GUIDE TO EVERYTHING FRACTIONS

A COMPLETE OVERVIEW FOR ALL THOSE WHO NEVER REALLY "GOT IT" THE FIRST TIME THROUGH.

A GUIDE FOR STUDENTS, ADULTS AND TEACHERS

JAMES TANTON

CONTENTS:

This material is based on work from the reference series *THINKING MATHEMATICS!*

Volume 1: ARITHMETIC = GATEWAY TO ALL

available at ww.lulu.com/jtanton.

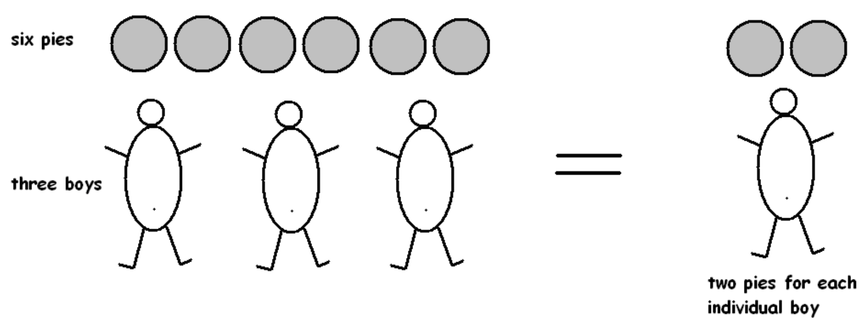
WHAT IS A FRACTION?

Simply put, a fraction is an answer to a division problem.

For example, suppose 6 pies are to be shared equally among 3 boys. This yields 2 pies per boy. We write:

$$\frac{6}{3} = 2$$

(We could, of course, also write $6 \div 3 = 2$ or $3\overline{)\,6\,}^{\,2}$.)



Here the fraction "$\frac{6}{3}$", our division problem, is equivalent to the number 2. It represents the number of pies one whole boy receives.

In the same way …

sharing 10 pies among 2 boys yields: $\frac{10}{2} = 5$ pies per boy.

sharing 8 pies among 2 boys yields: $\frac{8}{2} = 4$

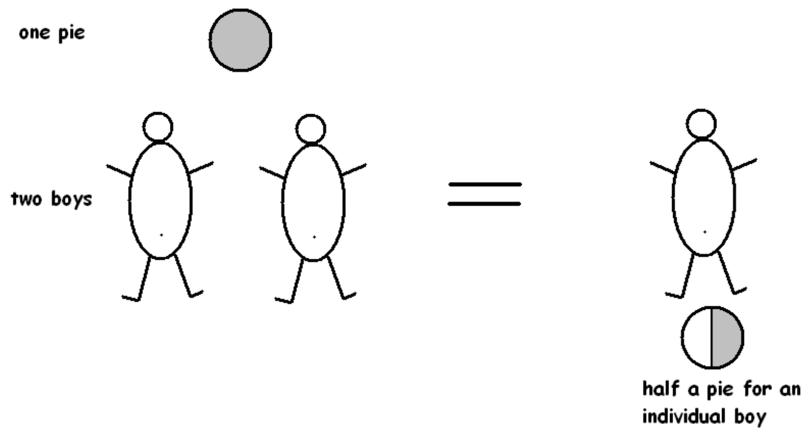
sharing 5 pies among 5 boys yields: $\frac{5}{5} = 1$

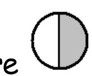
and

the answer to sharing 1 pie among 2 boys is $\frac{1}{2}$, which we call one half.

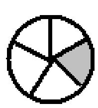
This final example is actually saying something! It also represents how fractions are usually taught to students:

If one pie is shared (equally) between two boys, then each boy receives a portion of a pie which we choose to call "half."



Thus students are taught to associate the number "$\frac{1}{2}$" to the picture .

In the same way, the picture  is said to represent "one third," that is, $\frac{1}{3}$. (And this is indeed the amount of pie an individual boy would receive if one pie is shared among three.)

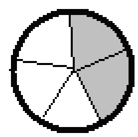
The picture  is called "one fifth" and is indeed $\frac{1}{5}$, the amount of pie an individual boy receives when one pie is shared among five.

And the picture  is called "three fifths" to represent $\frac{3}{5}$, the amount of pie an individual receives if three pies are shared among five boys.

EXERCISE 1: Draw a picture associated with the fraction $\frac{1}{6}$.

EXERCISE 2: Draw a picture associated with the fraction $\frac{3}{7}$. Is your picture really the amount of pie an individual boy would receive if three pies are shared among seven boys? Be very clear on this!

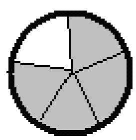
EXERCISE 3: Let's now do it backwards! Here is the answer to a division problem:



This represents the amount of pie an individual boy receives if some number of pies is shared among some number of boys.

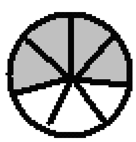
How many pies? __________
How many boys? __________

EXERCISE 4: Here is another answer to a division problem:



How many pies? __________
How many boys? __________

EXERCISE 5: Here is yet another answer to a division problem:



How many pies ? ________
How many boys? ________

EXERICSE 6: Leigh says that "$\frac{3}{5}$ is three times as big as $\frac{1}{5}$." Is this right? Is three pies shared among five boys three times as much as one pie shared among five boys? What do you think?

EXERCISE 7: Draw a picture for the answer to the division problem $\frac{4}{8}$. Describe what you notice about the answer.

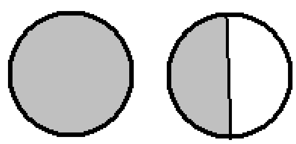
EXERCISE 8: Draw a picture for the answer to the division problem $\frac{2}{10}$. Describe what you notice about the answer.

EXERCISE 9: What does the division problem $\frac{1}{1}$ represent? How much pie does an individual boy receive?

EXERCISE 10: What does the division problem $\frac{5}{1}$ represent? How much pie does an individual boy receive?

EXERCISE 11: What does the division problem $\frac{5}{5}$ represent? How much pie does an individual boy receive?

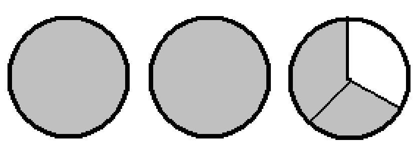
EXERCISE 12: Here is the answer to another division problem. This is the amount of pie an individual boy receives



How many pies were there in the division problem? _______

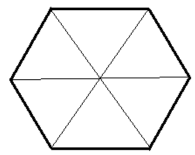
How many boys were there in the division problem? ______

EXERCISE 13: How many pies and how many boys for this answer?



Number of pies: _______
Number of boys: _______

EXERCISE 14: Many teachers have young students divide differently shaped pies into fractions. For example, a hexagonal pie is good for illustrating the fractions $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$ and $\frac{6}{6}$!



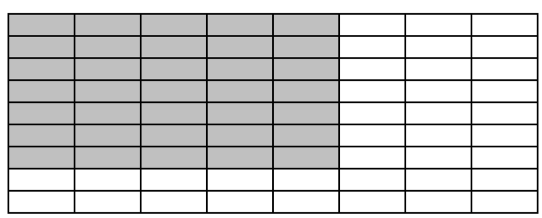
a) Why is this shape used? What does $\frac{1}{6}$ of a pie look like?

b) What does $\frac{6}{6}$ of a pie look like?

c) What shape pie would be good for illustrating the fractions $\frac{1}{8}$ up to $\frac{8}{8}$?

EXERCISE 15: TRICKY

Some rectangular pies are distributed to some number of boys. This picture represents the amount of pie an individual boy receives.

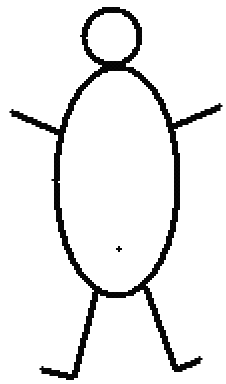


How many pies ? _______.

How many boys? _______.

THINKING EXERCISE: Let's be gruesome. Instead of dividing pies, we could divide boys!

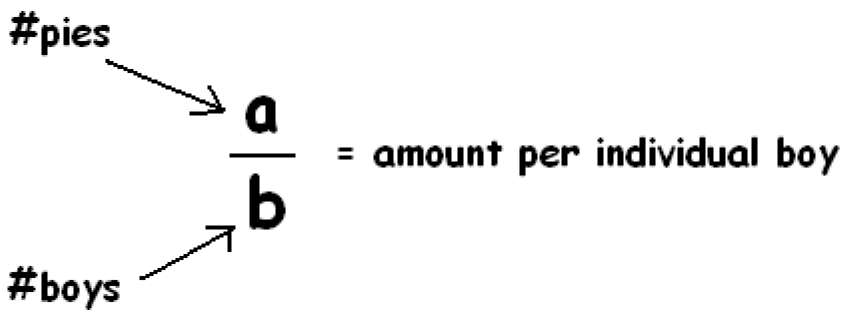
Here is one boy:



What would half a boy look like?
What would a third of a boy look like?
What would three fifths of a boy look like?

No need to write anything. Just try to imagine what these answers would be!

IN OUR MODEL … A fraction $\frac{a}{b}$ represents the amount of pie an individual boy receives when a pies are given to b boys.



(Note: We are assuming, for now, that both a and b are positive numbers.)

EXERCISE 16: What is $\frac{2}{2}$? $\frac{7}{7}$? $\frac{100}{100}$? What is $\frac{a}{a}$ for any positive whole number a?

EXERCISE 17: What is $\frac{1876}{1}$?

EXERCISE 18: "I have no pies to share among seven boys."
Use this to make a statement about a division problem and hence a statement about fractions.

SOME FANCY LANGUAGE …

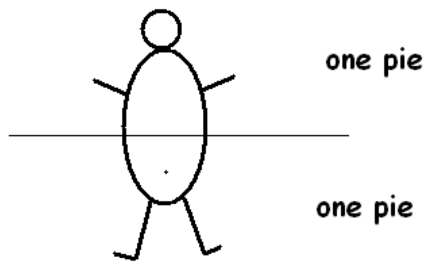
For a fraction $\frac{a}{b}$, the top number a (which, for us, is the number of pies) is called the <u>numerator</u> of the fraction, and the bottom number b (the number of pies), the <u>denominator</u> of the fraction. Most people insist that these numbers each be whole numbers, but they really don't have to be.

To see what I mean, let's have some fun!

QUESTION: What does $\frac{1}{\left(\frac{1}{2}\right)}$ represent?

This means assigning one pie to each "group" of half a boy. So how much would a whole boy receive?

Answer: Two pies!



We have:

$$\frac{1}{\left(\frac{1}{2}\right)} = 2$$

□

QUESTION: What does $\frac{1}{\left(\frac{1}{3}\right)}$ represent?

Answer: Distributing one pie to each "group" of a third of a boy yields the result of 3 whole pies for an individual boy.

$$\frac{1}{\left(\frac{1}{3}\right)} = 3$$

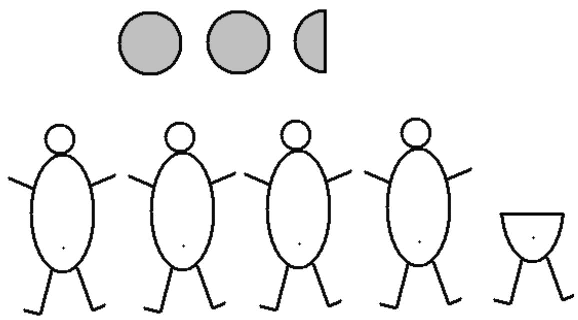
□

EXERCISE 19: What is the answer to $\frac{1}{\left(\frac{1}{6}\right)}$?

EXERCISE 20: The answer to $\frac{5}{\left(\frac{1}{2}\right)}$ is ten. Do you we why? [How many pies are given to half a boy?]

EXERCISE 21: What is the answer to $\frac{4}{\left(\frac{1}{3}\right)}$?

SCARY COMPLETELY-OPTIONAL CHALLENGE:
Two-and-a-half pies are to be shared equally among four-and-a-half boys!



How much pie does an individual (whole) boy receive?

This is a very tricky problem. Only attempt this if it seems fun to do so. We'll see a very easy way to think about these types of problems a little bit later in this packet.

THE KEY FRACTION RULE

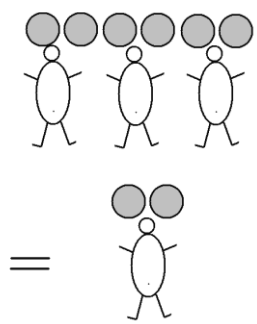
We have that $\frac{a}{b}$ is an answer to a division problem:

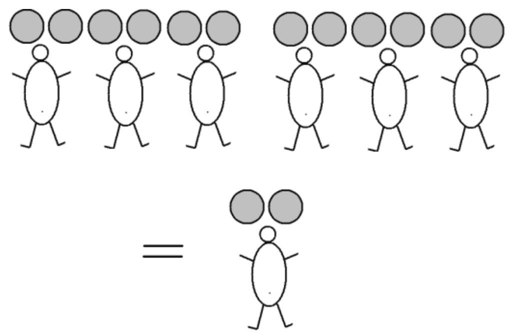
> $\frac{a}{b}$ represents the amount of pie an individual boy receives when a pies are distributed among b boys.

What happens if we double the number of pies and double the number of boys? Nothing! The amount of pie per boy is still the same:

$$\frac{2a}{2b} = \frac{a}{b}$$

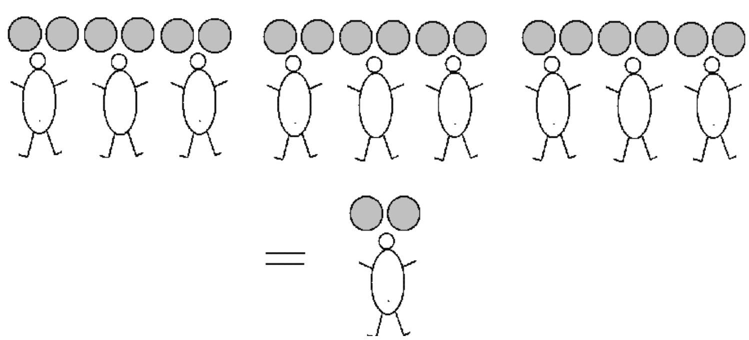
For example, as the picture shows, $\frac{6}{3}$ and $\frac{12}{6}$ both give two pies for each boy.





And tripling the number of pies and tripling the number of boys also does not change the final amount of pie per boy, nor does quadrupling each number, or one-trillion-billion-tupling the numbers!

$$\frac{6}{3} = \frac{12}{6} = \frac{18}{9} = \cdots \text{ = two pies per boy}$$



This leads us to want to believe a fraction rule:

FRACTION RULE: $\frac{xa}{xb}=\frac{a}{b}$ (for positive numbers at least).

For example,

$$\frac{3}{5}$$ (sharing three pies among five boys)

yields the same result as

$$\frac{3\times 2}{5\times 2}=\frac{6}{10}$$ (sharing six pies among ten boys),

and as

$$\frac{3\times 100}{5\times 100}=\frac{300}{500}$$ (sharing 300 pies among 500 boys).

Going backwards …

$$\frac{20}{32}$$ (sharing 20 pies among 32 boys)

is the same problem as:

$$\frac{5\times 4}{8\times 4}=\frac{5}{8}$$ (sharing five pies among eight boys).

Comment: Most people say we have <u>cancelled</u> or <u>taken</u> a common factor of 4 from the numerator and the denominator.

Mathematicians call this process <u>reducing</u> the fraction to simpler terms. (We've made the numerator and denominator each smaller!) Teachers tend to say that we are <u>simplifying</u> the fraction. (One has to admit that $\frac{5}{8}$ does look simpler than $\frac{20}{32}$.)

As another example $\frac{280}{350}$ can certainly be simplified by noticing that there is a common factor of 10 in both the numerator and the denominator:

$$\frac{280}{350} = \frac{28 \times 10}{35 \times 10} = \frac{28}{35}$$

We can go further as 28 and 35 are both multiples of 7:

$$\frac{28}{35} = \frac{4 \times 7}{5 \times 7} = \frac{4}{5}$$

Thus, sharing 280 pies among 350 boys gives the same result as sharing just 4 pies among 5 boys!

$$\frac{280}{350} = \frac{4}{5}$$

As 4 and 5 share no common factors, this is as far as we can go with this example (while staying with whole numbers!).

EXERCISE 22: MIX AND MATCH: On the top are some fractions that have not been simplified. On the bottom are the simplified answers, but in random order. Which simplified answer goes with which fraction? (Notice that there are less answers than questions!)

1. $\frac{10}{20}$ **2.** $\frac{50}{75}$ **3.** $\frac{24000}{36000}$ **4.** $\frac{24}{14}$ **5.** $\frac{18}{32}$ **6.** $\frac{1}{40}$

A. $\frac{2}{3}$ **B.** $\frac{9}{16}$ **C.** $\frac{12}{7}$ **D.** $\frac{1}{40}$ **E.** $\frac{1}{2}$

1. goes with ______
2. goes with ______
3. goes with ______
4. goes with ______
5. goes with ______
6. goes with ______

EXERCISE 23: Jenny says that $\frac{4}{5}$ does "reduce" further is you are willing to move away from whole numbers. She writes:

$$\frac{4}{5} = \frac{2\times 2}{2\frac{1}{2}\times 2} = \frac{2}{2\frac{1}{2}}$$

Is she right? Does sharing 4 pies among 5 boys yield the same result as sharing 2 pies among $2\frac{1}{2}$ boys?

What do you think?

ADDING AND SUBTRACTING FRACTIONS

Here are two very similar fractions: $\frac{2}{7}$ and $\frac{3}{7}$. What might it mean to add them? It might be tempting to say …

$\frac{2}{7}$ represents 2 pies being shared among 7 boys
$\frac{3}{7}$ represents 3 pies being shared among 7 boys

so $\frac{2}{7}+\frac{3}{7}$ probably represents sharing 5 pies among 14 boys, giving the answer $\frac{5}{14}$.

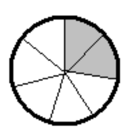
That is, it is very tempting to say that "adding fractions" means to "add pies and to add boys."

The trouble is that a fraction is not a pie, and a fraction is not a boy. (So adding pies and adding boys is <u>not</u> actually adding fractions.) A fraction is something different. It is related to pies and boys, but something more subtle. A fraction is an amount of pie <u>per</u> boy.

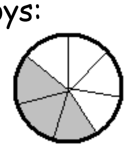
One can't add pies, one can't add boys. One must add instead the amounts individual boys receive.

Let's take it slowly:

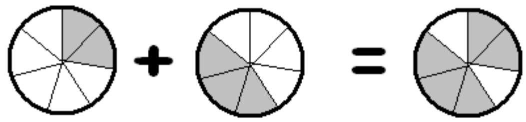
Consider the fraction $\frac{2}{7}$. Here is a picture of the amount an individual boy receives when two pies are given to seven boys:



Consider the fraction $\frac{3}{7}$. Here is a picture of the amount an individual boy receives when three pies are given to seven boys:

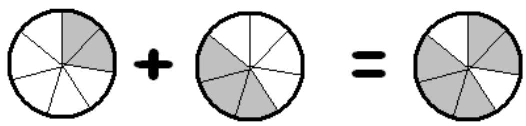


The sum $\frac{2}{7}+\frac{3}{7}$ corresponds to the sum:



The answer, from the picture, is $\frac{5}{7}$.

Most people read this as "Two sevenths plus three sevenths gives five sevenths" and think that the problem is just as easy as saying "two apples plus three apples gives five apples." And, in the end, they are right!



$$\frac{2}{7}+\frac{3}{7}=\frac{5}{7}$$

This is how the addition of fractions is first taught to students: Adding fractions with the same denominator seems just as easy as adding apples:

4 tenths + 3 tenths + 8 tenths = 15 tenths

$$\frac{4}{10}+\frac{3}{10}+\frac{8}{10}=\frac{15}{10}$$

(and, of course, $\frac{15}{10}=\frac{5\times 3}{5\times 2}$ simplifies to $\frac{3}{2}$).

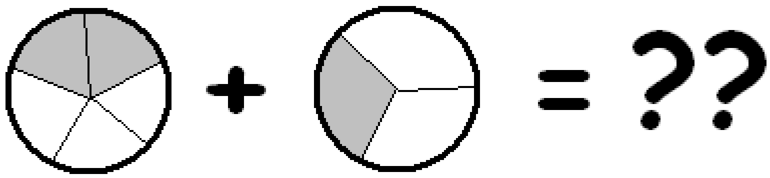
82 sixty-fifths + 91 sixty-fifths = 173 sixty-fifths

$$\frac{82}{65}+\frac{91}{65}=\frac{173}{65}$$

We are really adding amounts per boy, but the answers match the same way.

EXERCISE 24: Is subtraction of fractions manageable, at least for fractions with a common denominator? What is $\frac{400}{903}-\frac{170}{903}$, for example?

This approach to adding fractions suddenly becomes tricky if the denominators involved are not the same common value. For example, what is $\frac{2}{5}+\frac{1}{3}$?



Let's phrase this question in terms of pies and boys.

Suppose Poindexter is part of a team of a five boys that receives two pies, and then later part of a team of three boys that receives one pie. How much pie does Poindexter receive in total?

a) Do you see that this is the same problem as computing $\frac{2}{5}+\frac{1}{3}$?
b) What might be the best approach to answering this problem?

Think about the challenge before reading on. It is actually a very difficult problem!

If you have any thoughts, write them here. If you don't have any thoughts about how to do this, that's okay!

One way to think about answering this addition question is to write $\frac{2}{5}$ in a series of alternative forms using our fraction rule (that is, multiply the numerator and denominator each by 2, and then each by 3, and then each by 4, and so on) and to do the same for $\frac{1}{3}$ …

$$\begin{array}{ccc} \frac{2}{5} & + & \frac{1}{3} \\[2ex] \frac{4}{10} & & \frac{2}{6} \\[2ex] \frac{6}{15} & & \frac{3}{9} \\[2ex] \frac{8}{20} & & \frac{4}{12} \\[2ex] \frac{10}{25} & & \frac{5}{15} \\ \vdots & & \vdots \end{array}$$

… and then notice, that we can see two common denominators. We see that the problem $\frac{2}{5}+\frac{1}{3}$ is actually the same as $\frac{6}{15}+\frac{5}{15}$, and so has answer $\frac{11}{15}$.

$$\frac{2}{5}+\frac{1}{3} \quad = \quad \frac{6}{15}+\frac{5}{15} \quad = \quad \frac{11}{15}$$

As another example, let's compute $\frac{3}{8}+\frac{3}{10}$:

$$\begin{array}{cccc} \frac{3}{8} & + \ \frac{3}{10} & = & \frac{27}{40} \\ \frac{6}{16} & \frac{6}{20} & & \\ \frac{9}{24} & \frac{9}{30} & & \\ \frac{12}{32} & \frac{12}{40} & & \\ \frac{15}{40} & \frac{15}{50} & & \\ \vdots & \vdots & & \end{array}$$

COMMENT: Of course, one doesn't need to list all the equivalent forms of each fraction in order to find a common denominator. If you can see a common denominator right away (or can think of a method that always works), go for it!!!

EXERCISE 25: What is $\frac{1}{2}+\frac{1}{3}$? The answer is some number of sixths. How many sixths?

EXERCISE 26: What is $\frac{2}{5}+\frac{37}{10}$?

EXERCISE 27: What is $\frac{1}{2}+\frac{3}{10}$?

EXERCISE 28: What is $\frac{2}{3}+\frac{5}{7}$?

EXERCISE 29: What is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$?

EXERCISE 30: What is $\frac{3}{10}+\frac{4}{25}+\frac{7}{20}+\frac{3}{5}+\frac{49}{50}$?

Let's do subtraction.

EXERCISE 31: What is $\frac{7}{10}-\frac{3}{10}$?

EXERCISE 32: What is $\frac{7}{10}-\frac{3}{20}$?

EXERCISE 33: What is $\frac{1}{3}-\frac{1}{5}$?

EXERCISE 34: What is $\frac{2}{35}-\frac{2}{7}+\frac{2}{5}$?

EXERCISE 35: What is $\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}$?

Here's a good question!

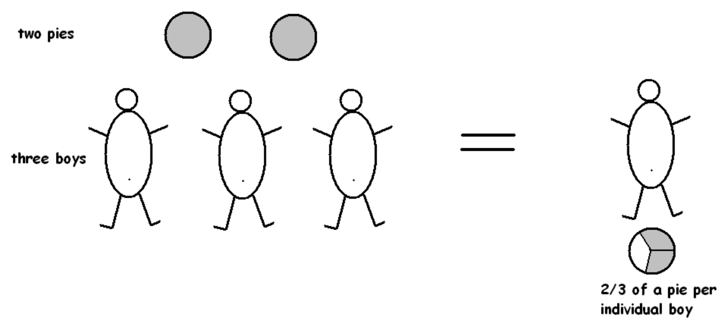
EXERCISE 36: Which is larger: $\frac{5}{9}$ or $\frac{6}{11}$?

What is a good way to approach this? Perhaps write each fraction with a common denominator?

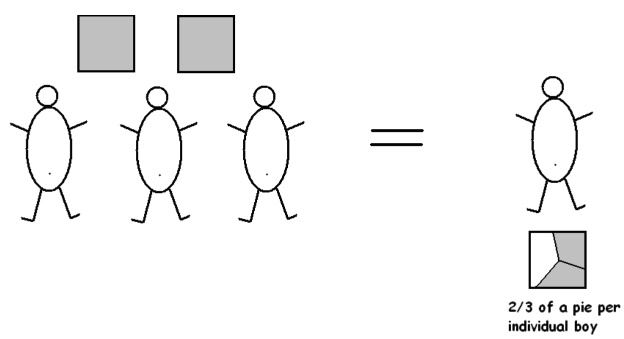
MULTIPLYING FRACTIONS

We saw in the previous sections that a fraction is simply an answer to a division problem – the amount of pie an individual boy receives when several pies are shared among several boys.

For example, $\frac{2}{3}$ is the result of sharing two pies among three boys.

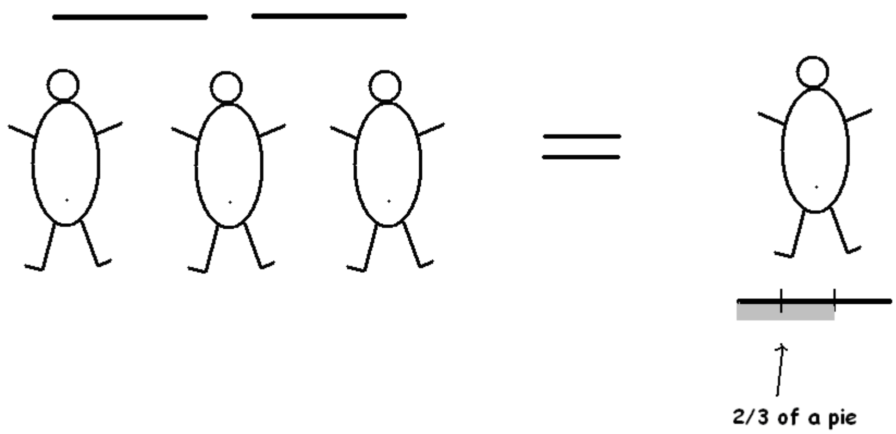


Pies do not have to be round. We can have square pies:

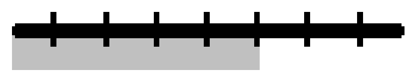


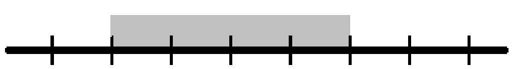
Or triangular pies, or hexagonal pies, or wibbly-wobbly pies.

In this section we shall work with straight-line pies!



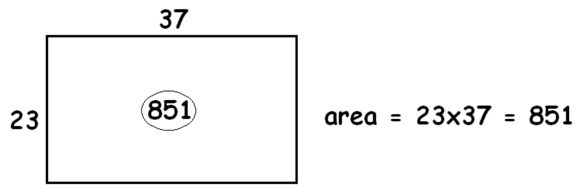
This picture  represents two thirds, that is $\frac{2}{3}$, of a straight line pie.

EXERCISE 37: How much pie per boy does  represent?

EXERCISE 38: How much pie per boy does  represent?

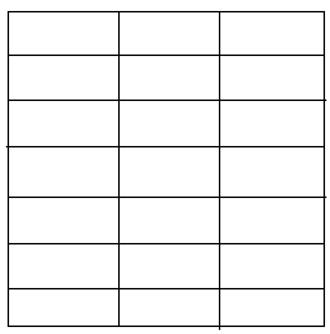
Let's try to multiply fractions.

In geometry, multiplication of numbers corresponds to an area problem. For example, the product 23×37 is the area of a 23-by-37 rectangle:

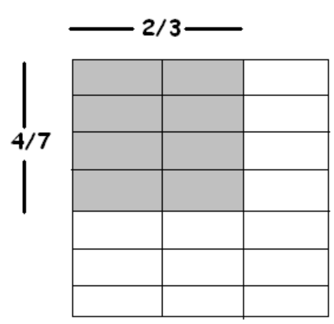


So the product of two fractions, say, $\frac{4}{7} \times \frac{2}{3}$ should also correspond to an area problem.

Let's again start a rectangle, but this time divide one side-length into sevenths (share the line segment among seven boys!) and the other side-length into thirds (share the side among three boys!).



We can now mark off four sevenths and two thirds.



and we see that the rectangle is divided into 21 pieces in all.

If we shade the region of interest to us we count 8 shaded pieces.

And, viewing this picture as a rectangular pie, we see that this picture corresponds to the fraction $\frac{8}{21}$.

The shaded region is the area we seek, the one that corresponds to the area problem $\frac{4}{7}\times\frac{2}{3}$. So we must have:

$$\frac{4}{7}\times\frac{2}{3}=\frac{8}{21}$$

THINKING EXERCISE:
The area problem $\frac{4}{7}\times\frac{2}{3}$ yielded a diagram with 21 small rectangles. Is it a coincidence that "21" happens to equal seven times three?

The area problem $\frac{4}{7}\times\frac{2}{3}$ yielded a diagram with 8 small shaded rectangles. Is it a coincidence that "8" equals four times two?

What do you think?

EXERCISE 39: Use this "rectangle method" to compute $\frac{3}{4}\times\frac{5}{6}$. Draw the picture to see the answer clearly,

EXERCISE 40: What is the answer to $\frac{3}{8}\times\frac{5}{10}$? What is the answer to $\frac{5}{11}\times\frac{7}{12}$?

EXERCISE 41: Compute the following products, simplifying each of the answers as much as possible:

a) $\frac{5}{8}\times\frac{3}{7}$ b) $\frac{4}{7}\times\frac{4}{8}$ c) $\frac{1}{2}\times\frac{1}{3}$ d) $\frac{2}{1}\times\frac{3}{1}$ e) $\frac{1}{5}\times\frac{5}{1}$

EXERCISE 42: Compute the following products. (Don't work too hard!)

a) $\frac{3}{4}\times\frac{1}{3}\times\frac{2}{5}$ b) $\frac{5}{5}\times\frac{7}{8}$ c) $\frac{88}{88}\times\frac{541}{788}$ d) $\frac{77876}{311}\times\frac{311}{77876}$

e) $\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{5}{6}\times\frac{6}{7}\times\frac{7}{8}\times\frac{8}{9}\times\frac{9}{10}$

(Make good use of the fraction rule $\frac{xa}{xb}=\frac{a}{b}$ before you do any arithmetic!)

THINKING EXERCISE: Ralph says that the fraction rule $\frac{xa}{xb}=\frac{a}{b}$ is "obvious" if you think in terms of multiplying fractions. He reasons as follows:

We know multiplying anything by 1 doesn't change the number:

$$1\times 4 = 4$$
$$1\times 3565 = 3565$$
$$1\times\frac{5}{7}=\frac{5}{7}$$

So $1\times\frac{a}{b}=\frac{a}{b}$.

Now $\frac{2}{2}$ *equals 1, so this means that* $\frac{2}{2}\times\frac{a}{b}$ *must still be* $\frac{a}{b}$. *So* $\frac{2a}{2b}=\frac{a}{b}$.
Now $\frac{3}{3}$ *equals 1, so this means that* $\frac{3}{3}\times\frac{a}{b}$ *must still be* $\frac{a}{b}$. *So* $\frac{3a}{3b}=\frac{a}{b}$.
And so on.

Do you agree with Ralph's insight?

How might Ralph explain why $\frac{xa}{xb}=\frac{a}{b}$ is true?

Many students are taught to multiply fractions by numbers by using the method of multiplying fractions by fractions. For example, to compute:

$$2\times\frac{3}{7}$$

think of "2" as $\frac{2}{1}$ and then compute:

$$\frac{2}{1}\times\frac{3}{7}$$

This has the answer: $\frac{6}{7}$.

The statement $2\times\frac{3}{7}=\frac{6}{7}$ can also be interpreted as:

Suppose three pies are shared among seven boys. (This is $\frac{3}{7}$ *.)*

To double the amount of pie each boy receives ($2\times\frac{3}{7}$ *) just double the number of pies (that is, make it* $\frac{6}{7}$ *).*

$$2\times\frac{3}{7}=\frac{6}{7}$$

Another way to think about this:

Double "three sevenths" is clearly "six sevenths."

THINKING EXERCISE: Do you like any of these interpretations?

As another example:

$$4\times\frac{3}{8}$$

equals

$$\frac{4}{1}\times\frac{3}{8} = \frac{12}{8}$$

and of course this simplifies $\frac{3\cdot 4}{2\cdot 4}=\frac{3}{2}$.

Here's another example of a product:

$$10\times\frac{2}{15} \quad =\frac{10}{1}\cdot\frac{2}{15} \quad =\frac{10\cdot 2}{15}$$

Rather than multiply out the numerator, let's break the numerator each into factors and simplify the fraction:

$$\begin{aligned}\frac{10\cdot 2}{15} &= \frac{2\cdot 5\cdot 2}{3\cdot 5}\\ &= \frac{2\cdot 2}{3}\\ &= \frac{4}{3}\end{aligned}$$

And a third example:

$$8\times\frac{212}{16}=\frac{8\cdot 212}{16}$$

Let's avoid work and notice that 16 is $8\cdot 2$. So:

$$\frac{8\cdot 212}{16}=\frac{8\cdot 212}{8\cdot 2}=\frac{212}{2}=106$$

EXERCISE 43: Compute each of the following, writing your answers in simplified form. Avoid extra arithmetic if you can!

a) $3\times\frac{2}{5}$ b) $17\times\frac{2}{3}$ c) $10\times\frac{1}{5}$ d) $\frac{3}{4}\times 4$ e) $11\times\frac{36}{33}$ f) $\frac{13}{12}\times 24$

g) $\frac{3}{7}\times\frac{7}{5}$ h) $\frac{5}{13}\times\frac{4}{7}\times\frac{13}{2}\times\frac{7}{10}$

EXERCISE 44: Compute $\frac{133}{112}\times 224$ in less than six seconds!

EXERCISE 45: a) Compute $6\times\frac{5}{6}$.

b) Compute $18\times\frac{7}{18}$.

c) What can you say about $b\times\frac{a}{b}$?

d) What can you say about $\frac{m}{n}\times n$?

<u>COMMENT</u>: REMEMBER PARTS c) and d) OF THIS QUESTION!

EXERCISE 46: Ibrahim was asked to compute:

$$\frac{18}{7}\times\frac{70}{36}$$

and, within three seconds, said that the answer was 5. He was right! How did he see this so quickly?

EXERCISE 47: What is the value of $\frac{39}{35}\times\frac{14}{13}$?

SOME JARGON

Many people like to name things. (The reason for this is not always clear!)

A fraction with a numerator smaller than its denominator is called a proper fraction. E.g. $\frac{45}{58}$ is a proper fraction.

A fraction with numerator larger than its denominator is called an improper fraction. E. g. $\frac{7}{3}$ is an improper fraction. (In the 1800s, these fractions were called vulgar fractions. Despite nineteenth-century views they are useful nonetheless!)

For some reason that doesn't really make sense, improper fractions are considered, well, improper by some teachers and students are made to write improper fractions as a combination of a whole number and a proper fraction.

Consider, for example, $\frac{7}{3}$. If seven pies are shared among three girls, then each girl will certainly receive 2 whole pies, leaving one pie over to share among the three girls. Thus, $\frac{7}{3}$ equals 2 plus $\frac{1}{3}$. People write:

$$\frac{7}{3} = 2\frac{1}{3}$$

and call the result $2\frac{1}{3}$ a mixed number. (One can also write $2+\frac{1}{3}$, which is what $2\frac{1}{3}$ really means, but most people choose to suppress the plus sign.)

As another example, consider $\frac{23}{4}$. The number 4 certainly "goes into" 23 five times and leaves a remainder of 3, which is still be divided by four. We have:

$$\frac{23}{4} = 5\frac{3}{4}$$

EXERCISE 48: Write each of the following as a mixed number. (For example, $\frac{32}{5}$ equals $6\frac{2}{5}$.)

a) $\frac{17}{3}$ b) $\frac{8}{5}$ c) $\frac{100}{13}$ d) $\frac{200}{199}$

Mathematically there is nothing wrong with an improper fraction and many mathematicians prefer improper fractions over mixed numbers.

Consider, for instance, the mixed number $2\frac{1}{5}$. This is really $2+\frac{1}{5}$.

For fun, let's write the number 2 as a fraction with denominator five:

$$2=\frac{2}{1}=\frac{2\times 5}{1\times 5}=\frac{10}{5}$$

So the number $2\frac{1}{5}$ is:

$$2+\frac{1}{5}=\frac{10}{5}+\frac{1}{5}=\frac{11}{5}$$

We've written the mixed number $2\frac{1}{5}$ as the improper fraction $\frac{11}{5}$.

EXERCISE 49: Convert each of these mixed numbers back into proper fractions:

a) $3\frac{1}{4}$ b) $5\frac{1}{6}$ c) $1\frac{3}{11}$ d) $200\frac{1}{200}$

COMMENT: Students are often asked to memorize the names "proper fraction," "improper fraction" and "mixed number" so that they can follow directions on tests and problem sets.

But, to a mathematician, these names are not at all important!!

There is no "correct" way to express an answer (assuming, that the answer is mathematically the right number!).

Just decide for yourself as you do your mathematics which type of fraction would be best to work with as you do your task.

DIVIDING FRACTIONS

Here is a nasty problem: $7\frac{2}{3}$ *pies are to be shared among* $5\frac{3}{4}$ *girls. How many pies per individual girl does this yield?*

Technically, we could just write down the answer as $\dfrac{7\frac{2}{3}}{5\frac{3}{4}}$ and be done! (This is indeed the correct fraction for the problem!) Is there a way to make this look friendlier?

Recall the key fraction rule:

$$\boxed{\frac{xa}{xb}=\frac{a}{b}}$$

Let's multiply the numerator and denominator of our answer each by a convenient choice of number. Right now we have the expression:

$$\frac{7\frac{2}{3}}{5\frac{3}{4}}=\frac{7+\frac{2}{3}}{5+\frac{3}{4}}$$

Let's multiply by 3. (Why three?)

$$\frac{\left(7+\frac{2}{3}\right)\times 3}{\left(5+\frac{3}{4}\right)\times 3}=\frac{21+2}{15+\frac{9}{4}}$$

(Recall that $\frac{a}{b}\times b$ equals a.)

Let's now multiply numerator and denominator each by 4. (Why four?)

$$\frac{21+2\ \ \times 4}{\left(15+\frac{9}{4}\right)\times 4}=\frac{84+8}{60+9}=\frac{92}{69}$$

We now see that the answer is $\frac{92}{69}$. Sharing $7\frac{2}{3}$ pies among $5\frac{3}{4}$ girls is the same as sharing 92 pies among 69 girls!

As another example, consider $\dfrac{3\frac{1}{2}}{1\frac{1}{2}}$.

Multiplying the numerator and denominator each by 2 should be enough to simplify the expression:

$$\frac{3\frac{1}{2}}{1\frac{1}{2}}=\frac{3+\frac{1}{2}}{1+\frac{1}{2}}=\frac{\left(3+\frac{1}{2}\right)\cdot 2}{\left(1+\frac{1}{2}\right)\cdot 2}=\frac{6+1}{2+1}=\frac{7}{3}$$

EXERCISE 50: What fraction is $\dfrac{4\frac{2}{3}}{5\frac{1}{3}}$ in disguise?

EXERCISE 51: What fraction is $\dfrac{2\frac{1}{5}}{2\frac{1}{4}}$ in disguise?

EXERCISE 52: What fraction is $\dfrac{1\frac{4}{7}}{2\frac{3}{10}}$ in disguise?

EXERCISE 53: What fraction is $\dfrac{\frac{3}{7}}{\frac{4}{5}}$ in disguise?

Without realizing it, we have just learned how to divide fractions.

For example, let's compute $\frac{3}{5} \div \frac{4}{7}$. Recall, that a fraction is just a division problem and here we are being asked about sharing $\frac{3}{5}$ of a pie among $\frac{4}{7}$ of a girl(!). That is, we are being asked to compute:

$$\frac{\frac{3}{5}}{\frac{4}{7}}$$

Let's multiply numerator and denominator each by 5:

$$\frac{\frac{3}{5} \times 5}{\frac{4}{7} \times 5} = \frac{3}{\frac{20}{7}}$$

Let's now multiply top and bottom each by 7:

$$\frac{3 \times 7}{\frac{20}{7} \times 7} = \frac{21}{20}$$

Done!

Let's do another. Let's consider $\frac{5}{9} \div \frac{8}{11}$, that is:

$$\frac{\frac{5}{9}}{\frac{8}{11}}$$

Let's multiply top and bottom each by 9 and by 11 at the same time. (Why not?)

$$\frac{\frac{5}{9} \times 9 \times 11}{\frac{8}{11} \times 9 \times 11} = \frac{5 \times 11}{8 \times 9}$$

(Do you see what happened here?)

and so:

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{5\times 11}{8\times 9} = \frac{55}{72}$$

EXERCISE 54: Compute each of the following:

a) $\frac{1}{2}\div\frac{1}{3}$ b) $\frac{4}{5}\div\frac{3}{7}$ c) $\frac{2}{3}\div\frac{1}{5}$

EXERCISE 55: Compute $\frac{45}{45}\div\frac{902}{902}$. Do you see what the answer simply must be?

EXERCISE 56: Compute $\frac{10}{13}\div\frac{2}{13}$. Any general comments about this one?

THINKING EXERCISE:

Consider the problem $\frac{5}{12} \div \frac{7}{11}$.

Janine wrote:

$$\frac{\frac{5}{12}}{\frac{7}{11}} = \frac{\frac{5}{12} \times 12 \times 11}{\frac{7}{11} \times 12 \times 11} = \frac{5 \times 11}{7 \times 12} = \frac{5}{12} \times \frac{11}{7}$$

and then stopped before completing her final step.

a) Check each step of her work here and make sure that she is correct in what she did up to this point.

Janine then exclaimed: "Dividing one fraction by another is the same as multiplying the first fraction with the second fraction upside down."

b) Do you see what Janine means by this from her example?

c) Is she right? Is dividing two fractions always the same as multiplying the two fractions with the second one turned upside down? What do you think?

Work out $\frac{\frac{3}{7}}{\frac{4}{13}}$. Is the answer the same as $\frac{3}{7} \times \frac{13}{4}$?

Work out $\frac{\frac{2}{5}}{\frac{3}{10}}$. Is the answer the same as $\frac{2}{5} \times \frac{10}{3}$?

Work out $\frac{\frac{a}{b}}{\frac{c}{d}}$. Is the answer the same as $\frac{a}{b} \times \frac{d}{c}$?

THINKING EXERCISE:

Some teachers have students solve fraction division by rewriting expressions via a common denominator. For example, to compute:

$$\frac{3}{4} \div \frac{2}{3}$$

rewrite the problem as:

$$\frac{9}{12} \div \frac{8}{12}$$

The claim is then made that the answer to the original problem is $9 \div 8 = \frac{9}{8}$.

a) Does $\frac{3}{4} \div \frac{2}{3}$ indeed equal $\frac{9}{8}$?

b) Work out $\frac{5}{4} \div \frac{7}{9}$ via the method of this section, and then again by the method described above. Are the answers indeed the same?

Why do you think this "common denominator method" works?

THINKING EXERCISE:

Work out $\frac{12}{15} \div \frac{3}{5}$ and show that it equals $\frac{4}{3}$.

Now notice that

$$12 \div 3 = 4$$
$$15 \div 5 = 3$$

and

$$\frac{12}{15} \div \frac{3}{5} = \frac{4}{3}$$

Is this a coincidence or does $\frac{a}{b} \div \frac{c}{d}$ always equal $\frac{a \div c}{b \div d}$?

ALGEBRA CONNECTIONS
(for those with upper high school mathematics experience)

In an advanced algebra course students are often asked to work with complicated expressions of the following ilk:

$$\frac{\frac{1}{x}+1}{\frac{3}{x}}$$

We can make it look friendlier by following exactly the same technique of the previous section. In this example, let's multiply the numerator and denominator each by x. (Do you see why this is a good choice?) We obtain:

$$\frac{\left(\frac{1}{x}+1\right)\times x}{\left(\frac{3}{x}\right)\times x}=\frac{1+x}{3}$$

and $\frac{1+x}{3}$ is much less scary.

As another example, given:

$$\frac{\frac{1}{a}-\frac{1}{b}}{ab}$$

one might find it helpful to multiply the numerator and the denominator each by a and then each by b:

$$\frac{\left(\frac{1}{a}-\frac{1}{b}\right)\times a\times b}{ab\times a\times b}=\frac{b-a}{a^2b^2}$$

and for

$$\frac{\frac{1}{(w+1)^2}-2}{\frac{1}{(w+1)^2}+5}$$

it might be good to multiply top and bottom each by $(w+1)^2$:

$$\frac{\frac{1}{(w+1)^2}-2}{\frac{1}{(w+1)^2}+5}=\frac{1-2(w+1)^2}{1+5(w+1)^2}$$

EXERCISE 57: (OPTIONAL) Make each of the following expressions look less scary:

a) $\dfrac{2-\frac{1}{x}}{1+\frac{1}{x}}$

b) $\dfrac{\frac{1}{x+h}+5}{\frac{1}{x+h}}$

c) $\dfrac{1}{\frac{1}{a}+\frac{1}{b}}$

d) $\dfrac{\frac{1}{x+a}-\frac{1}{x}}{a}$

MULTIPLYING AND DIVIDING BY NUMBERS BIGGER AND SMALLER THAN ONE

People say that multiplying a quantity by a number bigger than one makes the answer bigger. Is this true?

For instance, $\frac{5}{4}$ represents more than one pie. Does multiplying 100, for example, by $\frac{5}{4}$ give an answer bigger than 100?

Well … Yes:

$$\frac{5}{4}\times 100=\frac{500}{4}=125$$

Does multiplying any number, let's call it X, by $\frac{5}{4}$ give an answer larger than X?

The answer is yes, and here it is good to write $\frac{5}{4}$ as a mixed number, $1\frac{1}{4}$, to see why. (Ah … mixed numbers are good for something!)

$$\begin{aligned}\frac{5}{4}\times X &= \left(1+\frac{1}{4}\right)X\\ &= 1\cdot X+\frac{1}{4}\cdot X\\ &= X+\ more\end{aligned}$$

Yes, the answer is bigger than X.

EXERCISE 58: Show that multiplying a number by $\frac{8}{5}$ is sure to give a larger answer.

EXERCISE 59: Show that multiplying a number by $\frac{20}{9}$ is sure to give a larger answer.

Does multiplying a quantity by a number smaller than one makes the answer smaller?

Consider $\frac{4}{5}$, for instance. This represents less than one pie. Does multiplying 100 by it give a smaller answer?

$$\frac{4}{5}\times 100 = \frac{400}{5} = 80$$

Yes!

Does multiplying any number X by $\frac{4}{5}$ give an answer smaller than X?

The answer is yes but we need to be tricky and write $\frac{4}{5}$ as a mixed number in an unusual way!

Notice that $\frac{4}{5}+\frac{1}{5}=1$, and so $\frac{4}{5}=1-\frac{1}{5}$. Thus:

$$\begin{aligned}\frac{4}{5}\times X &= \left(1-\frac{1}{5}\right)X \\ &= X-\frac{1}{5}\cdot X \\ &= \text{ smaller than } X\end{aligned}$$

EXERCISE 60: Show that multiplying a number by $\frac{7}{8}$ is sure to give a smaller answer.

EXERCISE 61: Show that multiplying a number by $\frac{5}{9}$ is sure to give a smaller answer.

Now let's consider <u>dividing</u> a number by a quantity smaller than one. For example, will 100 divided by $\frac{4}{5}$ give an answer smaller or larger than 100? Let's see:

$$\frac{100}{\frac{4}{5}} = \frac{100 \times 5}{\frac{4}{5} \times 5} = \frac{500}{4} = 125$$

The answer is larger!

In general:

$$\frac{X}{\frac{4}{5}} = \frac{X \times 5}{\frac{4}{5} \times 5} = \frac{5X}{4} = \frac{5}{4} \times X$$

and we know that $\frac{5}{4} \times X$ will be larger than X. (We did this two pages ago!)

EXERCISE 62: Show that dividing a number X by $\frac{7}{9}$ will give an answer larger than X.

EXERCISE 63: Show that dividing a number X by $\frac{8}{5}$ will give an answer <u>smaller</u> than X.

FRACTIONS INVOLVING ZERO AND NEGATIVE NUMBERS

Sharing zero pies among seven boys gives zero pie per boy: $\frac{0}{7}=0$. And it seems reasonable to say:

$$\frac{0}{a}=0$$

for any positive number a. But what if things are flipped the other way round? Does $\frac{a}{0}$ make sense? Can we give meaning to $\frac{0}{0}$?

One checks whether or not a division problem is correct by performing multiplication. For example:

$\frac{6}{2}=3$ is correct because 3 times 2 is indeed 6.

$\frac{20}{5}=4$ is correct because 4 times 5 is indeed 20.

$\frac{18}{\frac{1}{2}}=36$ is correct because 36 times $\frac{1}{2}$ is indeed 18.

A THINKING QUESTION:

a) Cyril says that $\frac{5}{0}$ equals 2. Why is he not correct?

b) Ethel says that $\frac{5}{0}$ equals 17. Why is she not correct?

c) Wonhi says that $\frac{5}{0}$ equals 887231243. Why is he not correct?

d) Duane says that there is no answer to $\frac{5}{0}$. Explain why he is correct.

A SECOND THINKING QUESTION:

Cyril says that $\frac{0}{0}$ equals 2.

Ethel says that $\frac{0}{0}$ equals 17.

Wonhi says that $\frac{0}{0}$ equals 887231243.

Why do they each believe that they are correct?

What might Duane say here?

To answer these questions …

Notice that if $\frac{5}{0}=2$, as Cyril says, then we should have that 2 times 0 is five, according to the check. This is not correct.

In fact, the check shows that there is no number x for which $\frac{5}{0}=x$.

On the other hand, Cyril says that $\frac{0}{0}=2$ and he believes he is correct because it passes the check: 2 times 0 is indeed zero. But so too does $\frac{0}{0}=17$ and $\frac{0}{0}=887231243$ pass the check! In fact, $\frac{0}{0}=x$ passes the check for <u>any</u> number x.

The trouble with $\frac{a}{0}$ (with a not zero) is that there are no meaningful values to assign to it, and the trouble with $\frac{0}{0}$ is that there are too many possible values to give it!

In general, most people would say that dividing by zero is simply too problematic to be done! They say it is best to avoid doing so and never will allow zero as the denominator of a fraction. (But all is fine with 0 as a numerator.)

Could a fraction have negative entries?

To answer this question one must assume that one has some familiarity with negative numbers. (If not, look at CHAPTER 4 of *THINKING MATHEMATICS!* Volume 1.)

Mathematically, "-2" represents the opposite of "2", in the sense that adding 2 and -2 together gives zero. If 2 represents "two pies" then -2 must represent "two anti-pies," magical things that cancel actual pies!

One can also think of -2 as -1×2 if that helps.

So … what might $\frac{-2}{3}$ mean? Well, this is the result of sharing two anti-pies to among three boys. Each boy then receives two thirds of an anti-pie.

What does $\frac{2}{-3}$ mean? We could try to interpret this as sharing two pies among three anti-boys … but that seems to be pushing things a bit. Perhaps the thing to do is to make use of the fraction rule:

$$\frac{xa}{xb}=\frac{a}{b}$$

with the belief that it should work with all types of numbers, including negative ones.

Let's take $\frac{2}{-3}=\frac{2}{-1\times 3}$ and multiply the denominator and numerator each by -1:

$$\frac{2}{-3}=\frac{2}{-1\times 3}=\frac{2\times -1}{-1\times 3\times -1}=\frac{-2}{3}$$

which is back to sharing two anti-pies to three boys.

QUESTION: What is $-\frac{2}{3}$?

We might guess that this would be the opposite of sharing two pies among three boys, which, we might say, is sharing two anti-pies among three boys: $-\frac{2}{3} = \frac{-2}{3}$.

The mathematics agrees:

$$-\frac{2}{3} = -1 \times \frac{2}{3} = \frac{-1}{1} \times \frac{2}{3} = \frac{-2}{3}$$

We have:

$\boxed{\frac{-2}{3} \text{ and } \frac{2}{-3} \text{ and } -\frac{2}{3} \text{ are the same quantity in different guises.}}$

People call writing $\frac{-a}{b}$ as $-\frac{a}{b}$, and writing $\frac{a}{-b}$ as $-\frac{a}{b}$, as "pulling out a negative sign."

EXERCISE 64:

a) What is $\frac{-a}{-b}$?

b) What is $\frac{-8}{9} \times \frac{2}{-5}$?

A BRIEF INTRODUCTION TO EGYPTIAN FRACTIONS

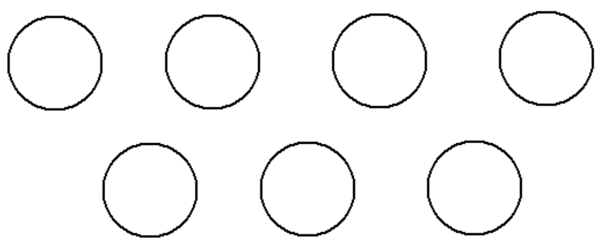
(See *THINKING MATHEMATICS!* Volume 1 for more.)

Scholars of ancient Egypt (ca. 3000 B.C.) were very practical in their approaches to mathematics and always sort answers to problems that would be of most convenience to the people involved. This led them to a curious approach to thinking about fractions.

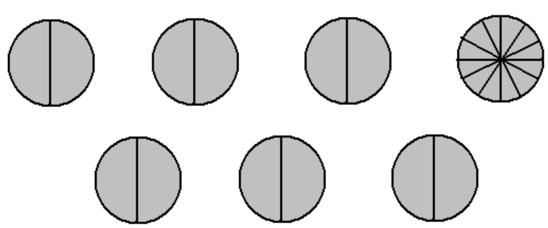
Consider the problem: *Share 7 pies among 12 boys.*

Of course, given our model for fractions, each boy is to receive the quantity "$\frac{7}{12}$" of pie. This answer has little intuitive feel.

But suppose we took this task as a very practical problem. Here are the seven pies:

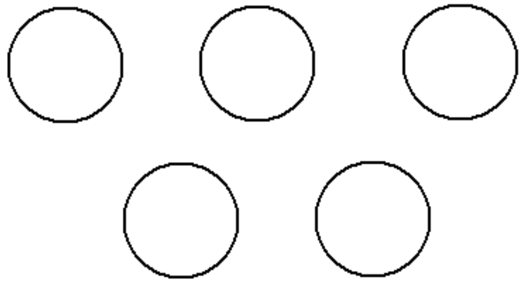


Is it possible to give each of the boys a whole pie? No. How about the next best thing – each boy half a pie? Yes! There are certainly 12 half pies to dole out. There is also one pie left over yet to be shared among the 12 boys. Divide this into twelfths and hand each boy an extra piece.



Thus each boy receives $\frac{1}{2}+\frac{1}{12}$ of a pie and it is indeed true that $\frac{7}{12}=\frac{1}{2}+\frac{1}{12}$.

EXERCISE 65:
a) How do you think the Egyptian's might have shared five pies among six girls?



b) How might they have shared 7 pies among 12 students?

The Egyptians insisted on writing all their fractions as sums of fraction with numerators equal to 1. For example:

$$\frac{3}{10} \text{ was written as } \frac{1}{4}+\frac{1}{20}$$

$$\frac{5}{7} \text{ was written as } \frac{1}{2}+\frac{1}{5}+\frac{1}{70}$$

That is, to share 3 pies among 10 students, the Egyptians said to give each student one quarter of a pie and one twentieth of a pie.

To share 5 pies among 7 students, the Egyptians suggested giving our half a pie, and one fifth of a pie, and one seventieth of a pie to each student.

EXERCISE 66: It is true that $\frac{4}{13}=\frac{1}{4}+\frac{1}{18}+\frac{1}{468}$. What does this say about how the Egyptians would have shared 4 pies among 13 girls?

The curious thing is that the Egyptians did not like to repeat fractions. Although it is obviously true that:

$$\frac{2}{5}=\frac{1}{5}+\frac{1}{5}$$

the Egyptians really did think it better to give each person receiving pie piece as large as possible, and so preferred the answer:

$$\frac{2}{5}=\frac{1}{3}+\frac{1}{15}$$

(even though it meant giving out a tiny piece of pie with that bigger piece).

EXERCISE 67: Consider the fraction $\frac{2}{11}$.

a) Show that $\frac{1}{5}$ is bigger than $\frac{2}{11}$.

b) Show that $\frac{1}{6}$ is smaller than $\frac{2}{11}$.

c) Work out $\frac{2}{11}-\frac{1}{6}$.

Use c) to write $\frac{2}{11}$ the Egyptian way.

EXERCISE 68: Consider the fraction $\frac{2}{7}$.

a) What is the biggest fraction $\frac{1}{N}$ that is still smaller than $\frac{2}{7}$?

b) Write $\frac{2}{7}$ the Egyptian way.

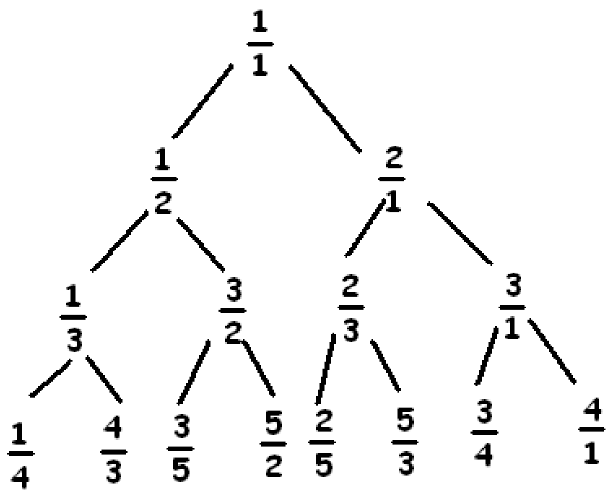
EXERCISE 69: CHALLENGE

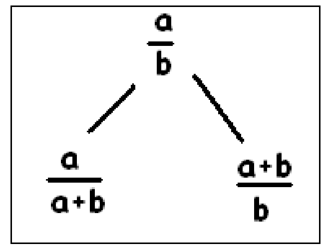
a) Write $\frac{17}{20}$ the Egyptian way.

b) Write $\frac{3}{7}$ the Egyptian way.

A CURIOUS FRACTION TREE

Here is something fun to think about. Consider the following "fraction tree:"





Do you see how it works? Do you see that each fraction has two "children"? The left child is always a number smaller than 1 and the right child is always a number larger than 1.

Do you see how the box to the upper right gives the method for computing the two children of the fraction?

a) Continue the drawing the fraction tree for another two rows.

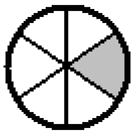
b) Explain why the fraction $\frac{13}{20}$ will eventually appear in the tree. (It might be easier to figure out what $\frac{13}{20}$s parent is by first noticing that $\frac{13}{20}$ is a "left child." What is its grandparent? What is its great grand parent?)

c) Explain why the fraction $\frac{13}{20}$ cannot appear twice in the tree.

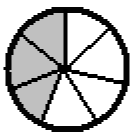
d) Will the fraction $\frac{456}{777}$ eventually appear in the tree? Could it appear twice?

SOLUTIONS

1.



2.

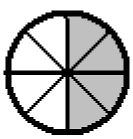


This really does represent the amount of pie an individual boy receives when 3 pies are shared among 7 boys.

3. 2 pies, 5 boys.   **4**. 4 pies, 5 boys   **5**. 4 pies, 7 boys

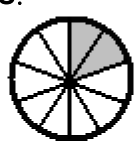
6. Yes. This is correct thinking.

7.



$\frac{4}{8}$ is the same answer as $\frac{1}{2}$

8.

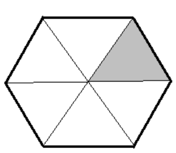


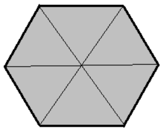
$\frac{2}{10}$ is the same answer as $\frac{1}{5}$

9. $\frac{1}{1}=1$ One pie per boy!   **10**. $\frac{5}{1}=5$ Five pies for one (lucky) boy!

11. $\frac{5}{5}=1$ One pie per boy.   **12**. 3 pies, 2 boys   **13**. 8 pies, 3 boys

14.

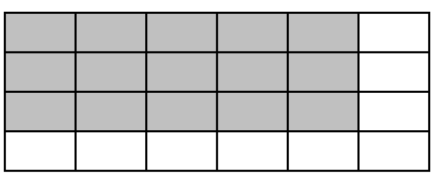
a) $\frac{1}{6}=$  It is easy to divide a hexagon into six equal parts.

b) $\frac{6}{6}=$  c) Octagon

15. 35 pies, 72 boys **16**. $\frac{a}{a}=1$ **17**. $\frac{1876}{1}=1876$

18. $\frac{0}{7}=0$ (Zero pie per boy.) **19**. 6

20. Five pies for each half = 10 pies for a whole boy
21. Four pies for each third = 12 pies for a whole boy
22. 1(E) 2(A) 3(A) 4(C) 5(B) 6(D)
23. This is actually correct thinking.
24. $\frac{400}{903}-\frac{170}{903}=\frac{230}{903}$ **25**. $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$ **26**. $\frac{2}{5}+\frac{37}{10}=\frac{4}{10}+\frac{37}{10}=\frac{41}{10}$
27. $\frac{1}{2}+\frac{3}{10}=\frac{5}{10}+\frac{3}{10}=\frac{4}{5}$ **28**. $\frac{2}{3}+\frac{5}{7}=\frac{14}{21}+\frac{15}{21}=\frac{29}{21}$ **29**. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{4}{8}+\frac{2}{8}+\frac{1}{8}=\frac{7}{8}$
30. $\frac{30}{100}+\frac{16}{100}+\frac{35}{100}+\frac{60}{100}+\frac{98}{100}=\frac{239}{100}$ **31**. $\frac{7}{10}-\frac{3}{10}=\frac{2}{5}$ **32**. $\frac{14}{20}-\frac{3}{20}=\frac{11}{20}$
33. $\frac{5}{15}-\frac{3}{15}=\frac{2}{15}$ **34**. $\frac{2}{35}-\frac{10}{35}+\frac{14}{35}=\frac{6}{35}$ **35**. $\frac{8}{16}-\frac{4}{16}-\frac{2}{16}-\frac{1}{16}=\frac{1}{16}$
36. $\frac{5}{9}=\frac{55}{99}$ and $\frac{6}{11}=\frac{54}{99}$ so $\frac{5}{9}$ is larger. **37**. $\frac{5}{8}$ **38**. $\frac{4}{9}$
39.



$\frac{3}{4}\times\frac{5}{6}=\frac{15}{24}$

40. $\frac{15}{80}$; $\frac{35}{121}$ **41**. a) $\frac{15}{56}$ b) $\frac{2}{7}$ c) $\frac{1}{6}$ d) 1!
42. a) $\frac{3\times1\times2}{4\times3\times5}=\frac{1\times2}{4\times5}=\frac{1\times1}{2\times5}=\frac{1}{10}$ b) $\frac{5}{5}\times\frac{7}{8}=1\times\frac{7}{8}=\frac{7}{8}$ c) 1!!! d) $\frac{1}{10}$
43. a) $\frac{6}{5}$ b) $\frac{34}{3}$ c) 2 d) 3 e) $\frac{36}{3}=12$ f) $13\times2=26$ g) $\frac{3}{5}$ h) $\frac{5\times4}{2\times10}=\frac{2}{2}=1$
44. $\frac{133}{112}\times224=133\times2=266$ **45**. a) 5 b) 7 c) a d) m
46. $\frac{18}{7}\times\frac{70}{36}=\frac{18\times70}{7\times36}=\frac{10}{2}=5$ **47**. $3\times\frac{14}{35}=\frac{42}{35}$

48. a) $5\frac{2}{3}$ b) $1\frac{3}{5}$ c) $7\frac{9}{13}$ d) $1\frac{1}{199}$ **49.** a) $\frac{13}{4}$ b) $\frac{31}{6}$ c) $\frac{14}{11}$ d) $\frac{40001}{200}$

50. $\frac{7}{8}$ **51.** $\frac{44}{45}$ **52.** $\frac{110}{161}$ **53.** $\frac{15}{28}$ **54.** a) $\frac{3}{2}$ b) $\frac{28}{15}$ c) $\frac{10}{3}$ **55.** $1 \div 1 = 1$!!!

56. $\dfrac{\frac{10}{13}}{\frac{2}{13}} = \frac{10}{2} = 5$ **57.** a) $\frac{2x-1}{x+1}$ b) $\frac{1+5\ \ x+h}{1} = 1+5\ \ x+h$ c) $\frac{ab}{b+a}$

d) $\frac{x-\ \ x-a}{ax\ \ x-a} = \frac{-a}{ax\ \ x-a} = -\frac{1}{x\ \ x-a}$

58. $\frac{8}{5} \cdot X = \left(1+\frac{3}{5}\right)X = X + \frac{3}{5}X = X + more$

59. $\frac{20}{9} \cdot X = \left(1+\frac{11}{9}\right)X = X + \frac{11}{9}X = X + more$

60. $\frac{7}{8} \cdot X = \left(1-\frac{1}{8}\right)X = X - \frac{1}{8}X = less\ than\ X$

61. $\frac{5}{9} \cdot X = \left(1-\frac{4}{9}\right)X = X - \frac{4}{9}X = less\ than\ X$

62. $\dfrac{X}{\frac{7}{9}} = \frac{9X}{7} = \frac{9}{7} \cdot X = more\ than\ X$ **63.** $\dfrac{X}{\frac{8}{5}} = \frac{5X}{8} = \frac{5}{8} \cdot X = less\ than\ X$

64. a) $\frac{a}{b}$ b) $\frac{16}{45}$

65. a) $\frac{1}{2} + \frac{1}{3}$ Half a pie and a third of a pie to each girl

b) $\frac{1}{2} + \frac{1}{12}$ Half a pie and a twelfth of a pie to each girl.

66. One quarter of a pie and one 18^{th} of a pie and one 468^{th} of a pie to each girl.

67. a) $\frac{1}{5} = \frac{11}{55}$ and $\frac{2}{11} = \frac{10}{55}$ so $\frac{1}{5}$ is larger. b) $\frac{1}{6} = \frac{11}{66}$ and $\frac{2}{11} = \frac{12}{66}$ so $\frac{1}{6}$ is smaller.

c) $\frac{2}{11} - \frac{1}{6} = \frac{12}{66} - \frac{11}{66} = \frac{1}{66}$ d) $\frac{2}{11} = \frac{1}{6} + \frac{1}{66}$

68. a) $\frac{1}{4}$ b) $\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$. (Other answers are possible.)

69. a) $\frac{17}{20} = \frac{1}{2} + \frac{1}{3} + \frac{1}{60}$ (Other answers are possible.) b) $\frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}$ (Other answers are possible.)

FRACTION TREE: Every reduced fraction does appear in the tree exactly once.

HONESTY STATEMENT: THE REAL REASON WHY FRACTIONS ARE SO HARD

PERSONAL COMMENTARY

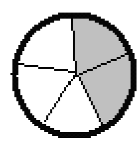
I have met several professional mathematicians who have expressed to me great admiration for K-8 teachers in their abilities to teach so-called "elementary" topics. They recognize that basic concepts are fundamentally hard, profoundly hard, and nigh-on impossible to actually pin down. What is a fraction? A mathematician will answer: "No clue!"

The approach I've taken in this packet actually illustrates the problem. It may seem that I have offered clarity and insight, but I haven't really. I've cheated a bit. Not because I wanted to, but because I was forced to.

Here are the discrepancies:

1. I started off by saying that a fraction is "simply" the answer to a division problem. In our case, the answer to a problem of sharing pies among boys. So in the opening sections of these notes, a fraction is some entity with units "pie per boy."

However, when I drew answers to division problems I just drew pie:

$\frac{2}{5} =$ 

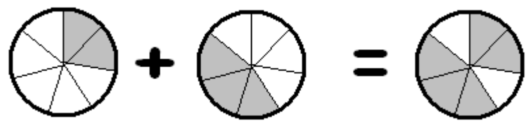
These pictures suggest that fractions are actually portions of pie. (That is, the answers are quantities with units being "pie", not "pie per boy.") All curricula do this and give the impression that a fraction is an amount of pie.

In my notes I loosely tried to cover this discrepancy by reminding the reader that these pictures are to be interpreted as "the amount of pie an individual boy receives." Already matters are a tad murky.

Thinking in terms of "pie per boy" was important in establishing the fundamental rule: $\frac{xa}{xb} = \frac{a}{b}$.

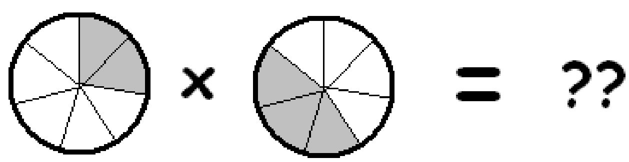
2. I was insistent that pictures of pie represented "amounts per boy" when it came to adding fractions. To make sense of the addition rule for fractions we have to point out that we are not adding pies and adding boys separately, but doing something more subtle: adding portions of pie individual boys receive. (Very confusing!)

Most curricula just have students combine pie (not "pie per boy") as the following picture plainly suggests:



We are caught between two interpretations.

3. When it comes to multiplying fractions, everything must be thrown out of the window! There is no meaning to multiplying portions of pie, or even multiplying portions of pie per boy.



So what is a fraction now if we are being asked to multiply them?

We are forced to switch models and now think of fractions as simply "portions." I chose to go with "portions of line segments" that allowed me to invoke an area model to make sense of multiplication.

Most people wouldn't think twice about it, but this switch of gear is fundamentally perturbing: Does a fraction have anything to do with pie or pies per boy or not? If the answer is that a fraction is more of an abstract concept that applies simultaneously to pies and boys and to portions of line segments, then what is that concept exactly? What is a fraction really?

The problem is that there is no single model that makes sense of fractions in all contexts.

And think about our poor young students. We keep switching concepts and models, and speak of fractions in each case as though all is naturally linked and obvious. All is <u>not</u> obvious and all <u>is</u> absolutely confusing.

So how are young students meant to have a firm grasp on fractions?

WHAT MATHEMATICIANS DO …

Mathematicians are honest and admit that there is no firm statement of what a fraction is. It is fundamentally an abstract concept that really cannot be pinned down with one concrete model.

On one level one can say--dryly--that a fraction is simply a pair of numbers a and b (with b not zero) written in the following form:

$$\frac{a}{b}$$

But even this is not quite right. A whole infinite collection of pairs of numbers represent the same fraction (for example, $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}=\cdots$). So a single fraction is actually a whole infinite class of pairs of numbers dubbed "equivalent."

This is a hefty shift of thinking: The notion of a "number" has changed from being a specific combination of symbols (for example, 23) to a whole class of combinations of symbols that are deemed equivalent.

Mathematicians then define the addition of fractions to be given by the daunting rule: $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ (obviously motivated by the "pies per boy" model), but must worry about proving that choosing different representations for $\frac{a}{b}$ and for $\frac{c}{d}$ lead to the same final answer. (For example, it is not immediately obvious that $\frac{2}{3}+\frac{4}{5}$ and $\frac{4}{6}+\frac{40}{50}$ give answers that are equivalent.)

They also define the product of fractions as: $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$ and again prove that all is consistent with different choices of representations.

Then mathematicians establish that the axioms of an arithmetic system hold with these definitions and carry on from there!

This is abstract, dry and not at all the best first encounter to offer students on the topic of fractions. And, moreover, this approach completely avoids the question as to what a fraction really means in the "real world." But it is the best one can do if one is to be completely honest. The definitions are certainly motivated by the type of work we did in this pamphlet, but in the end one can't explain why these rules are the way they are.

SO … what is a fraction, really?

Like I said … no clue!

And this lack of answer leaves teachers, sadly, in a very awkward position when working with students.