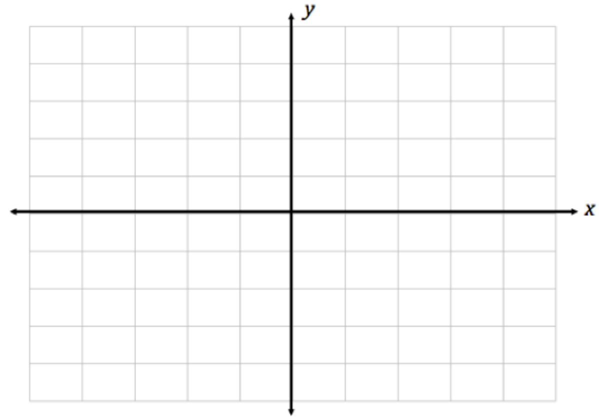
   a. r_{x-axis}

   b. r_{y-axis}

   c. $r_{y=x}$

   d. $r_{x=-1}$

   e. $r_{y=2}$

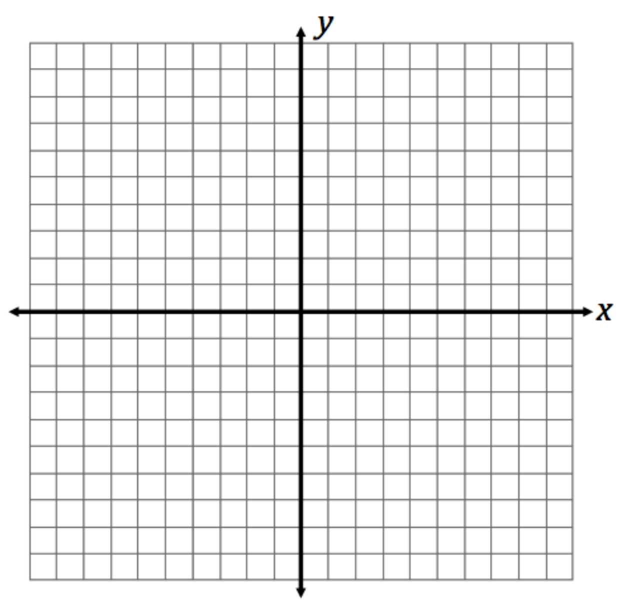


2. Given ΔABC with vertices *A* (2, 1), *B* (3, 4) and *C* (-4, 5).

   a. Graph ΔABC on the axes provided.

   b. Graph and state the coordinates of $\Delta A'B'C'$, the image of ΔABC reflected over the *x*-axis.

   c. Graph and state the coordinates of $\Delta A''B''C''$, the image of $\Delta A'B'C'$ reflected over the line $y = x$.

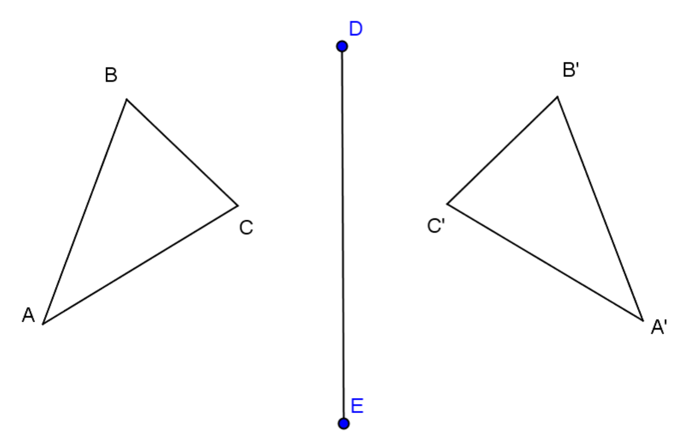


Lesson 2: Reflections off the Coordinate Plane

Opening Exercise

You will need a compass and a straightedge

As shown in the diagram to the right, ΔABC is reflected across DE and maps on to $\Delta A'B'C'$.



a. Use your straightedge to draw in segments AA', BB' and CC'.

b. Use your compass to measure the distances from the pre-image point to DE and from the image point to DE. What do you notice about these distances?

c. What is the relationship between segment DE and each of the segments that were drawn in *part a*?

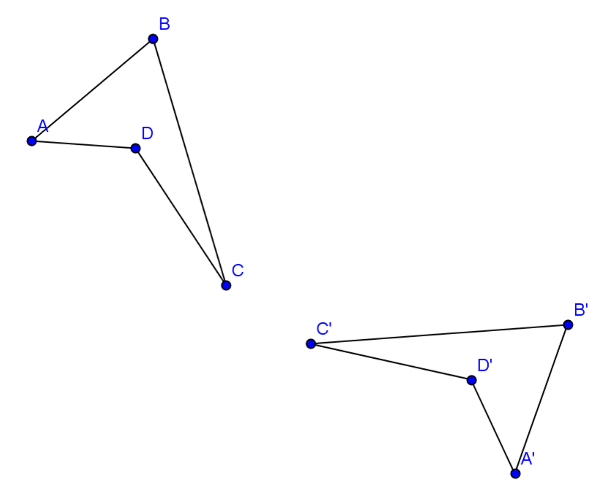
Example 1

You will need a compass and a straightedge

We now know that the line of reflection is the *perpendicular bisector* of the segments connecting the pre-image to the image point. We are going to use this, along with our knowledge of constructions, to construct the line of reflection.

a.	Connect any point to its image point.

b.	Draw the perpendicular bisector of this segment.



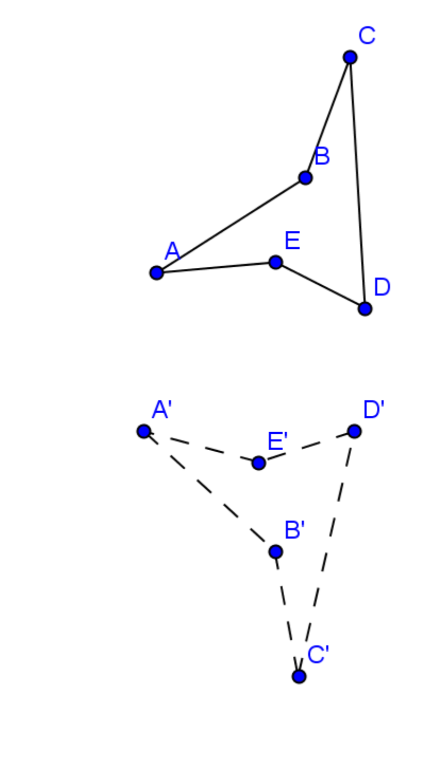
This is the line of reflection! Each point and its image point are equidistant from this line!!! Selecting a second pair of points and constructing its perpendicular bisector can verify this.

Exercises

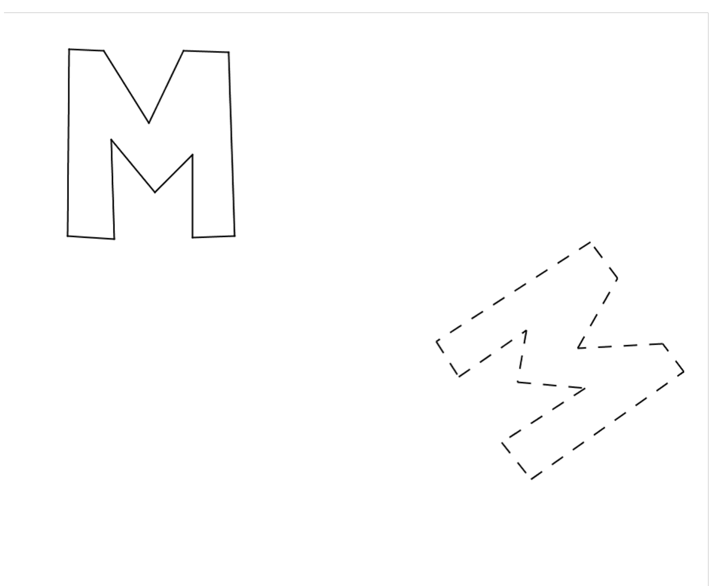
You will need a compass and a straightedge

Construct the line of reflection for each image and its pre-image.

1.



2.

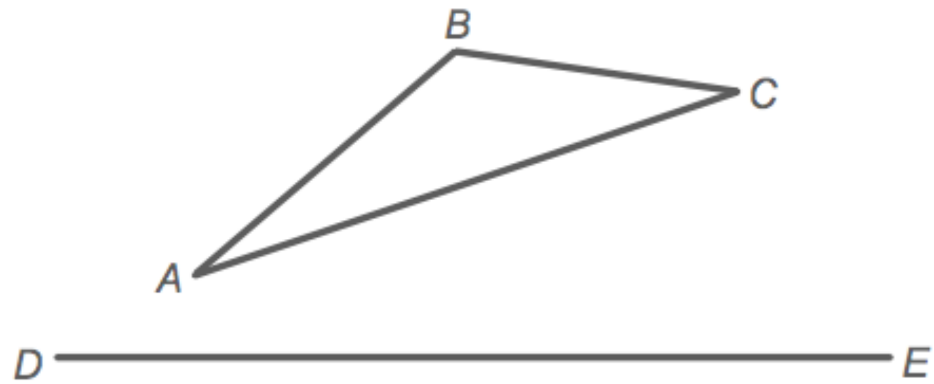


Example 2

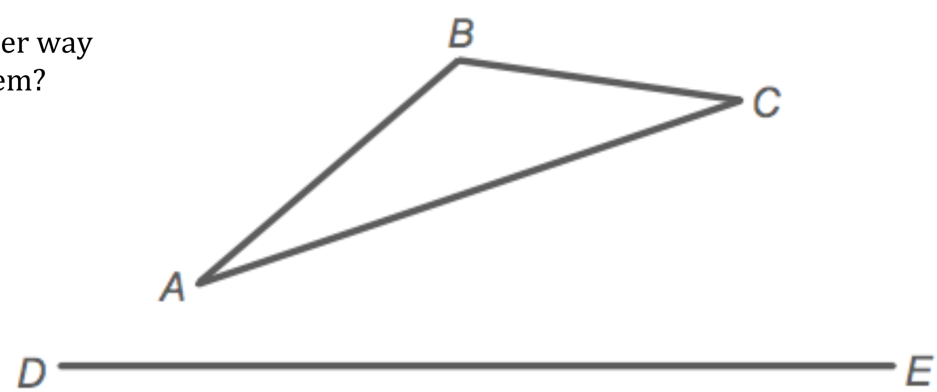
You will need a compass and a straightedge

Using our knowledge of perpendicular bisectors we are going to reflect an object over a given line.

Reflect ΔABC over DE.



Can we think of another way to do this same problem?

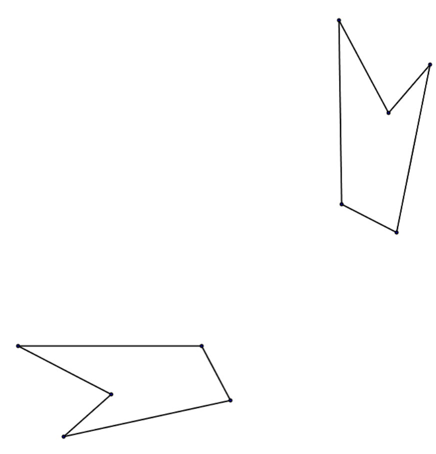


Homework

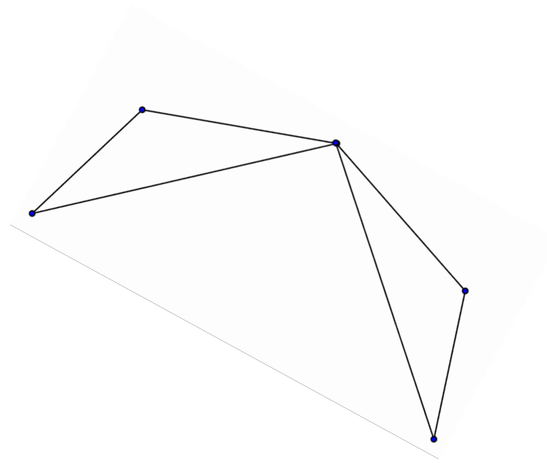
You will need a compass and a straightedge

In 1-2, construct the line of reflection.

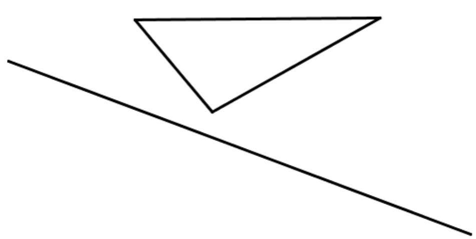
1.



2.



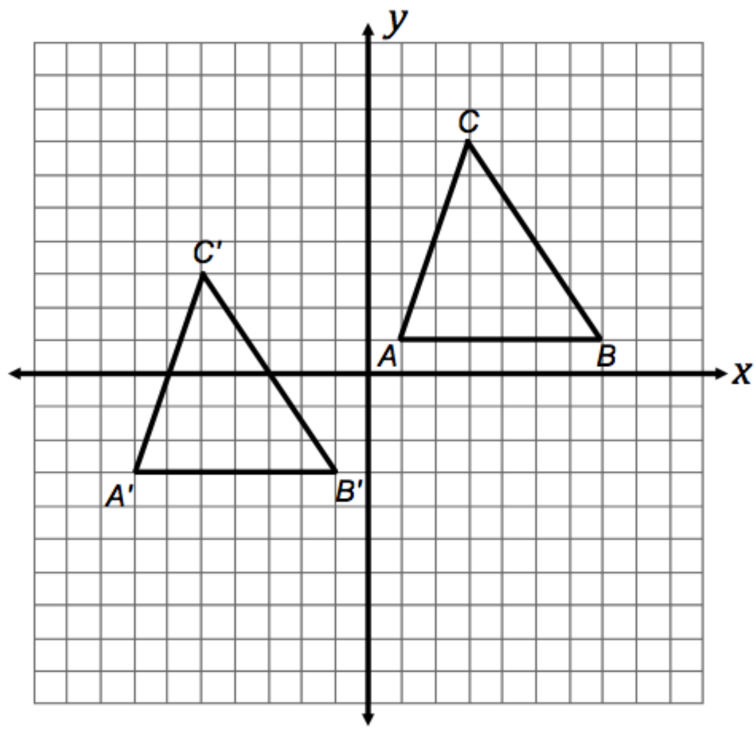
3. Reflect the given figure across the line of reflection provided.



Lesson 3: Translations on the Coordinate Plane

Opening Exercise

Another type of rigid motion is called a *translation*, or slide, in which every point of a figure is moved the same distance in the same direction.



Describe how to translate ΔABC to its image $\Delta A'B'C'$.

We can also write this translation using two different types of notation:

Is the size of the object preserved under a translation?

Is the order of the vertices the same?

Vocabulary

An **isometry** is a transformation that does not change in size. These include all of the rigid motions: reflections, translation and rotations.

A **direct isometry** preserves size and the order (**orientation**) of the vertices.

An **opposite isometry** preserves the size, but the order of the vertices changes.

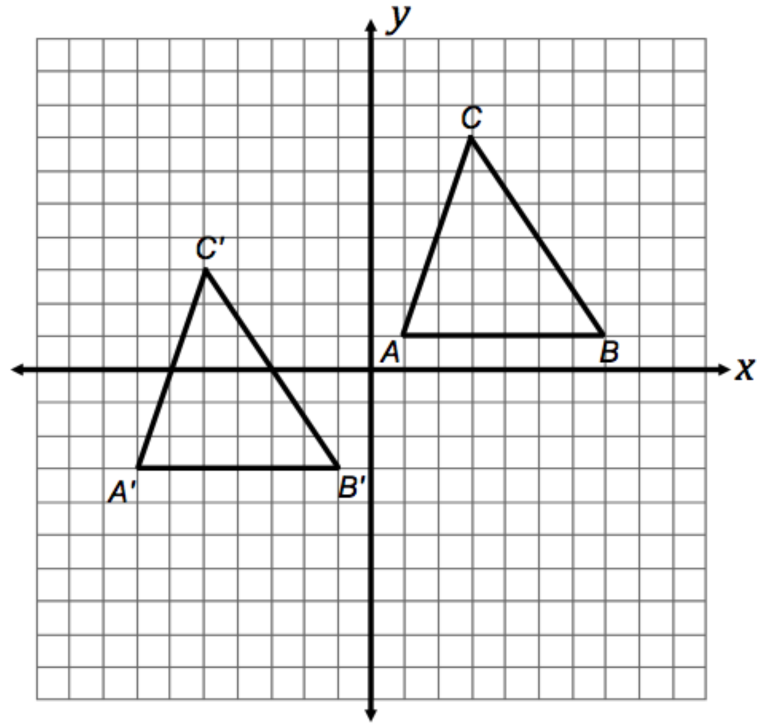
A *translation* would be an example of which isometry?

A *reflection* would be an example of which isometry?

Example 1

Using the same translation as the Opening Exercise, we are going to explore the path the image follows.

Using rays, connect the pre-image points with the image points. What do you notice about the rays you have drawn?

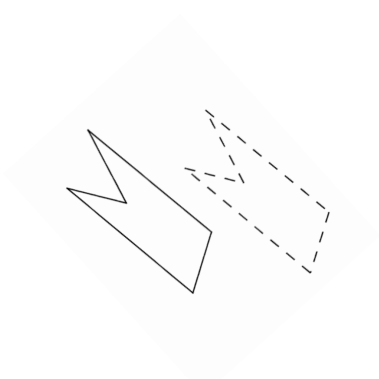


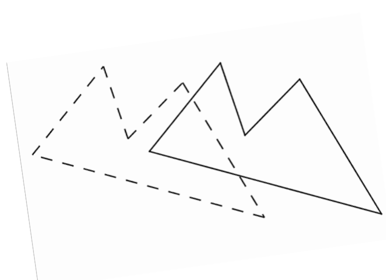
This ray is called a **vector**. A vector is a directed line segment that has both length and direction.

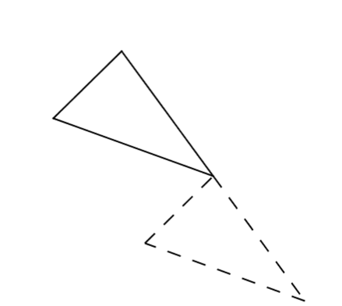
Exercises

You will need a straightedge

Draw the vector that defines each translation below.







Translation Rule: $T_{a,b}(x, y) \rightarrow$

Exercises

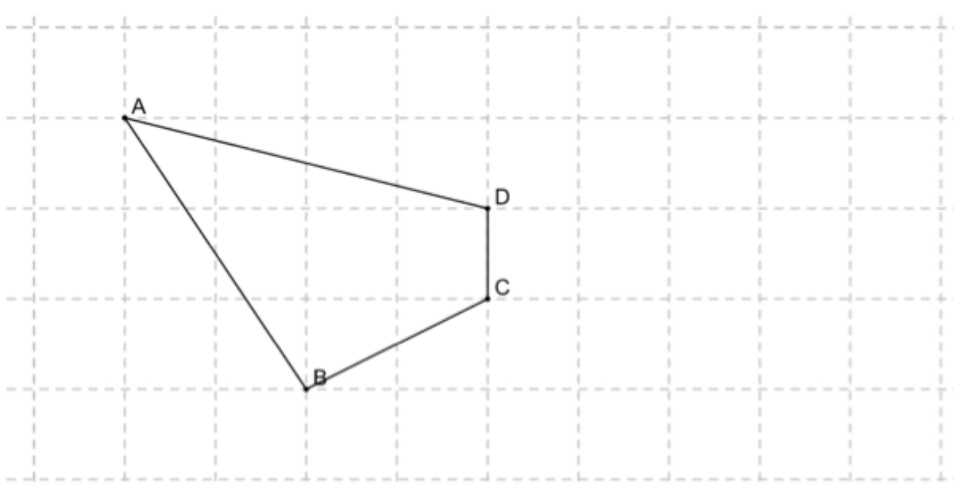
1. Determine the coordinates of the image of the point (5, -3) under $T_{-2,-1}$.

2. Determine the coordinates of the image of the point (-8, -3) under the translation $(x, y) \rightarrow (x+4, y-1)$

3. Determine the translation that maps the point (-5, 5) to the point (7, 1).

4. A translation maps the point (-2, 5) to the point (-4, -4). What is the image of (1,4) under the same translation?

5. Translate the image one unit down and three units right. Draw the vector that defines the translation.



Homework

1. Determine the coordinates of the image of the point (-2, 2) under $T_{-2,6}$.

2. Determine the coordinates of the image of the point (2, -3) under the translation $(x, y) \to (x-4, y+2)$

3. If translation $T_{x,y}$ maps point P (-3, 1) on to point P' (5, 5), find x and y.

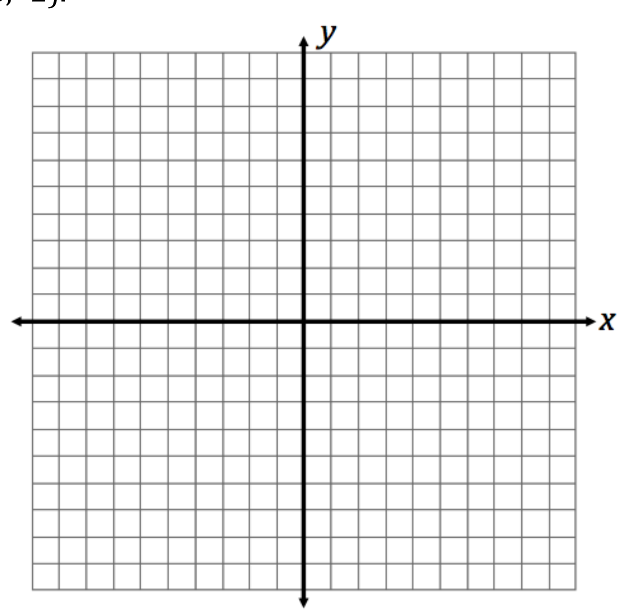
4. A translation maps the point (3, 1) to the point (-4, 2). What is the image of (4,-1) under the same translation?

5. ΔABC has vertices A (1, 1), B (2, 3) and C (6, -2).

a. Graph ΔABC

b. Graph $\Delta A'B'C'$, the image of ΔABC after the translation $(x, y) \to (x-2, y-6)$

c. Draw the vector that defines the translation.

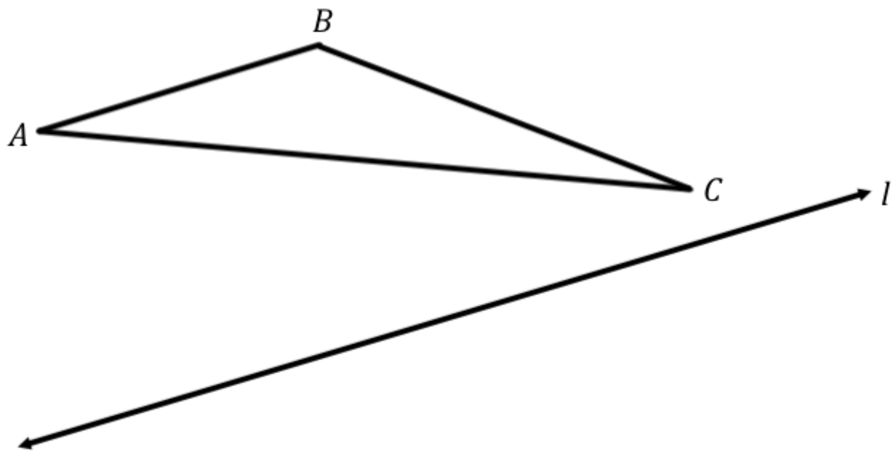


Lesson 4: Translations off the Coordinate Plane

Opening Exercise

You will need a compass and a straightedge

Construct $\Delta A'B'C'$, the image of ΔABC after a reflection over line *l*.



Example 1

You will need a compass and a straightedge

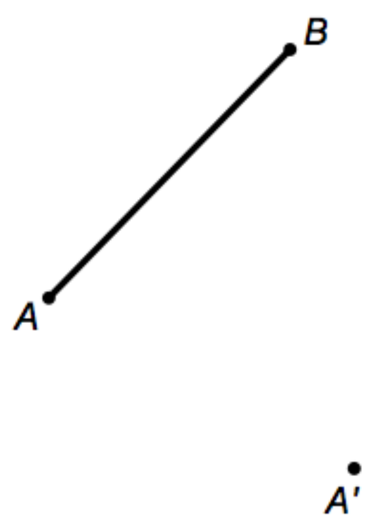
As we learned in our last lesson, a translated object follows the path of a vector. If we were to connect each pre-image point with its image point, we would have congruent and parallel segments.

In the diagram below, segment *AB* is translated to produce *A'B'.*

a. Draw the vector that defines this translation.

b. Using your compass, locate *B*'

c. Construct segment *A'B'*



Example 2

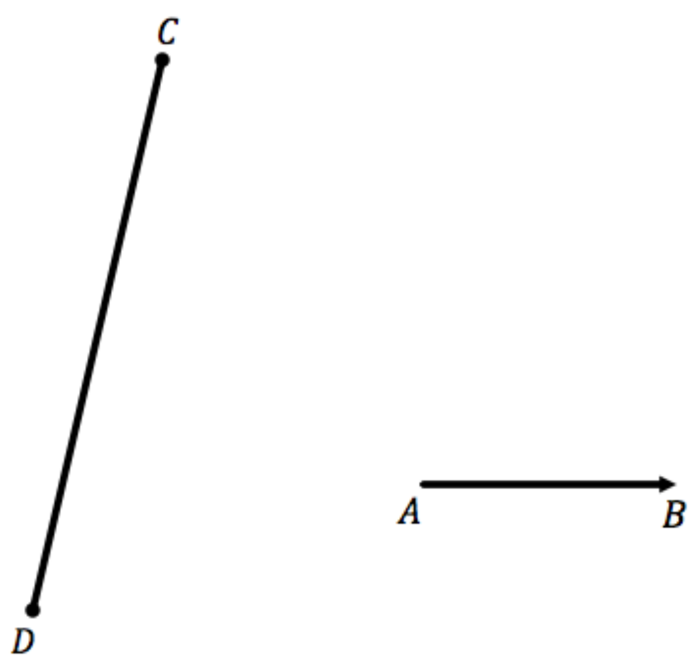
You will need a compass and a straightedge

Apply $T_{\overrightarrow{AB}}$ to segment $\overline{CD}$.

Vectors tell us 2 things: distance & direction

Step 1: Length
Using your compass, measure how far $\overline{CD}$ will travel based on vector $\overrightarrow{AB}$.

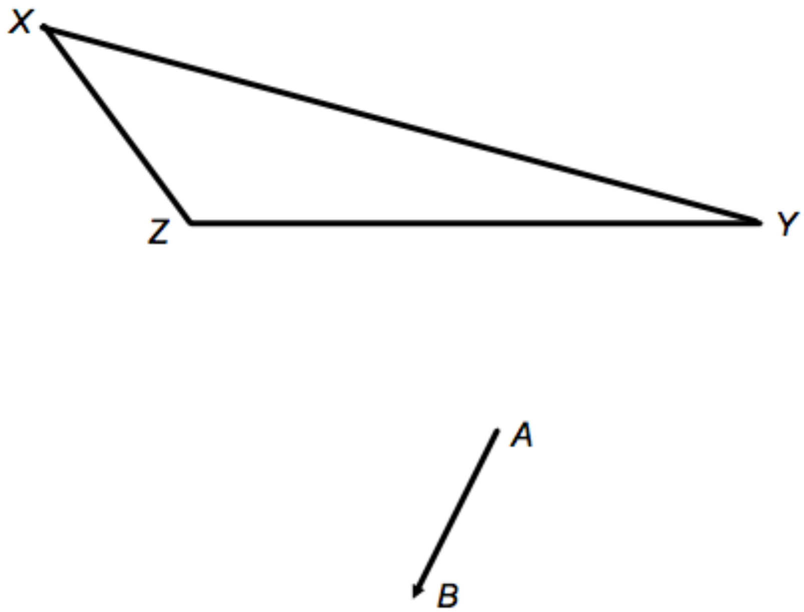
Step 2: Direction
How will you slide point C to its new point?



Example 3

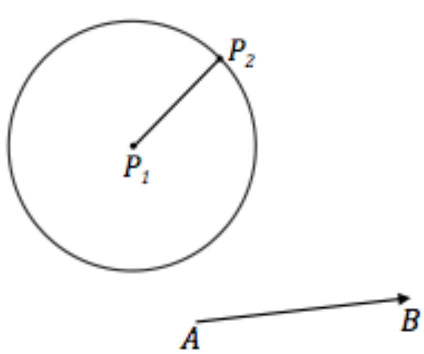
You will need a compass and a straightedge

Apply $T_{\overrightarrow{AB}}$ to ΔXYZ .



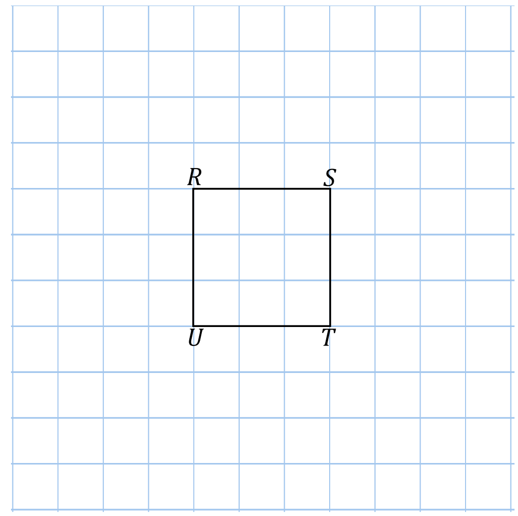
Example 4

Use your compass and straightedge to apply $T_{\overrightarrow{AB}}$ to the circle below:



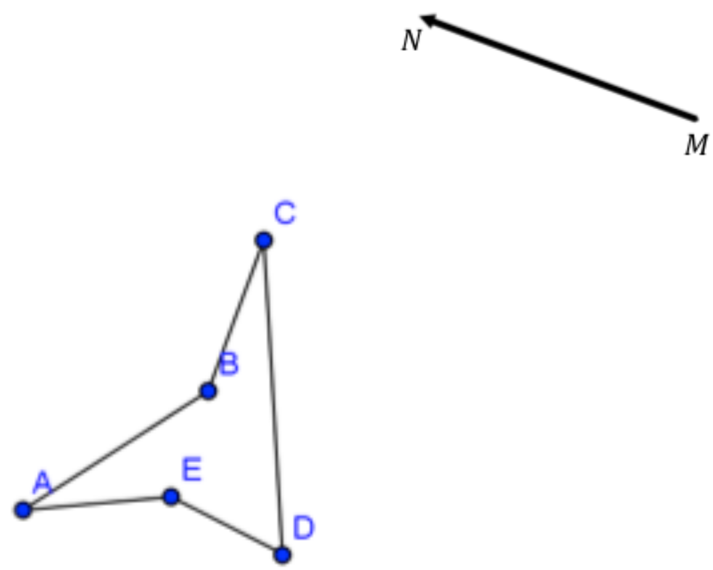
Homework

1. *a.* Translate the figure 2 units down and 3 units left



b. Draw the vector that defines the translation

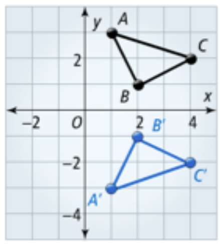
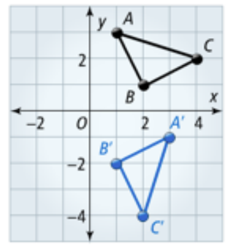
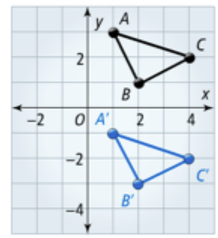
2. Use your compass apply $T_{\overline{MN}}$ to pentagon *ABCDE*.



Lesson 5: Rotations on the Coordinate Plane

Opening Exercise

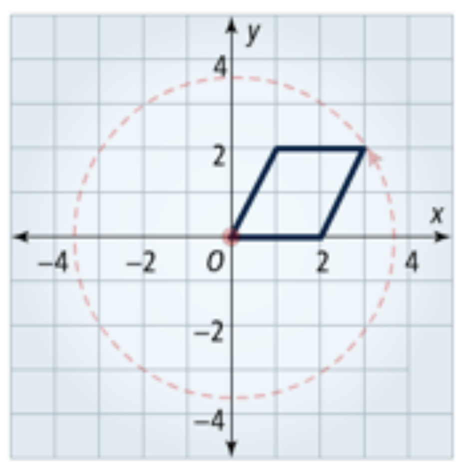
Fill in the table below to identify the characteristics and types of rigid motions being applied to create $\Delta A'B'C'$, the image of ΔABC.

			
Type of Rigid Motion:			
Is size preserved?			
Is orientation preserved?			
Which type of isometry?			

Vocabulary

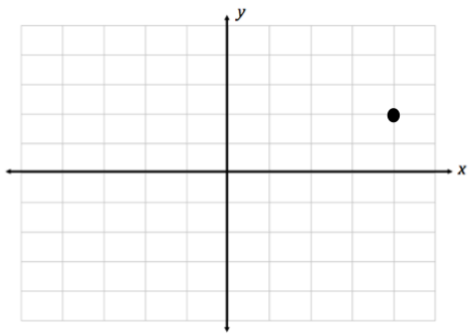
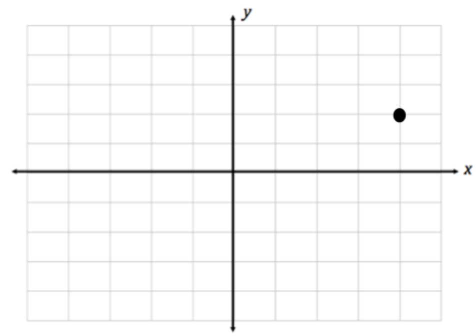
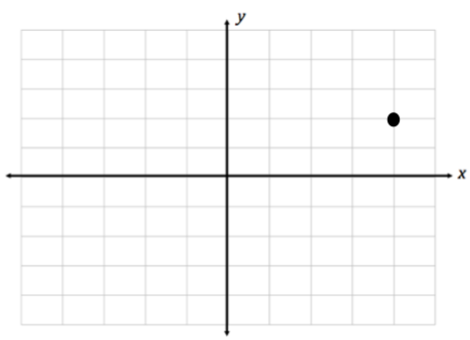
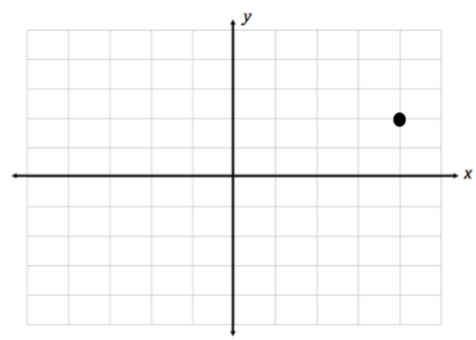
A **rotation** is a rigid motion that turns a figure about a fixed point called the **center of rotation.**

The **angle of rotation** is the number of degrees the figure rotates. A positive angle of rotation turns the figure counterclockwise (a negative angle of rotation can be used for clockwise rotations).



Example 1

Take the point (4, 2) and rotate it as stated. Plot the new point and state its coordinates.

Rotation of 90°  Coordinates of the New Point: ____________	**Rotation of 180°**  Coordinates of the New Point: ____________
Rotation of 270°  Coordinates of the New Point: ____________	**Rotation of 360°**  Coordinates of the New Point: ____________

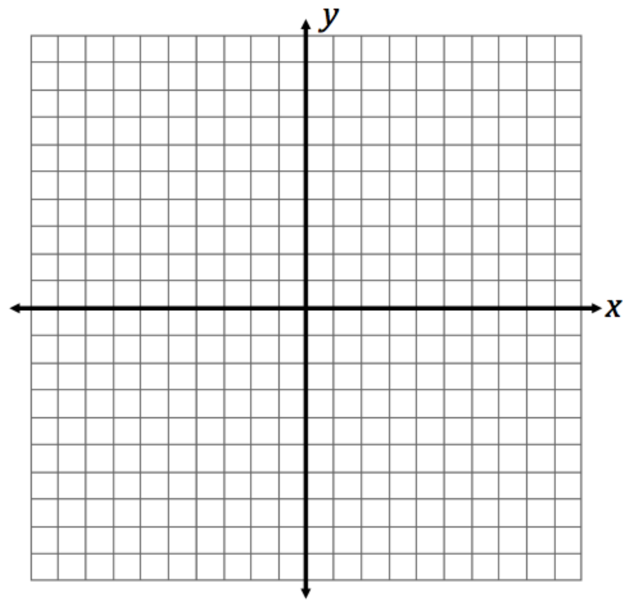
Summary of the Rules:

$$R_{O,90^\circ}: \quad (x, y) \to$$
$$R_{O,180^\circ}: \quad (x, y) \to$$
$$R_{O,270^\circ}: \quad (x, y) \to$$

Example 2

Given ΔCOW with vertices C (-1, 2), O (-1, 5) and W (-3, 3).

a. Graph ΔCOW on the axes provided.



b. Graph and state the coordinates of $\Delta C'O'W'$, the image of ΔCOW after a rotation of 180°

c. What type of isometry is the image?

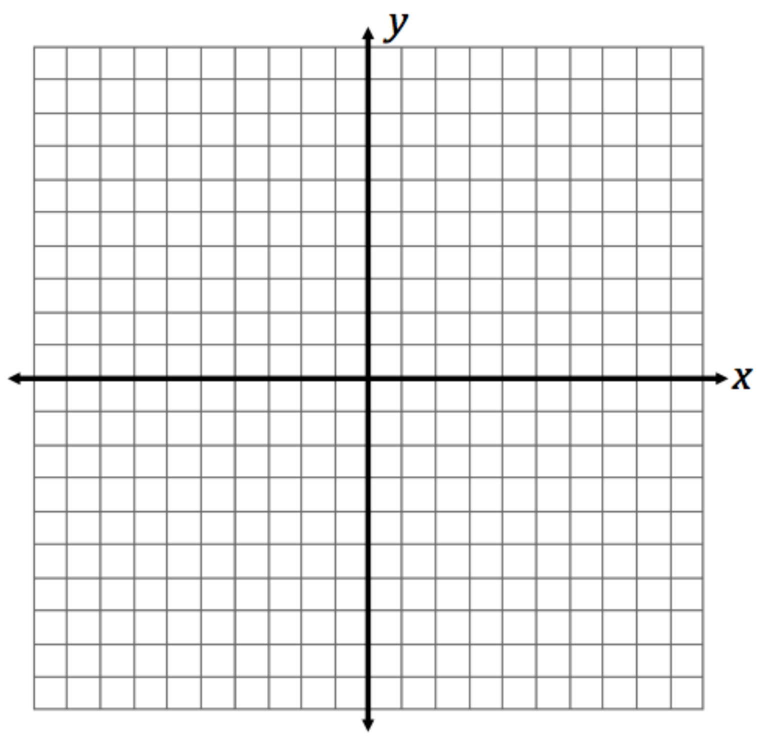
Example 3

We are now going to rotate an image around a point other than the origin.

Given ΔABC with vertices A (2, 3), B (0, 6) and C (2, 6).

a. Graph ΔABC on the axes provided below.

b. Graph and state the coordinates of $\Delta A'B'C'$, the image of ΔABC after a rotation of $90°$ about the point (-1, 2).



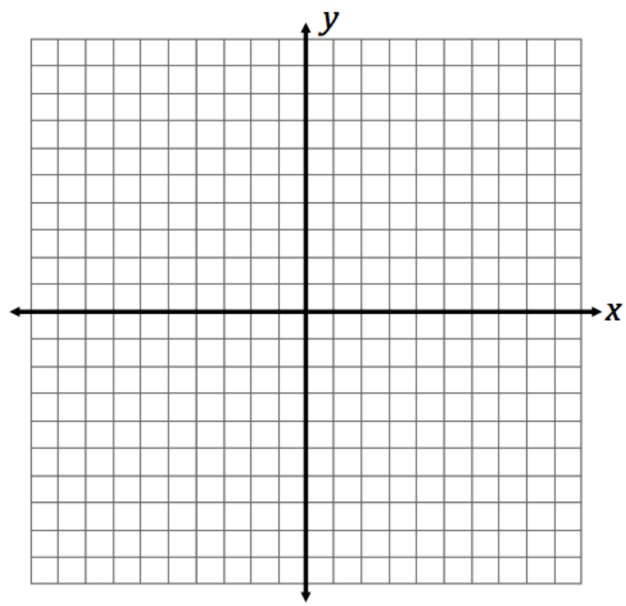
<u>To rotate an image about a point other than the origin</u>:

Step 1: Translate the rotation point to the origin.
Step 2: Translate the pre-image using the same translation as Step 1.
Step 3: Rotate the image following the rules of rotations.
Step 4: Translate the image the opposite direction as the translation from Step 1.

Exercises

1. Given ΔABC with vertices A (4, 0), B (2, 3) and C (1, 2).

   a. Graph ΔABC on the axes provided.



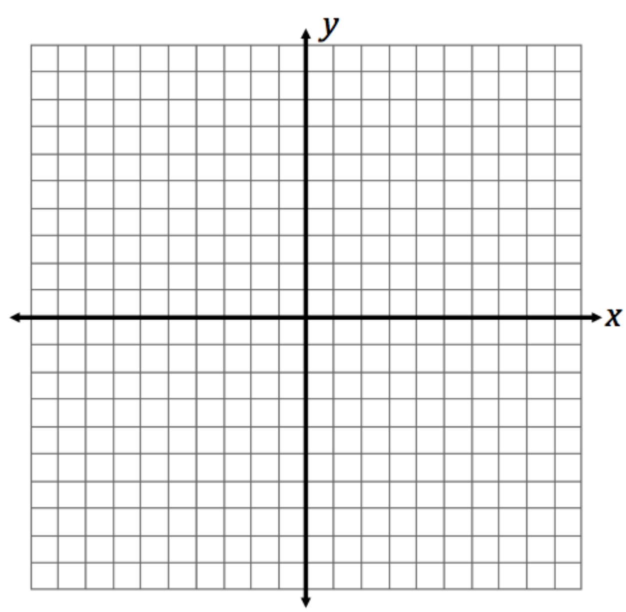
   b. Graph and state the coordinates of $\Delta A'B'C'$, the image of ΔABC after r_{y-axis}.

   c. Graph and state the coordinates of $\Delta A''B''C''$, the image of $\Delta A'B'C'$ after r_{x-axis}.

   d. This twice-reflected object is the same as which single transformation?

2. Given ΔDOG with vertices D (-3, 5), O (4, 6) and G (0, 2).

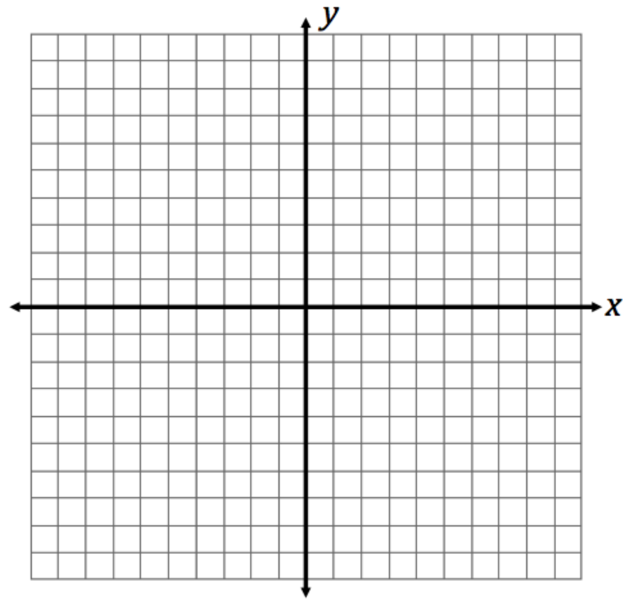
   a. Graph ΔDOG on the axes provided.



   b. Graph and state the coordinates of $\Delta D'O'G'$, the image of ΔDOG after a rotation of $90°$ about the point (3, -2).

Homework

1. Given ΔCAT with vertices C (-1, 5), A (-3, 1) and T (-2, -2).



a. Graph ΔCAT on the axes provided.

b. Graph and state the coordinates of $\Delta C'A'T'$, the image of ΔCAT after $R_{O,270°}$.

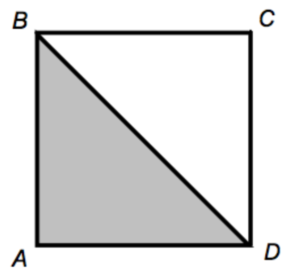
c. Graph and state the coordinates of $\Delta C''A''T''$, the image of $\Delta C'A'T'$ after r_{x-axis}.

d. Which type of isometry is the image in part *c*? Explain your answer.

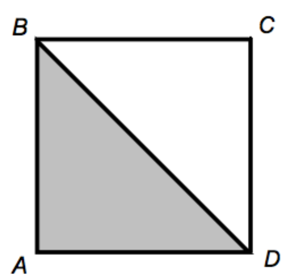
Lesson 6: Rotations off the Coordinate Plane

Opening Exercise

Rotate the figure below about point A. Show the image of the figure after rotations of $90°$, $180°$ and $270°$.



Now rotate the figure about point D using the same rotations of $90°$, $180°$ and $270°$.



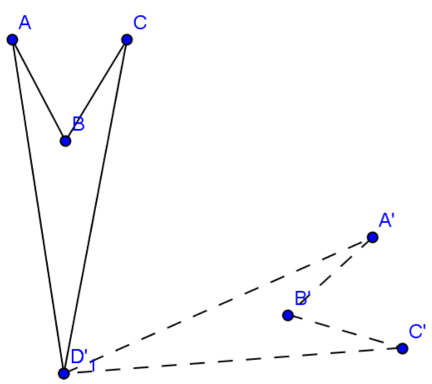
Notice the difference between the two images created!!!

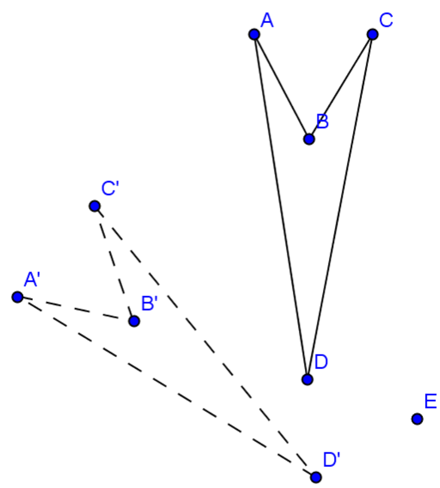
Example 1

Given the two figures below, we are going to use a protractor to measure the angle of rotation.

To find the angle of rotation:

Step 1: Identify the point that is the center of rotation.
Step 2: Measure the angle formed by connecting corresponding vertices to the center point of rotation.
Step 3: Check your answer using a different set of corresponding vertices.



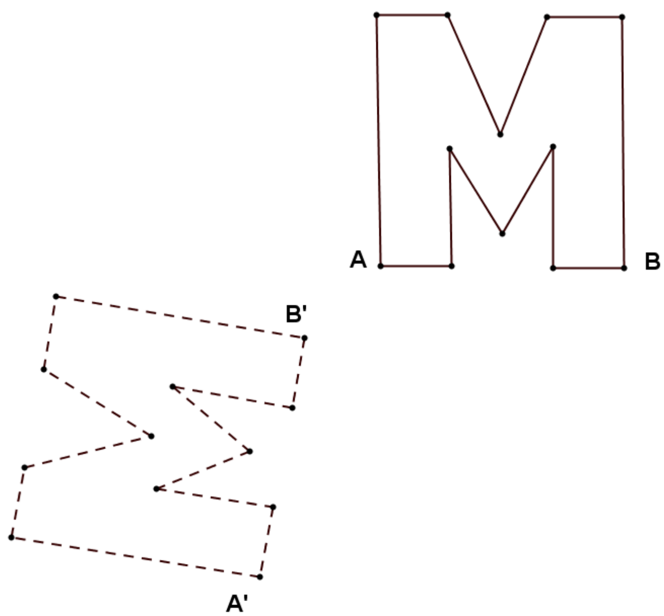


Center of Rotation:
Angle of Rotation:

Center of Rotation:
Angle of Rotation:

Example 2

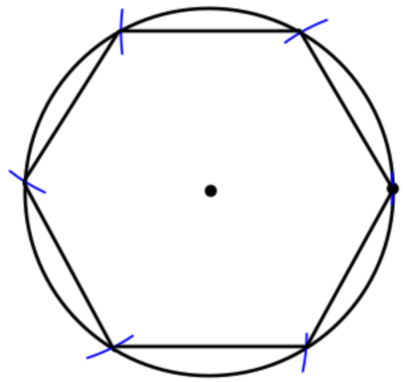
We are now going to locate the center of rotation.



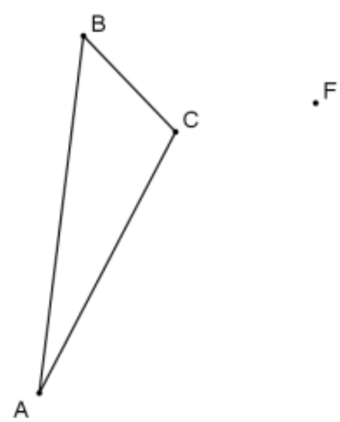
Step 1: Construct the perpendicular bisector of segment *AA'*.
Step 2: Construct the perpendicular bisector of segment *BB'*.
Step 3: The point of intersection of the perpendicular bisectors is the point of rotation. Label this point *P*.

Example 3

In Unit 1, we looked at the construction of a hexagon by using equilateral triangles.



Using this same concept, we are going to rotate ΔABC 60° around point F using a compass and straightedge only.

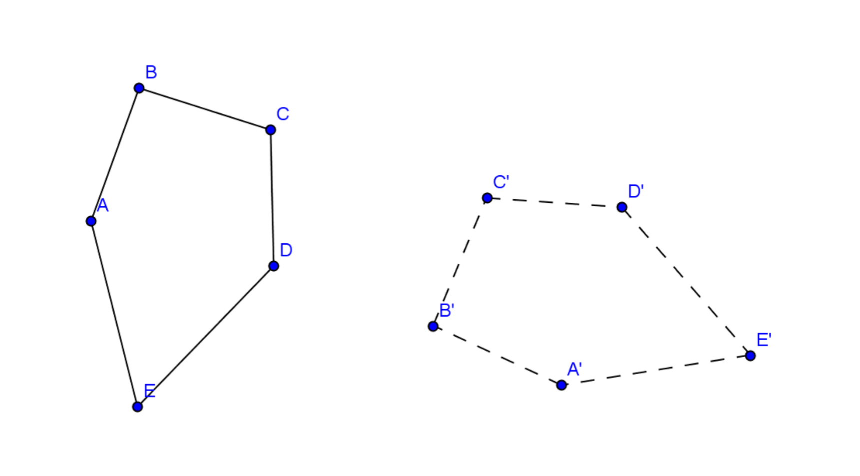


How could we rotate this image 120°?

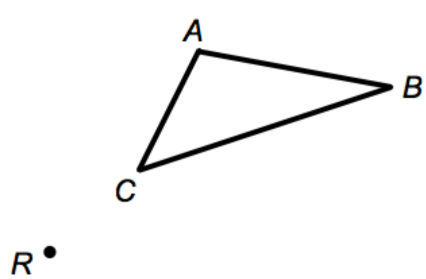
How could we rotate this image 90°?

Homework

1. Find the center of rotation and the angle of rotation for the transformation below:



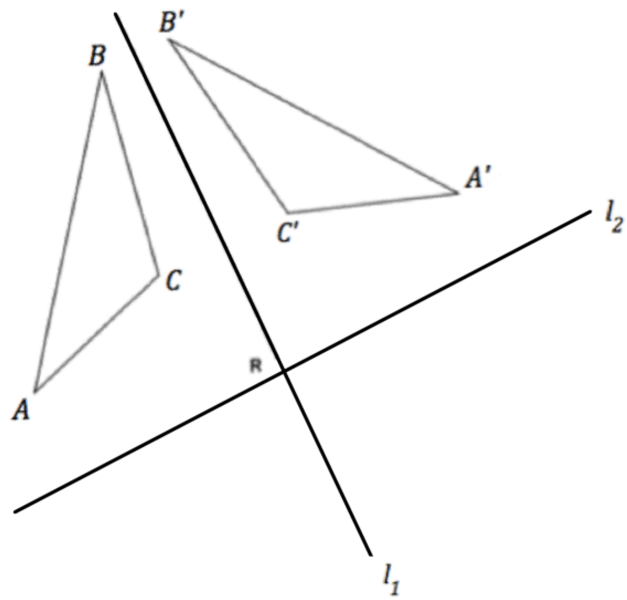
2. Rotate ΔABC $120°$ around point R using a compass and straightedge only.



Lesson 7: Types of Symmetry

Opening Exercise

$\Delta A'B'C'$ is the image of ΔABC after a reflection across line l_1.



a. Reflect the image across line l_2.

b. What is the relationship between the original triangle and the twice-reflected image?

c. What does point *R* represent?

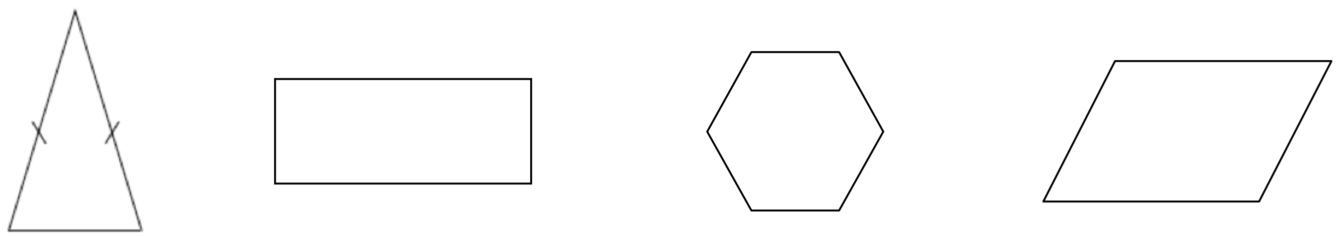
d. How could we determine the angle of rotation?

Reflecting a figure twice over intersecting lines will give the same result as a rotation about the point of intersection!

Example 1

Looking at the Opening Exercise, we can see that the lines of reflection are also **lines of symmetry**. The line of symmetry is equidistant from all corresponding pairs of points.

In the figures below, sketch all the lines of symmetry:

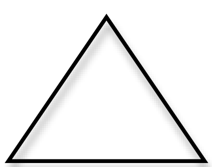
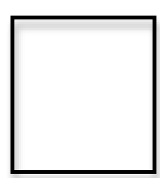
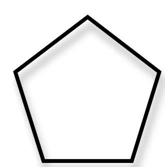
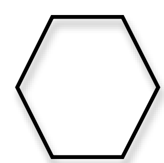


Example 2

Rotational Symmetry is a rotation that maps a figure back on to itself.

In **regular polygons** (polygons in which all sides and angles are congruent) the number of rotational symmetries is equal to the number of sides of the figure.

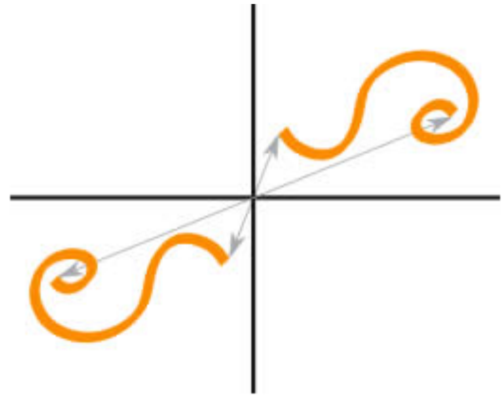
How can we find the angles of rotation?

	Equilateral Triangle	**Square**	**Regular Pentagon**	**Regular Hexagon**
				
# of sides				
Angles of Rotation				

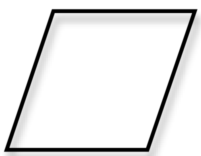
A rotation of 360° will always map a figure back on to itself. This is called the **identity transformation.**

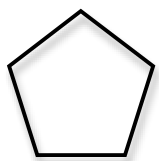
Example 3

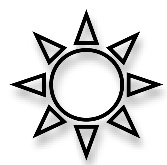
When reflecting an object through a point, the image and the pre-image create **point symmetry**. With point symmetry, the object will look exactly the same upside down! This can be seen by reflecting an object through the origin as pictured to the right.



Which of the objects below have point symmetry?



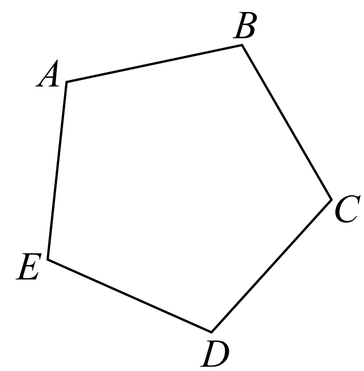




Exercises

Using regular pentagon *ABCDE* pictured to the right, complete the following:

1. Draw all line of symmetry.



2. Locate the center of rotational symmetry.

3. Describe all symmetries explicitly.

    a. What kinds are there?

    b. How many are rotations?

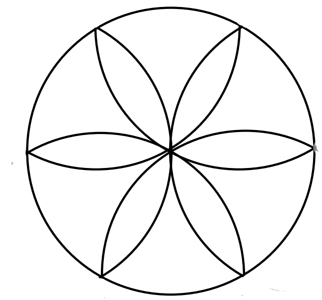
    c. What are the angles of rotation?

    d. How many are reflections?

Homework

1. Using the figure to the right, describe all the symmetries explicitly.

   a. How many are rotations?

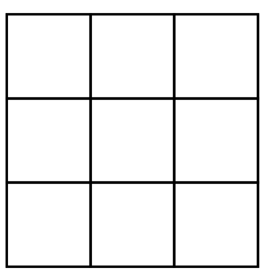
   b. What are the angles of rotation?

   c. How many are reflections?

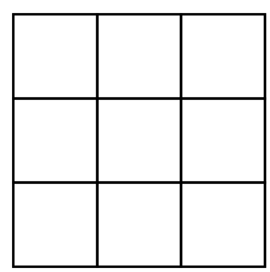
   d. Shade the figure so that the resulting figure only has 3 possible rotational symmetries.

2. Using the figures provided, shade exactly 2 of the 9 smaller squares so that the resulting figure has:

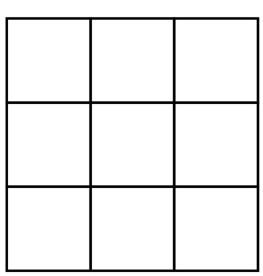
Only one vertical and one horizontal line of symmetry



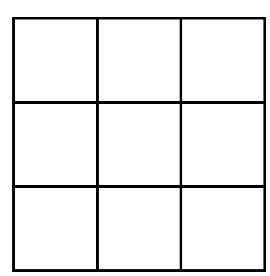
Only two lines of symmetry about the diagonals



Only one horizontal line of symmetry

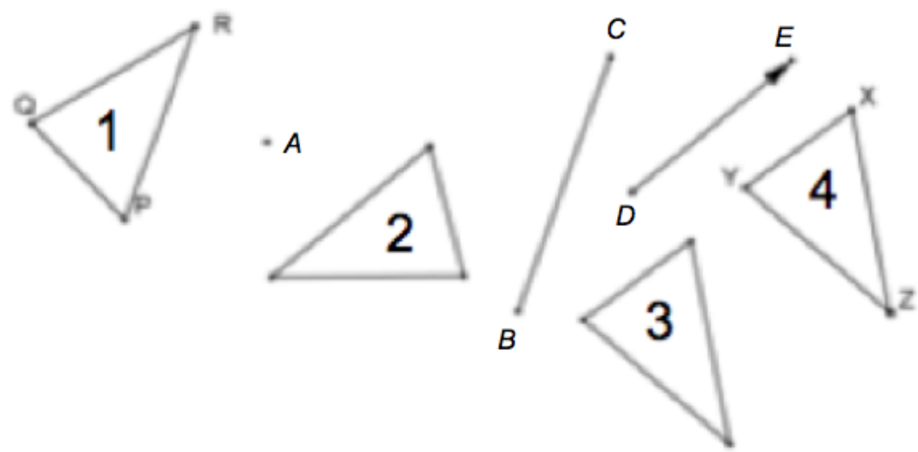


No line of symmetry



Lesson 8: Compositions of Rigid Motions

Opening Exercise



Looking at the picture above, describe the type of rigid motion that takes place to go from:

$\Delta 1 \rightarrow \Delta 2$:

$\Delta 2 \rightarrow \Delta 3$:

$\Delta 3 \rightarrow \Delta 4$:

When a series of rigid motions takes place with one rigid motion building off another (as shown above) this is called a **composition**.

The symbol used for compositions:

When performing or writing a composition, you must work from right to left!

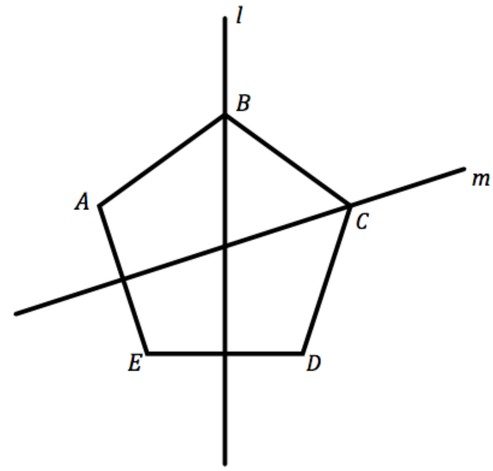
Two different ways to write the composition pictured above:

1.

2.

Example 1

Pentagon *ABCDE* is pictured to the right where *l* and *m* are lines of symmetry. Evaluate the following compositions:



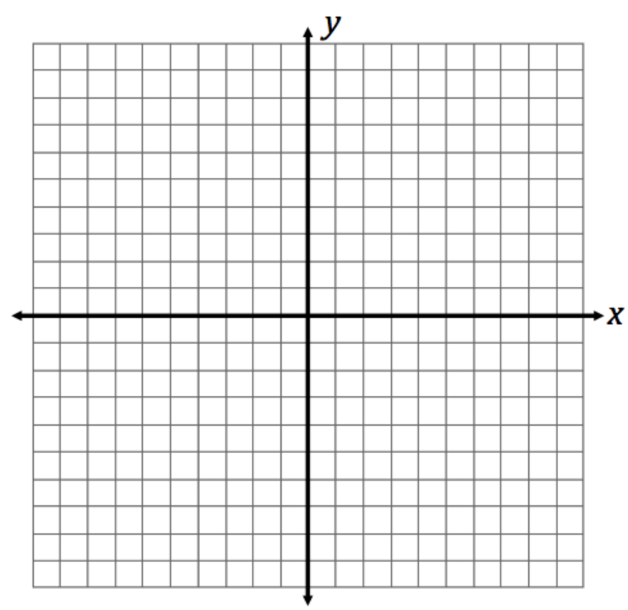
a. $r_l \circ r_m\,(E)$

b. $r_m \circ r_l\left(\overline{BA}\right)$

c. $r_m \circ R_{72^\circ}\,(B)$

Example 2

The coordinates of ΔABC are $A(-2, 2)$, $B(3, 5)$ and $C(4, 2)$. Graph and state the coordinates of $\Delta A''B''C''$, the image of ΔABC after the composition $r_{y-axis} \circ T_{2,-3}$.

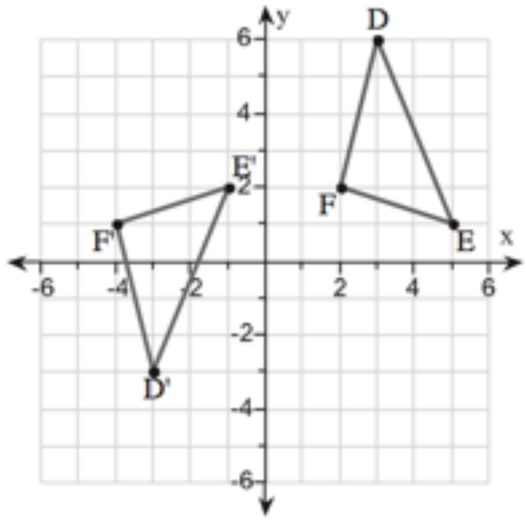


A composition of a translation and a reflection is called a **glide reflection**.

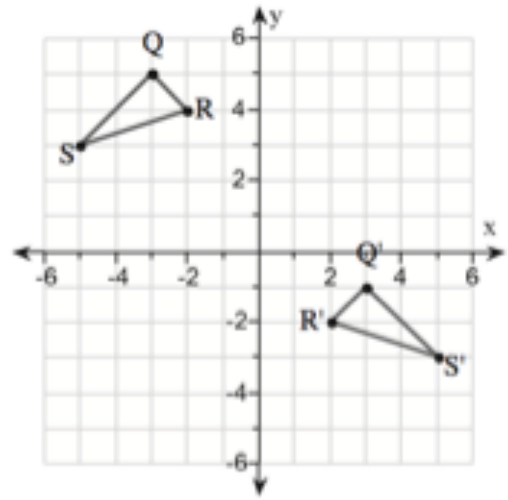
Example 3

Using the diagrams, write the rule of the composition:

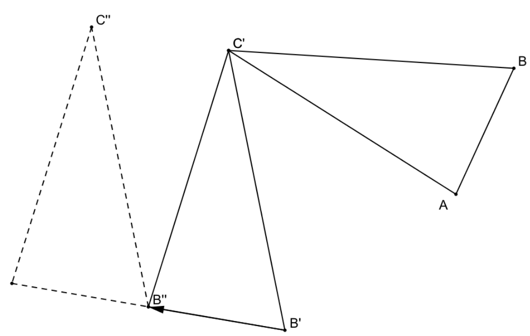
a.



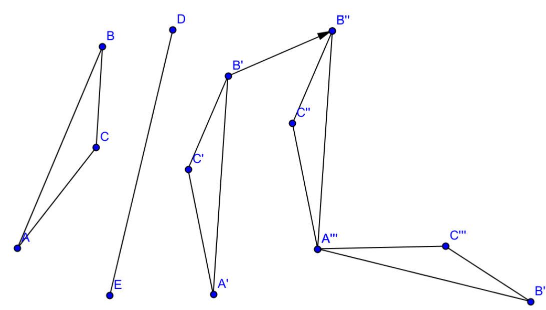
b.



c.



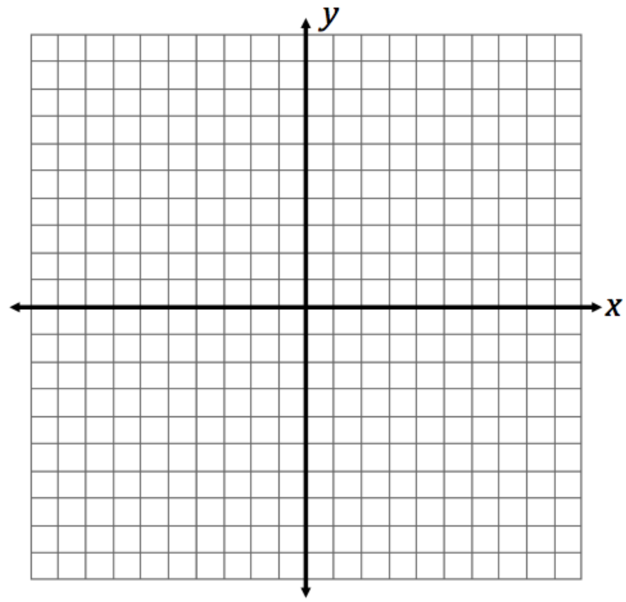
d.



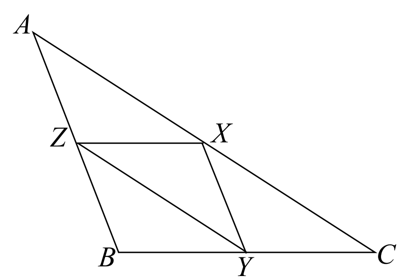
e. In each of the compositions shown in parts *a*-*d*, is the image congruent to the pre-image? Explain.

Homework

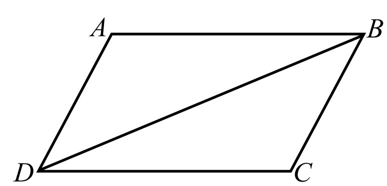
1. The coordinates of ΔABC are A(-1, 1), B(-5, 3) and C(-2, 7). Graph and state the coordinates of $\Delta A"B"C"$, the image of ΔABC after the composition $T_{-5,1} \circ R_{O,180°}$.



2. In the diagram pictured, all of the smaller triangles are congruent to each other. What rigid motion(s) map *ZB* onto *AZ*?



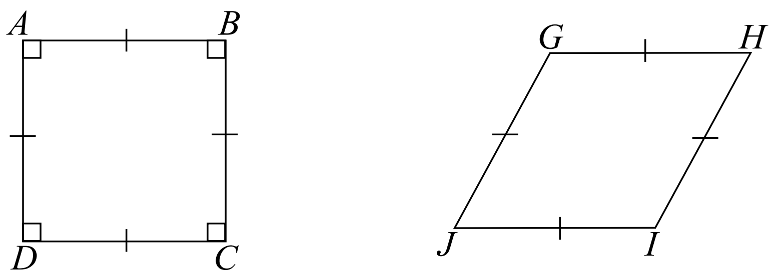
3. In the diagram pictured, what rigid motion(s) map *CD* onto *AB*?



Lesson 9: Congruence and Correspondence

Opening Exercise

Pictured below are square *ABCD* and rhombus *GHIJ*. Are they congruent? Explain.



Vocabulary

When figures are **congruent**, this means that there is a rigid motion (or a composition of rigid motions) that maps the pre-image onto the image. This rigid motion is called a congruence.

Example 1

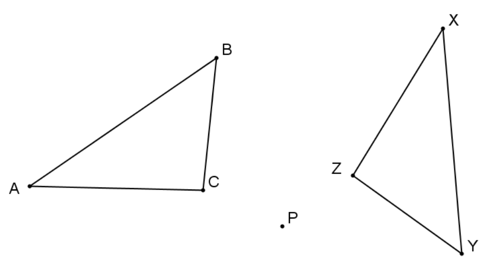
Under this definition of congruence, describe why the figures in the Opening Exercise are not congruent.

Vocabulary

A **correspondence** between two triangles is a pairing of each vertex of one triangle with one and only one vertex of another triangle. This pairing can be expanded to figures other than triangles and could also involve sides.

Example 2

In the figure below, the triangle on the left has been mapped to the one on the right by a rotation of $240°$ about *P*. Identify all six pairs of corresponding parts (angles and sides).



Corresponding angles	Corresponding sides
$\angle A \rightarrow$	$AB \rightarrow$
$\angle B \rightarrow$	$AC \rightarrow$
$\angle C \rightarrow$	$BC \rightarrow$

a. Is $\Delta ABC \cong \Delta XYZ$? Explain.

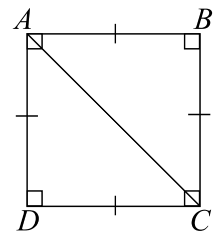
b. What rigid motion mapped ΔABC onto ΔXYZ? Write the transformation in function notation.

Important Discovery!

Rigid motions produce congruent figures and therefore, congruent parts (angles and sides). As a result, we can say that *corresponding parts of congruent figures are congruent*.

Exercises

1. $ABCD$ is a square, and AC is one diagonal of the square. ΔABC is a reflection of ΔADC across segment AC.



a. Complete the table below identifying the corresponding angles and sides.

Corresponding angles	Corresponding sides
$\angle BAC \rightarrow$	$AB \rightarrow$
$\angle ABC \rightarrow$	$BC \rightarrow$
$\angle BCA \rightarrow$	$AC \rightarrow$

b. Are the corresponding sides and angles congruent? Justify your response.

c. Is $\Delta ABC \cong \Delta ADC$? Justify your response.

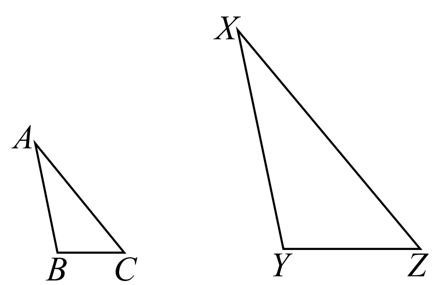
2. Each side of ΔXYZ is twice the length of each side of ΔABC.

a. Fill in the blanks below so that each relationship between lengths of sides is true.

_________ $\times 2 =$ _________

_________ $\times 2 =$ _________

_________ $\times 2 =$ _________



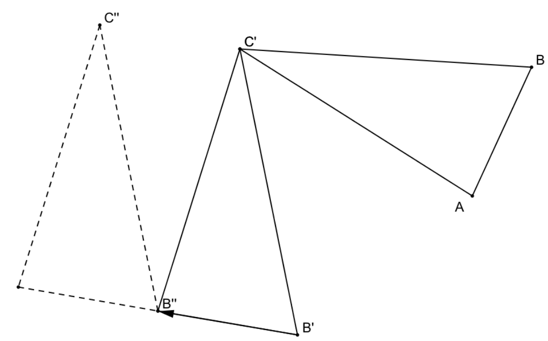
b. Is $\Delta ABC \cong \Delta XYZ$? Justify your response.

Important Discovery!

Corresponding parts do not always result in congruent figures.

Example 3

The figure to the right shows a series of rigid motions performed on ΔABC to produce the dotted triangle.



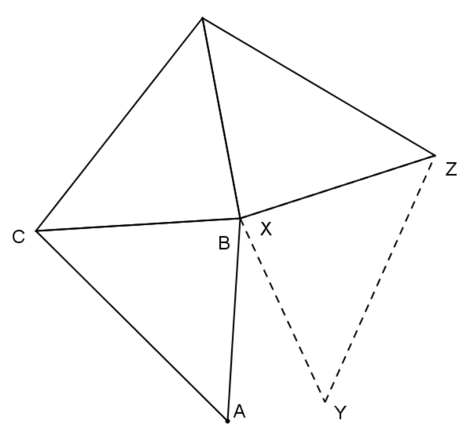
a. Identify the rigid motions.

b. Label vertex *A''*.

c. Is $\Delta A''B''C'' \cong \Delta ABC$? Justify your answer.

Example 4

Complete the table based on the series of rigid motions performed on ΔABC to produce the dotted triangle.



Sequence of rigid motions	
Composition in function notation	
Sequence of corresponding sides	
Sequence of corresponding angles	
Triangle congruence statement	

Homework

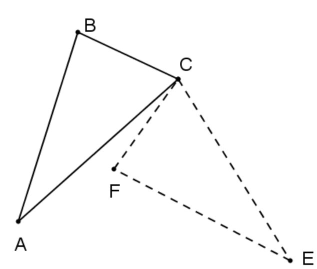
1. Using your understanding of congruence, explain the following:

   a. Why is a triangle not congruent to a quadrilateral?

   b. Why is an isosceles triangle not congruent to a scalene triangle?

2. Draw a diagram with two triangles in which all three corresponding angles are congruent but the corresponding sides are not congruent.

3. In the figure below, the triangle on the left has been mapped to the one on the right by a rotation of 80° about vertex *C*. Identify all six pairs of corresponding parts (angles and sides).



Corresponding angles	Corresponding sides
$\angle A \rightarrow$	$AB \rightarrow$
$\angle B \rightarrow$	$AC \rightarrow$
$\angle C \rightarrow$	$BC \rightarrow$

Write the rigid motion in function notation.