On Equivalent Words in the Free Group on Two Generators

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Abstract

We characterize minimal words in the free group on two generators and prove various results for words that are effectively "new" with respect to word length. Several properties of single-word equivalence classes are given. The size and behavior of equivalence classes is also discussed in relation to the automorphisms between words of an equivalence class.

1 Introduction

In 1936 J.H.C. Whitehead proved that if two words in a free group are equivalent under an automorphism, then they are equivalent under a finite sequence of a certain class of automorphisms [5],[6]. Furthermore he showed that the lengths of the words obtained after applying each such "Whitehead automorphism" in this sequence are strictly decreasing until the minimal length is attained, after which the automorphisms leave the length fixed. Whitehead's theorem has led to several studies of equivalent words. The following notation and definitions will be used in discussing these pursuits.

- $F_n = F(x_1, x_2, \ldots, x_n)$ denotes the free group on the generators $x_1, x_2, \ldots, x_n$ (in which no relation exists except the trivial one between an element and its inverse). We typically write the explicit generators of F_n as $a, b, \ldots$ and their inverses $\overline{a}, \overline{b}, \ldots$

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- A word is an element $w \in F_n$. The identity word is represented by 1.
- $w \sim v$ designates equivalence between two words $w, v \in F_n$ under some automorphism $S \in Aut\ F_n$.
- $|w|$ denotes the length of the word w (after any adjacent inverses are cancelled). The length of the identity word is 0.
- The set $\{a, b, \ldots, x_n, \overline{a}, \overline{b}, \ldots, \overline{x_n}\}$ of generators and their inverses is referred to as L_n.

Definition 1.1 *A* ***cycle*** *is an inner automorphism $S \in Aut\ F_n$. A* ***cyclic word*** *w represents the equivalence class of w under all cycles, i.e. the final $n \leq |w|$ letters of w can be fronted to obtain the same cyclic word. (The initial letter is thus adjacent to the final letter.)*

We begin studying equivalence classes by examining cyclic words, as this reduces the number of words we must consider. We also restrict ourselves to minimal words, for each word in F_n has a representation of minimal length.

Definition 1.2 *A cyclic word $w \in F_n$ is* ***minimal*** *if $|w| \leq |S(w)|\ \forall S \in Aut\ F_n$.*

Automorphisms that fix the length of a word are useful in finding equivalence classes of minimal words.

Definition 1.3 *An automorphism S is* ***level*** *on a word w if $|S(w)| = |w|$.*

Definition 1.4 *A* ***Whitehead Type I automorphism*** *is a permutation $S \in Aut\ F_n$ acting on L_n such that $S(\overline{x}) = \overline{S(x)}\ \forall x \in L_n$.*

We typically refer to Type I automorphisms simply as permutations.

Definition 1.5 *A* ***cyclic permutation*** *is a cycle composed with a permutation. A cyclic permutation of a word w is the image of w under a cyclic permutation. $[w]$ denotes a cyclic permutation of w.*

Using Type I automorphisms we now create a more expansive equivalence relation for words.

Definition 1.6 *A minimal word w is* ***reduced under cyclic permutations****, i.e.* ***RCP****, if, of all cyclic permutations $[w]$ of w, w itself appears first in the lexicographic ordering specified by $L_n = \{a, b, \ldots, x_n, \overline{a}, \overline{b}, \ldots, \overline{x_n}\}$.*

Note that we do not consider nonminimal words to be RCP.

Example 1.7 *In F_3, $aabcb\overline{c} = [\overline{c}\overline{a}\overline{b}\overline{a}b\overline{c}]$, and moreover $aabcb\overline{c}$ is RCP.*

Lau [1] notes the following.

Remark 1.8 *An RCP word begins with the string a^n, and furthermore this is the longest single-letter substring in the word.*

Definition 1.9 *Let $x \in L_n$ and $A \subset L_n$. $S = (A, x)$ represents a* ***Whitehead Type II automorphism****, defined on $y \in L_n$ as*

$$S(y) = \begin{cases} yx & \text{if} \quad y \in A,\ \overline{y} \notin A,\ y \notin \{x, \overline{x}\}, \\ \overline{x}y & \text{if} \quad y \notin A,\ \overline{y} \in A,\ y \notin \{x, \overline{x}\}, \\ \overline{x}yx & \text{if} \quad y, \overline{y} \in A, \\ y & \text{otherwise.} \end{cases}$$

Our definition of Type II automorphisms is slightly simpler than that used by past researchers because membership of x in A is immaterial in the actual images under S; we therefore omit the conditions $x \in A$ and $\overline{x} \notin A$. Generally, we take $x, \overline{x} \notin A$.

Definition 1.10 *A* ***one-letter automorphism*** *is a Type II automorphism $S = (A, x)$ with the set A containing only a single letter.*

Example 1.11 *The one-letter automorphism $(\{a\}, b)$ maps $a \to ab$ and $\overline{a} \to \overline{b}\overline{a}$ and leaves $b, \overline{b}$ fixed.*

Example 1.12 *Let $S = (\{a, b, \overline{a}, c\}, b) \in Aut\ F_3$. $S(ba\overline{c}) = b\overline{b}ab\overline{b}\overline{c} = a\overline{c}$. ($ba\overline{c}$ is therefore nonminimal.)*

We now give Whitehead's theorem.

Theorem 1.13 *(Whitehead) If $w, v \in F_n$ such that $w \sim v$ and v is minimal, then there exists a sequence $S_1, S_2, \cdots, S_m$ of Type I and Type II automorphisms such that $S_m \cdots S_2 S_1(w) = v$ and for $k \leq m$, $|S_k \cdots S_2 S_1(w)| \leq |S_{k-1} \cdots S_2 S_1(w)|$, with strict inequality unless $S_{k-1} \cdots S_2 S_1(w)$ is minimal.*

2 Minimality

Definition 2.1 *A* ***syllable*** *is a substring of a cyclic word. We write a word in* ***syllable form*** *as a list of its two-letter syllables, where the last syllable is composed of the last letter and the initial letter. Every letter in a word is thus a member of exactly two such syllables.*

Syllable form allows letter adjacencies to be counted easily, and it is useful in characterizing minimality. When considering the effect of an automorphism on the syllable representation of a word, only additions and cancellations *between* the two letters of each syllable are counted (to avoid redundancy).

Notation 2.2 *$(xy)_w$ denotes the number of occurrences of the two-letter syllables xy and $\overline{yx}$ in a word w. More generally, $(v)_w$ denotes the number of occurrences of the substrings v and v^{-1} in w.*

Definition 2.3 *An* **x-*string*** *is a syllable of a word w of the form x^n that is not a substring of x^{n+1} in w (i.e. its length is maximal). A minimal word is* ***alternating*** *if the length of its longest x-string is 1. $\lambda(w)$ denotes the length of the longest x-string in a cyclic word w.*

Example 2.4 *The syllable form of $w = aa\overline{b}\overline{b}\overline{a}bab\overline{b}$ is $(aa)\left(a\overline{b}\right)\left(\overline{b}\overline{b}\right)\left(\overline{b}\overline{a}\right)(\overline{a}b)(ba)\left(a\overline{b}\right)\left(\overline{b}a\right)$. One can determine, for example, that $(aa)_w = 1$, $(bb)_w = 1$, $(ab)_w = 1$, $(\overline{a}b)_w = 2$, and $(bb\overline{a}\overline{a})_w \equiv \left(aa\overline{b}\overline{b}\right)_w = 1$. Additionally $\lambda(w) = 2$.*

We now restrict ourselves to F_2 for the remainder of the paper and adopt the convention that $x, y \in L_2$, $x \notin \{y, \overline{y}\}$. By observing that each a-string and $\overline{a}$-string has as its neighbors b or $\overline{b}$ and that each b-string and $\overline{b}$-string likewise has as its neighbors a or $\overline{a}$, we prove the following theorem.

Theorem 2.5 *In a cyclic word w, $(xy)_w = (yx)_w$.*

Proof. Each occurrence of the syllables xy and $\overline{x}y$ in the syllable representation of w must be followed by either yx or $y\overline{x}$ (with possibly an intermediate y-string). Similarly each occurrence of the syllables $y\overline{x}$ and $\overline{y}\overline{x}$ in w must be followed by either $\overline{x}y$ or $\overline{x}\overline{y}$ (with possibly an intermediate $\overline{x}$-string). Thus, letting $\{xy\}_w$ be the number of occurrences of the syllable xy in w, we have

$$\begin{aligned} \{xy\}_w + \{\overline{x}y\}_w &= \{yx\}_w + \{y\overline{x}\}_w \\ \{y\overline{x}\}_w + \{\overline{y}\overline{x}\}_w &= \{\overline{x}y\}_w + \{\overline{x}\overline{y}\}_w . \end{aligned}$$

By definition, $\{xy\}_w + \{\overline{y}\overline{x}\}_w = (xy)_w$ and $\{yx\}_w + \{\overline{x}\overline{y}\}_w = (yx)_w$. We use these relations and add the above equations to obtain $(xy)_w = (yx)_w$. ■

Corollary 2.6 *If w is a cyclic word, then $(ab)_w = \left(\overline{a}\overline{b}\right)_w$ and $\left(a\overline{b}\right)_w = (\overline{a}b)_w$.*

The following theorem describes the effect of a one-letter automorphism on a word.

Theorem 2.7 *Let $S = (\{y\}, x)$ and $v = S(w)$, where w is a cyclic word. Then*

$$\begin{aligned} (yy)_v &= (y\overline{x}y)_w \\ (yx)_v &= (yx)_w + (yy)_w \\ (y\overline{x})_v &= (y\overline{x})_w - (y\overline{x}y)_w \\ (xx)_v &= (yx)_w - (yx\overline{y})_w + (xx)_w - (y\overline{x}\overline{x})_w . \end{aligned}$$

Proof. yy and $\overline{y}\overline{y}$ only appear in v as a result of cancellations in $y\overline{x}y$ and $\overline{y}x\overline{y}$ respectively in w. yx and $\overline{x}\overline{y}$ remain fixed under S but also arise from yy and $\overline{y}\overline{y}$ respectively. Similarly $y\overline{x}$ and $x\overline{y}$ remain fixed unless they appear in $y\overline{x}y$ and $\overline{y}x\overline{y}$ respectively. Finally, xx and $\overline{x}\overline{x}$ arise from yx and $\overline{x}\overline{y}$ respectively unless they appear in $yx\overline{y}$ and $\overline{y}\overline{x}y$ respectively but also stay fixed unless they appear $y\overline{x}\overline{x}$ and $xx\overline{y}$ respectively. ■

The following is a simplified version of a more general theorem (Theorem 7) given by Rapaport [2]. The left side of the inequality counts the number of cancellations in w under the automorphism $(\{y\}, x)$, while the right side counts the number of additions under that automorphism.

Theorem 2.8 *(Rapaport) w is minimal if and only if for all $x, y \in L_2$ with $x \notin \{y, \overline{y}\}$, $(y\overline{x})_w \leq (yx)_w + (yy)_w$.*

The following result characterizes level automorphisms.

Lemma 2.9 *$(\{y\}, x)$ is a level automorphism on w if and only if $(y\overline{x})_w = (yx)_w + (yy)_w$.*

Proof. The automorphism $(\{y\}, x)$ causes cancellations in w only in the syllables $y\overline{x}$ and $x\overline{y}$. The total number of cancellations is therefore $(y\overline{x})_w$. Similarly $(\{y\}, x)$ causes additions to w in the syllables yx, and $\overline{xy}$, yy, and $\overline{yy}$, totaling $(yx)_w + (yy)_w$. A level automorphism fixes the length of the word, so $(y\overline{x})_w = (yx)_w + (yy)_w$, and conversely this equality implies that $(\{y\}, x)$ is a level automorphism. ■

Remark 2.10 *The automorphism $(\{y, \overline{y}\}, x)$ cycles by one place words of the forms $y^e v \overline{x}^e$ and $x^e v \overline{y}^e$ for $e \in \{1, -1\}$ and leaves all other words fixed.*

Because two-letter automorphisms are cycles, it suffices to consider one-letter automorphisms when determining the minimality of a word in F_2. There are eight one-letter automorphisms, but each of these can be written as the product of a cycle and another one-letter automorphism:

Lemma 2.11 $(\{y\}, x) = (\{y, \overline{y}\}, x)(\{\overline{y}\}, \overline{x})$.

Therefore we need only consider the four automorphisms $(\{a\}, b)$, $\left(\{a\}, \overline{b}\right)$, $(\{b\}, a)$, and $(\{b\}, \overline{a})$ in characterizing minimal words. Sanchez [3] gives a characterization in the same vein of thought as the following, but the current presentation is a much easier test to implement in practice.

Theorem 2.12 *w is minimal if and only if $\left|(ab)_w - \left(a\overline{b}\right)_w\right| \leq \min\left((aa)_w, (bb)_w\right)$.*

Proof. By Rapaport's theorem, w is minimal if and only if $(ab)_w \leq \left(a\overline{b}\right)_w + (aa)_w$, $\left(a\overline{b}\right)_w \leq (ab)_w + (aa)_w$, $(ba)_w \leq (b\overline{a})_w + (bb)_w$, and $(b\overline{a})_w \leq (ba)_w + (bb)_w$. Combining these gives $\left|(ab)_w - \left(a\overline{b}\right)_w\right| \leq (aa)_w$ and $\left|(ba)_w - (b\overline{a})_w\right| \leq (bb)_w$. By Corollary 2.6, these yield the result. ■

Corollary 2.13 $(ab)_w \leq \min\left((aa)_w, (bb)_w\right)$ *for minimal words w with no inverses.*

Corollary 2.14 *If w and v are minimal words with the same initial letter, then wv is minimal.*

Proof. The inequality of Theorem 2.12 holds for w and for v, so it holds for wv by addition because the syllables included are preserved. ■

3 Root Words

A result of Corollary 2.14 was observed by Virnig [4], namely that if w is RCP, then aw is also RCP. This motivates the following definition.

Definition 3.1 *An RCP word v is a **descendant** of an RCP word w if $v = [a^n [w]]$, $n \geq 1$, where $[w]$ begins with an a-string.*

A descendant is simply a word obtained by increasing the length of an x-string in another word (and possibly applying a cyclic permutation to achieve RCP form). (As we consider descendants we also have ancestors, but words do not necessarily have unique ancestors. For example, $aabb\overline{aa}abb$ has ancestors $aabba\overline{bb}$ and $aabaa\overline{bb}$.) Descendants are necessarily minimal by Corollary 2.14.

We are interested, then, in characterizing words that cannot be obtained through descendancy. These will be the essentially new minimal words of a given length. Since they provide the basis of all other words we refer to them as "root" words.

Definition 3.2 *A **root word** is a minimal word that is not a descendant of any minimal word.*

The definition of root words is natural enough, but it is not immediately evident that they will be simple to work with. After all, in order to verify that a word is a root word from the definition we must check that shortening any x-string in the word does not result in a minimal word. However, there is a useful characterization of root words: They are simply the words for which Theorem 2.12 holds for equality.

Theorem 3.3 *w is a root word if and only if $\left|(ab)_w - \left(a\overline{b}\right)_w\right| = (aa)_w = (bb)_w$.*

Proof. By Theorem 2.12, we have $\left|(ab)_w - \left(a\overline{b}\right)_w\right| \leq (aa)_w$ and $\left|(ab)_w - \left(a\overline{b}\right)_w\right| \leq (bb)_w$. w is a root word if and only if decrementing $(aa)_w$ or $(bb)_w$ by shortening some a-string or b-string of length ≥ 2 in w causes w to lose minimality, so one of the inequalities fails under these conditions. Therefore both hold for equality if and only if w is a root word. ■

Corollary 3.4 *w is a root word if and only if $|(yx)_w - (y\overline{x})_w| = (xx)_w = (yy)_w$ for any $x, y \in L_2$, $x \notin \{y, \overline{y}\}$.*

From this characterization we can derive many properties of root words.

Corollary 3.5 *If w is a root word, then the weights of the generators in w are equal.*

Proof. The only two-letter syllables in w with unequal generator weights are aa, $\overline{aa}$, bb, and $\overline{bb}$, but by the previous theorem $(aa)_w = (bb)_w$. ■

Corollary 3.6 *w is a root word if and only if w^n is a root word.*

Proof. Multiplying w by itself preserves equality in the Theorem 3.3. Likewise, taking nth roots of w^n preserves equality. ■

Theorem 3.7 *If w is a root word, then $|w|$ is divisible by 4.*

Proof. $(aa)_w + (bb)_w + (ab)_w + \left(\overline{a}\overline{b}\right)_w + \left(a\overline{b}\right)_w + \left(\overline{a}b\right)_w = |w|$ because these syllables and their inverses constitute the set of two-letter syllables. By Corollary 2.5 and Theorem 3.3 this simplifies to $2\,(aa)_w + 2\,(ab)_w + 2\left(a\overline{b}\right)_w = |w|$. By Theorem 3.3 we also have $(aa)_w = \left|(ab)_w - \left(a\overline{b}\right)_w\right|$, so $2\left|(ab)_w - \left(a\overline{b}\right)_w\right| + 2\,(ab)_w + 2\left(a\overline{b}\right)_w = |w|$. If $(ab)_w \geq \left(a\overline{b}\right)_w$ then $4\,(ab)_w = |w|$, and if $(ab)_w < \left(a\overline{b}\right)_w$ then $4\left(a\overline{b}\right)_w = |w|$. In either case $|w|$ is divisible by 4. ■

Remark 3.8 *If w is a minimal alternating word, then w contains $\overline{a}$ or $\overline{b}$.*

Proof. Assume w contains no instances of either $\overline{a}$ or $\overline{b}$. Then $w = (ab)^n$. The automorphism $\left(\{a\}, \overline{b}\right)$ maps $w \to a^n$, precluding the minimality of w. ■

We now show that all minimal alternating words are root words.

Theorem 3.9 *The following are equivalent:*

(1) w is a minimal alternating word.
(2) w is an alternating root word.
(3) All four one-letter automorphisms are level on w.

Proof. Assume (1). If w is a minimal alternating word, then $(aa)_w = (bb)_w = 0$, so $(ab)_w = \left(a\overline{b}\right)_w$ by Theorem 2.12. Therefore, by Theorem 3.3, w is a root word, so we have (2).

Assume (2) and let $S = (\{x\}, y)$. Because $(aa)_w = (bb)_w = 0$ and $(ab)_w = \left(a\overline{b}\right)_w$, we have $(xy)_w = (x\overline{y})_w$ and $(xx)_w = 0$ $\forall x, y \in L_2$, $x \notin \{y, \overline{y}\}$. Therefore the number of cancellations caused by S is equal to the number of additions, and the length of w does not change. Thus we have (3).

Let all one-letter automorphisms be level on w in accordance with (3). This implies $(ab)_w - \left(a\overline{b}\right)_w = (aa)_w$, $\left(a\overline{b}\right)_w - (ab)_w = (aa)_w$, $(ab)_w - \left(a\overline{b}\right)_w = (bb)_w$, and $\left(a\overline{b}\right)_w - (ab)_w = (bb)_w$ by Lemma 2.9, so $(aa)_w = (bb)_w = 0$ and $(ab)_w = \left(a\overline{b}\right)_w$. Therefore w is minimal and alternating, so (1) holds. ■

We can also say something about non-alternating root words.

Lemma 3.10 *If w is a non-alternating root word, then exactly two of the four one-letter automorphisms are level on w.*

Proof. From Theorem 3.3, $\left|(ab)_w - \left(a\overline{b}\right)_w\right| = (aa)_w = (bb)_w$. If $(ab)_w > \left(a\overline{b}\right)_w$, then $\left(\{a\}, \overline{b}\right)$ and $(\{b\}, \overline{a})$ are level automorphisms of w by Lemma 2.9. If $(ab)_w < \left(a\overline{b}\right)_w$, then $(\{a\}, b)$ and $(\{b\}, a)$ are level automorphisms of w. ■

The following result gives an upper bound on the length of x-strings in root words.

Lemma 3.11 *If w is a root word, then $\lambda(w) \leq \frac{|w|}{4} + 1$.*

Proof. Let w be a root word with $\lambda(w) > \frac{|w|}{4} + 1$. We can assume that the longest x-string is an a-string, so $(aa)_w \geq \frac{|w|}{4} + 1$. $(ab)_w + (a\bar{b})_w > |(ab)_w - (a\bar{b})_w| = (aa)_w \geq \frac{|w|}{4} + 1$, so substituting for $(aa)_w$ and $(ab)_w + (a\bar{b})_w$ into $2\left((aa)_w + (ab)_w + (a\bar{b})_w\right) = |w|$ (from the proof of Theorem 3.7) gives $|w| + 4 \leq |w|$. This is a contradiction. ∎

Furthermore, there exists a root word of length $4n$ that achieves $\lambda(w) = \frac{|w|}{4} + 1$, namely $a^{n+1}(ba)^{n-1}b^{n+1}$, which can be shown to be a root word by Theorem 3.3.

Of all properties of root words discussed this far, the following result perhaps most concretely justifies their study, for regardless of how they are represented (i.e. in RCP form or any other), a root word is still a root word.

Theorem 3.12 *If w is a root word, then all $v \sim w$ are root words for $|v| = |w|$.*

Proof. Let $S = (\{y\}, x)$ be a level automorphism on w. (We are only concerned with level automorphisms because if an automorphism S increases the length of w, then $S(w)$ is not minimal.) Let $v = S(w)$. By Theorem 2.7, $(yy)_v = (y\bar{x}y)_w$ and $(xx)_v = (yx)_w - (yx\bar{y})_w + (xx)_w - (y\bar{x}\bar{x})_w$, so $(yy)_v - (xx)_v = (y\bar{x}y)_w - (yx)_w + (yx\bar{y})_w - (xx)_w + (y\bar{x}\bar{x})_w$, which simplifies by $(y\bar{x})_w = (y\bar{x}y)_w + (y\bar{x}\bar{y})_w + (y\bar{x}\bar{x})_w$ to $(yy)_v - (xx)_v = (y\bar{x})_w - (yx)_w - (xx)_w$. By $(yy)_w = (xx)_w$ from Corollary 3.4 and $(y\bar{x})_w = (yx)_w + (yy)_w$ from Lemma 2.9, $(y\bar{x})_w - (yx)_w - (xx)_w = 0$, so $(yy)_v = (xx)_v$. By Theorem 2.7, $|(yx)_v - (y\bar{x})_v| = |(yx)_w + (yy)_w - (y\bar{y}\bar{x})_w + (y\bar{x}y)_w| = |(yx)_w - (yx)_w + (y\bar{y}\bar{x})_w - (y\bar{x})_w + (y\bar{x}y)_w|$ $= |(y\bar{x}y)_w| = (yy)_v$. Therefore, by Corollary 3.4, v is a root word. Root words thus remain root words under level one-letter automorphisms. By Whitehead's Theorem, each such $v \sim w$ is connected to w by a chain of one-letter automorphisms, cycles, and permutations that leaves the length of w unchanged. Therefore each v is a root word. ∎

4 Equivalence Classes

Throughout this section, the term "equivalence class" is taken to mean "equivalence class of RCP elements." The size of the equivalence class of a word w is therefore the number of RCP words equivalent to w.

Lemma 4.1 *If $(\{y\}, x)$ is a level automorphism on a minimal word w, then the following automorphisms are also level on w:*

(1) $(\{y\}, \bar{x})$ if and only if $(xx)_w = 0$,
(2) $(\{x\}, y)$ if and only if w is a root word,
(3) $(\{x\}, \bar{y})$ if and only if w is an alternating root word.

Proof. Since $(\{y\}, x)$ is level on w, we have (✠) $(y\bar{x})_w = (yx)_w + (yy)_w$ by Lemma 2.9.

Suppose $(\{y\}, \bar{x})$ is level on w. Then $(yx)_w = (y\bar{x})_w + (yy)_w$ by Lemma 2.9. Adding this to (✠) yields $(yy)_w = 0$, and reversing this argument gives (1).

Suppose $(\{x\}, y)$ is level on w. Then $(x\bar{y})_w = (xy)_w + (xx)_w$ by Lemma 2.9. We use Theorem 2.7 to obtain $(y\bar{x})_w = (yx)_w + (xx)_w$. Subtracting this from (✠) yields $(yy)_w = (xx)_w$, and we have $|(y\bar{x})_w - (yx)_w| = (yy)_w$ from (✠), so w is a root word. Reversing this argument gives (2).

Suppose $(\{x\}, \overline{y})$ is level on w. Then we have $(xy)_w = (x\overline{y})_w + (xx)_w$ by Lemma 2.9. We use Theorem 2.7 to obtain $(yx)_w = (y\overline{x})_w + (xx)_w$. Adding this to ($\maltese$) yields $(yy)_w = (xx)_w$, and $|(y\overline{x})_w - (yx)_w| = (yy)_w$ from ($\maltese$), so w is a root word. Since $(yy)_w = (xx)_w$, $(yx)_w = (y\overline{x})_w + (yy)_w$. Adding this to ($\maltese$) yields $(yy)_w = 0$, so w is alternating. Conversely, if w is alternating, then all one-letter automorphisms are level on w. Therefore (3) holds. ∎

Remark 4.2 *If $S = (\{y\}, x)$ is a level automorphism on a minimal word w, then $(\{y\}, \overline{x})$ is level on $S(w)$.*

Conjecture 4.3 *If w is not a root word, then there exists $v \sim w$ such that $|v| = |w|$, $(aa)_v \neq 0$, and $(bb)_v \neq 0$.*

If this conjecture is true, then we are able to show that all words in a non-root word class are connected by a single one-letter automorphism.

Corollary 4.4 *If w is not a root word and has equivalence class size ≥ 2, then for all $v \sim w$ there exists some one-letter automorphism $S = (\{y\}, x)$ and integer n such that $v = [S^n(w)]$.*

Proof. Let w' be the word in the equivalence class with $(aa)_{w'} \neq 0$ and $(bb)_{w'} \neq 0$ given by the previous conjecture. There is at most one level automorphism on w' by Lemma 4.1. Call this automorphism $S = (\{y\}, x)$. Let $v_1 \sim w'$ such that $v_1 = TS'(w')$ for some one-letter automorphism S' and some cyclic permutation T. Then $S' = S$ because S is the only level automorphism on w'. Inductively, suppose $v_n \sim w'$ such that $v_n = S^n(w')$ for some n. Assume $v_{n+1} \sim v_n$ by a single one-letter automorphism followed by a cyclic permutation. By Lemma 4.1, the only possible level one-letter automorphisms on v_n are S and $S^{-1} = (\{y\}, \overline{x})$, so $v_{n+1} = T_{n+1}S^{n\pm1}(w')$ by Remark 4.2. It follows that for any $v \sim w'$ $\exists n \geq 0$ such that $v = T_nS^n(w')$. Therefore given a non-root word w we can reach any $v \sim w$ by some integer power of S along the chain we have established with w' as an "endpoint." If v happens to follow w in this chain then $n > 0$, and if v precedes w then $n < 0$. ∎

Next we prove some properties of root word classes, culminating in a bound on the size of any root word class. First, however, we require a few lemmas:

Lemma 4.5 *Let $S = (\{y\}, x)$ and let w be a minimal alternating word. $S(w) = w$ if and only if $S(w)$ is alternating.*

Proof. If $S(w) = w$, then trivially $S(w)$ is alternating.

Suppose $S(w)$ is alternating. Then $(yy)_{S(w)} = 0$, so by Theorem 2.7, $(y\overline{x}y)_w = 0$. The only two-letter syllables in w that cause cancellations under S are $y\overline{x}$ and $x\overline{y}$. Since $(y\overline{x}y)_w = 0$, every $y\overline{x}$ appears in $y\overline{xy}$ and every $x\overline{y}$ appears in $yx\overline{y}$, but $S(y\overline{xy}) = y\overline{xy}$ and $S(yx\overline{y}) = yx\overline{y}$. Therefore S causes no cancellations in w and no additions (since by Theorem 3.9 all automorphisms on w are level), so $S(w) = w$. ∎

Lemma 4.6 $(\{x\}, \overline{y})(\{y\}, x) = (\{x, \overline{x}\}, \overline{y})\, S(\{x\}, y)$, *where S is the automorphism that maps $x \to \overline{y}$ and $y \to x$.*

Proof. The two automorphisms can be shown to be equal by listing the images of the letters x and y and their inverses under each. ∎

Theorem 4.7 *Two equivalent root words are connected by a chain of at most two one-letter automorphisms.*

Proof. Let $v \sim w$ be root words connected by a chain of no fewer than 3 one-letter automorphisms S_i such that $TS_3S_2S_1(w) = v$, where $S_1 = (\{y\}, x)$, T is a cyclic permutation, and S_1, S_2, and S_3 are level automorphisms on w, $S_1(w)$, and $S_2S_1(w)$ respectively. We show that a contradiction results if S_2 is any of the four one-letter automorphisms given by Lemma 2.11.

If $S_2 = (\{y\}, \overline{x})$, then $S_2S_1(w) = w$ and we have $TS_3(w) = v$, so v and w are connected by a chain of a single one-letter automorphism.

If $S_2 = (\{x\}, \overline{y})$, then by Lemma 4.6 $TS_3S_2S_1(w) = TS_3(\{x, \overline{x}\}, \overline{y})T_1(\{x\}, y)(w)$, where T_1 maps $x \to \overline{y}$ and $y \to x$. Let $S_3 = (\{r\}, s)$ for some $r, s \in L_2$, $r \notin \{s, \overline{s}\}$, and let $S_3' = \left(\left\{T_1^{-1}(r)\right\}, T_1^{-1}(s)\right)$. Then $S_3 = T_1S_3'T_1^{-1}$. We ignore the cycle $(\{x, \overline{x}\}, \overline{y})$ because we are operating on cyclic words, so substituting for S_3 gives $TS_3S_2S_1(w) = T\left(T_1S_3'T_1^{-1}\right)T_1(\{x\}, y)(w) = TT_1S_3'(\{x\}, y)(w)$. Therefore v and w are connected by a chain of two one-letter automorphisms.

Suppose $S_2 = (\{y\}, x)$. By Lemma 4.1 $(yy)_{S_1(w)} = 0$ because $(\{y\}, \overline{x})$ is also level on $S_1(w)$ by Remark 4.2. Since $S_1(w)$ is a root word, $S_1(w)$ is an alternating word. $S_2S_1(w) \neq S_1(w)$ because otherwise we would not need S_2, so by Lemma 4.5 $S_2S_1(w)$ is not alternating. There are only two possibilities for S_3 by Lemma 4.1, namely $(\{y\}, \overline{x})$ and $(\{x\}, \overline{y})$, because $S_2S_1(w)$ is not alternating. As shown above for $S_2 = (\{y\}, \overline{x})$ and for $S_2 = (\{x\}, \overline{y})$, in either case we can decrease the length of the chain of one-letter automorphisms between $S_1(w)$ and $TS_3S_2S_1(w)$.

The reasoning used above to show that $S_2S_1(w)$ is not alternating for $S_2 = (\{y\}, x)$ is valid also for $S_2 = (\{x\}, y)$. The two possibilities for S_3 are $(\{x\}, \overline{y})$ and $(\{y\}, \overline{x})$: If $S_3 = (\{x\}, \overline{y})$ then $S_3S_2S_1(w) = S_1(w)$, and if $S_3 = (\{y\}, \overline{x})$ then by Lemma 4.6 $TS_3S_2S_1(w) = T(\{y, \overline{y}\}, \overline{x})T_2(\{y\}, x)S_1(w) = TT_2(\{y\}, x)S_1(w)$. Therefore there are fewer than three one-letter automorphisms in the chain between w and v. ∎

We now show that an equivalence class cannot have two distinct alternating RCP words.

Theorem 4.8 *There is at most one alternating RCP word in an equivalence class.*

Proof. Suppose w and v are distinct alternating RCP words in an equivalence class. By Theorem 3.9 w and v are root words, and by Theorem 4.7 they are connected by a chain of at most 2 one-letter automorphisms. There must be more than one one-letter automorphism in the chain by Lemma 4.5, so we have $TS_2S_1(w) = v$, where $S_1 = (\{y\}, x)$, T is a cyclic permutation, and S_1 and S_2 are level automorphisms on w and $S_1(w)$ respectively. Since $S_1(w)$ is not alternating by Lemma 4.5, there are two possibilities for S_2 (by Lemma 4.1). If $S_2 = (\{y\}, \overline{x})$, then $w = v$, which is a contradiction. If $S_2 = (\{x\}, \overline{y})$, we use Lemma 4.6 to show that $TS_2S_1(w) = TT_1(\{x\}, y)(w)$ where T_1 maps $x \to \overline{y}$ and $y \to x$. This means that there is a single automorphism between S_2 and S_1, which is also a contradiction. ∎

This allows the following important result.

Theorem 4.9 *The size of a root word class is at most* 5.

Proof. Suppose there is no alternating word in a root word equivalence class W. We construct the graph G in which vertices are distinct RCP words $w \in W$ and two vertices w and v are adjacent if there is a single one-letter automorphism S such that $S(w) = [v]$. G is connected by Whitehead's Theorem, and the diameter of the graph is 2 because W is a root word class (by Theorem 4.7). Each vertex has degree at most 2 because, by Lemma 4.1, each word in W has at most 2 level automorphisms. Therefore there are at most 3 vertices in G (because the simple graph of three vertices and two edges is maximal under these conditions). Thus, there are at most 3 distinct RCP members in W.

It suffices, then, to consider root word classes W of size 5 that contain an alternating word. By the previous theorem, there is only one alternating word in W, and this alternating word has four level one-letter automorphisms by Theorem 3.9. We construct the graph G as above with maximum vertex degree 4 and only one vertex achieving degree 4. We add another vertex to G, retaining its connectedness and the maximum degree number of 4. We therefore connect the new vertex to a vertex whose degree is not 4, but this creates two vertices, the distance between which is 3. This is a contradiction by Theorem 4.7 because then there is a chain of no fewer than three one-letter automorphisms between two root words. Therefore there can be no more than 5 members in a root word class. ■

Conjecture 4.10 *The size of a root word equivalence class is either* 1, 2, 3, *or* 5.

This observation holds for words of length ≤ 16. From the proof of Theorem 4.9, a root word class with 4 members must contain an alternating word. Thus it would suffice to show that an alternating word cannot have exactly 3 distinct images under the four one-letter automorphisms.

5 Isolated Words

Equivalence classes consisting of a single RCP word are particularly amenable to analysis.

Definition 5.1 *A word is* ***isolated*** *if there is only one RCP element in its equivalence class.*

Isolated words have either no level automorphisms or every level one-letter automorphism has the effect of a cyclic permutation on that word (though the automorphism need not necessarily be a permutation).

Example 5.2 *The only level one-letter automorphisms on* $w = aabb\bar{a}\bar{b}\bar{a}\bar{b}$ *are* $S_1 = (\{a\}, \bar{b})$ *and* $S_2 = (\{b\}, \bar{a})$. $S_1(w) = a\bar{b}abb\bar{a}\bar{a}\bar{b}$ *and* $S_2(w) = aab\bar{a}\bar{b}\bar{a}\bar{b}\bar{b}$ *are cyclic permutations of* w*, so* w *is isolated.*

Remark 5.3 $S(w) = [w]$ *if and only if each syllable* v *of length* $n \leq |w| + 1$ *that appears in* w *satisfies* $(v)_w = (S(v))_{S(w)}$.

It is interesting to note the similarities between the characterizations of minimal words and root words. There is an additional statement we can make in the same general form, and these three can be combined into a "measure of minimality." If the inequality of Theorem 2.12 holds for equality then we have a root word, and if it holds strictly we have an isolated word. (It might be possible to define this measure in such a way that it is preserved under level automorphisms.)

Theorem 5.4 *If w satisfies $\left|(ab)_w - \left(a\overline{b}\right)_w\right| < \min\left((aa)_w, (bb)_w\right)$, then w and the descendants of w are isolated.*

Proof. We have $\left|(ab)_w - \left(a\overline{b}\right)_w\right| < (aa)_w$ and $\left|(ab)_w - \left(a\overline{b}\right)_w\right| < (bb)_w$, so there are no level automorphisms on w because otherwise equality would hold for one of these inequalities. Therefore w is isolated. The descendants of w are also isolated because increasing any x-strings in w only increases the right side of the inequality. ■

The words that satisfy the previous theorem have no level automorphisms. However, there are isolated words with level one-letter automorphisms. Such words are mapped to cyclic permutations of themselves by their level one-letter automorphisms. We now look at some conditions on these words, starting with those that are fixed by a level automorphism.

Lemma 5.5 *Let $S = (\{y\}, x)$. $S(w) = w$ if and only if $w = x^n$ or $w = \prod x^{m_i} y^e \overline{x}^{n_i} \overline{y}^e$, $e \in \{1, -1\}$.*

Proof. Note that S fixes the syllable counts of w. By Theorem 2.7,

$$\begin{aligned}(yy)_w &= (yy)_{S(w)} = (y\overline{x}y)_w \\ (yx)_w &= (yx)_{S(w)} = (yx)_w + (yy)_w .\end{aligned}$$

From these equations we have $(y\overline{x}y)_w = (yy)_w = 0$. Because for each $y\overline{x}\overline{x}y$ in w we obtain $y\overline{x}y$ in $S(w)$, $(y\overline{x}\overline{x}y)_w \leq (y\overline{x}y)_w = 0$. Generally $(y\overline{x}^n y)_w \leq (y\overline{x}^{n-1}y)_w$, so it follows inductively that $(y\overline{x}^n y)_w = 0$. For every y in w there is a cancellation under S because the length of w is fixed under S. $(\overline{y}\overline{x}^n\overline{y})_w = 0$ because $\overline{y}\overline{x}^n\overline{y}$ would cause additions that have no corresponding cancellations. Therefore either w contains no instance of y (so w is x^n) or $(y\overline{x}^n\overline{y})_w \neq 0$ so w is of the form $\prod x^{m_i} y^e \overline{x}^{n_i} \overline{y}^e$. If we apply S to w in these forms, we find $S(w) = w$. ■

Corollary 5.6 *There is no word w such that the effect of the one-letter automorphism $S = (\{y\}, x)$ on w is that of the permutation $x \to \overline{x}$ and $y \to \overline{y}$.*

Proof. If such an automorphism S exists, for some word w, then S fixes the syllable counts of w, and the rest of the proof of the previous lemma holds. However, this implies that $w = x^n$ or $w = \prod x^{m_i} y^e \overline{x}^{n_i} \overline{y}^e$. Since S fixes these words we reach a contradiction. ■

Lemma 5.7 *There is no word w such that the effect of the one-letter automorphism $S = (\{y\}, x)$ on w is that of the permutation $x^e \to y$ and $y \to x^e$, $e \in \{1, -1\}$.*

Proof. Suppose that the effect of S on w is that of the automorphism that maps $y \to x^e$ and $x^e \to y$. Then by Theorem 2.7 and because the syllables in $S(w)$ correspond to those in w, we have

$$\begin{aligned}(xx)_w &= (yy)_{S(w)} = (y\overline{x}y)_w \\ (yx)_w &= (yx)_{S(w)} = (yx)_w + (yy)_w \\ (y\overline{x})_w &= (y\overline{x})_{S(w)} = (y\overline{x})_w - (y\overline{x}y)_w .\end{aligned}$$

From these equations we have $(xx)_w = (y\overline{x}y)_w = (yy)_w = 0$. Therefore w is an alternating word, and thus $S(w)$ is as well. By Lemma 4.5 $S(w) = w$, which is a contradiction. ■

Lemma 5.8 *Let $S = (\{y\}, x)$ and $S(w) = [w]$. If the effect of S on w is that of the permutation $x \to y^e$ and $y \to \overline{x}^e$, $e \in \{1, -1\}$, then w is a root word that satisfies $(yx\overline{y})_w + (y\overline{x}\overline{x})_w = (xy\overline{x})_w + (\overline{x}yy)_w$.*

Proof. By Theorem 2.7 and because the syllables in $S(w)$ correspond to those in w, we have

$$
\begin{aligned}
&(1)\ \ (xx)_w = (yy)_{S(w)} = (y\overline{x}y)_w \\
&(2)\ \ (y\overline{x})_w = (yx)_{S(w)} = (yx)_w + (yy)_w \\
&(3)\ \ (yx)_w = (y\overline{x})_{S(w)} = (y\overline{x})_w - (y\overline{x}y)_w \\
&(4)\ \ (yy)_w = (xx)_{S(w)} = (yx)_w - (yx\overline{y})_w + (xx)_w - (y\overline{x}\overline{x})_w .
\end{aligned}
$$

From (3) and (1) we have $(xx)_w + (yx)_w = (y\overline{x})_w$. This implies that $(\{x\}, y)$ is a level automorphism, so w is a root word by Lemma 4.1. Also $(y\overline{x}y)_w = (xyx)_{S(w)}$ because S is a permutation. Because xyx appears in $S(w)$ whenever there is xy in w unless it appears in $xy\overline{x}$ and whenever there is yy in w unless it appears in $\overline{x}yy$, we deduce that $(xyx)_{S(w)} = (xy)_w - (xy\overline{x})_w + (yy)_w - (\overline{x}yy)_w$. $(yy)_w = (xx)_w$ because w is a root word, so $(y\overline{x}y)_w = (yx)_w - (xy\overline{x})_w + (xx)_w - (\overline{x}yy)_w$, which gives $(yx)_w = (xy\overline{x})_w + (\overline{x}yy)_w$ because $(xx)_w = (y\overline{x}y)_w$. Then from (3) we have $(yx\overline{y})_w + (y\overline{x}\overline{x})_w = (xy\overline{x})_w + (\overline{x}yy)_w$. ■

By the previous three lemmas we have the following theorem.

Theorem 5.9 *If a word w is isolated, then at least one of the following is true for some x, y and $e \in \{1, -1\}$:*

(1) w has no level automorphism.
(2) $w = x^n$.
(3) $w = \prod x^{m_i} y^e \overline{x}^{n_i} \overline{y}^e$.
(4) w is a non-alternating root word with $(yx\overline{y})_w + (y\overline{x}\overline{x})_w = (xy\overline{x})_w + (\overline{x}yy)_w$.
(5) $(\{y\}, x)$ has the effect of the mapping $x \to x^e$ and $y \to \overline{y}^e$ on w.

Proof. Note that (1), (2), and (3) are also sufficient conditions for isolation.

If (1) holds, we are done. If (1) does not hold, then w has a level automorphism $(\{y\}, x)$ that has the effect of a permutation on w. We then have six possibilities for the effect of $(\{y\}, x)$: $x \to x$ and $y \to y$, $x \to \overline{x}$ and $y \to \overline{y}$, $x \to x$ and $y \to \overline{y}$, $x \to \overline{x}$ and $y \to y$, $x^e \to y$ and $y \to x^e$, and $x \to y^e$ and $y \to \overline{x}^e$. We look at each case.

$x \to x$ and $y \to y$: By Lemma 5.5 $w = x^n$ or $w = \prod x^{m_i} y^e \overline{x}^{n_i} \overline{y}^e$ so (2) or (3) holds.

$x \to \overline{x}$ and $y \to \overline{y}$: By Corollary 5.6 this case never occurs.

$x^e \to y$ and $y \to x^e$: By Lemma 5.7, this case never occurs.

$x \to y^e$ and $y \to \overline{x}^e$: By Lemma 5.8, w is a root word that satisfies $(yx\overline{y})_w + (y\overline{x}\overline{x})_w = (xy\overline{x})_w + (\overline{x}yy)_w$. Therefore (4) holds.

$x \to x$ and $y \to \overline{y}$ or $x \to \overline{x}$ and $y \to y$: These cases fulfill (5). ■

If the following conjecture is in fact true, then case (5) above never occurs, allowing us to strengthen the previous theorem.

Conjecture 5.10 *There is no automorphism $S = (\{y\}, x)$ and word w such that the effect of S on w is that of the automorphism that maps $x \to x^e$ and $y \to \overline{y}^e$.*

If there is such an automorphism and a word w, we have $(yy)_w = (y\overline{x}y)_w$.

Remark 5.11 *The only isolated RCP alternating words are* $\left(ab\overline{a}\overline{b}\right)^n$.

Proof. Assume a minimal alternating word w is not of the form $\left[\left(ab\overline{a}\overline{b}\right)^n\right]$; then w contains a syllable of the form xyx. By Theorem 3.9 all one-letter automorphisms are level on w, so let $S = (\{x\}, \overline{y})$. $S(xyx) = xx$, so $S(w)$ has an x-string of at least length 2. Therefore $S(w) \neq [w]$, but since $S(w) \sim w$, w is not isolated. It follows that w must be of the form $\left[\left(ab\overline{a}\overline{b}\right)^n\right]$, the RCP representative of which is $\left(ab\overline{a}\overline{b}\right)^n$.

We now show that $\left(ab\overline{a}\overline{b}\right)^n$ is in fact isolated. Applying the four one-letter automorphisms considered as a result of Lemma 2.11 indeed gives back $\left(ab\overline{a}\overline{b}\right)^n$. ■

6 Observations and Conjectures

In this section we look at several conjectures and observations that may direct further research. The first was conjectured by Lau [1] for F_n, and it remains unproven.

Conjecture 6.1 *If words w and v are RCP with $|w| = |v|$ and $aw \sim av$, then $w \sim v$.*

This conjecture cannot be generalized to read "descendants of equivalent words are equivalent," for the simple case of $aba\overline{b}$ is a counterexample as $aaba\overline{b} \nsim aab\overline{a}b$ though both are descendants of $aba\overline{b}$.

Lau also conjectured that the number of equivalence classes of word length n is strictly increasing. Indeed this seems likely, as the number of isolated words increases with word length and there are infinitely many root words (because certain classes are predictable at each length $4m$).

Several patterns were observed in equivalence classes of words in the free group on two generators. In the spirit of word descendancy, we define descendancy of equivalence classes.

Definition 6.2 *An equivalence class V of words of length n is a* ***descendant*** *of an equivalence class W of words of length $m < n$ if each word $v \in V$ is a descendant of some word $w \in W$.*

Remark 6.3 *Not all equivalence classes are descendants of another equivalence class.*

This can be seen by tracing each of the words in the following equivalence class back to their root word ancestors.

Example 6.4 *Let* $V = \left\{aaabaabbb, aabaab\overline{a}bb, aabab\overline{a}b\overline{a}b\right\}$ *(class 9 : 81 as listed in Table 3 in the appendix). $aaabaabbb$ has ancestors in classes 8 : 37 and 8 : 38, and $aabaab\overline{a}bb$ has ancestors in classes 8 : 32 and 8 : 40. These ancestors are root words, so they are the ancestors of shortest length.*

Remark 6.5 *The equivalence classes of w and w^n have the same size because the level automorphisms on w have exactly analogous effects on w^n.*

The following observation gives some concept of equivalence class growth from words of length n to length $n+1$.

Conjecture 6.6 *Equivalence classes with no root words increase in size by at most one word from any generation to the next.*

For example, the following classes have the desired properties.

Remark 6.7 *For $n \geq 3$, the RCP words $a^{n-m}ba^e\overline{b}a^m\overline{b}a^eb$, $0 \leq m \leq \lfloor \frac{n}{2} \rfloor$, comprise a single equivalence class and are connected by a single automorphism, namely $(\{\overline{b}\}, a)$, for each $e \in \{1, -1\}$.*

The next theorem discusses the two types of words observed among the larger equivalence classes of each word length.

Theorem 6.8 *$w = a^nbba^{e_1m_1}\overline{b}a^{e_2m_2}b$ and $v = a^nba^{e_1m_1}\overline{b}\overline{b}a^{e_2m_2}b$, $n+m_1+m_2 \geq 5$, each have $n+1$ distinct RCP words in their equivalence class for each $e \in \{1, -1\}$, and these words can be obtained by repeatedly applying the automorphism $(\{\overline{b}\}, a)$ to w and to v.*

Proof. The sequence of words obtained by applying $(\{\overline{b}\}, a)$ to w is $a^nbba^{e_1m_1}\overline{b}a^{e_2m_2}b$, $a^{n-1}b\overline{a}ba^{e_1m_1}\overline{b}a^{e_2m_2}b$, $\ldots$, $ab\overline{a}^{n-1}ba^{e_1m_1}\overline{b}a^{e_2m_2}b$, $b\overline{a}^nba^{e_1m_1}\overline{b}a^{e_2m_2}b$. w has syllable representation $(aa)^{n-1}(ab)(bb)(ba^{e_1})(a^{e_1}a^{e_1})^{m_1-1}(a^{e_1}\overline{b})(\overline{b}a^{e_2})(a^{e_2}a^{e_2})^{m_2-1}(a^{e_2}b)(ba)$. $(ab)_w = 2$, $(a\overline{b})_w = 1$, $(aa)_w = n+m_1+m_2-3$, and $(bb)_w = 1$, so $(\{\overline{b}\}, a)$ is the only one-letter level automorphism by Lemma 4.1. The words in the sequence have syllable representation $(aa)^{n-1-k}(ab)(b\overline{a})(\overline{a}\overline{a})^k(\overline{a}b)(ba^{e_1})(a^{e_1}a^{e_1})^{m_1-1}(a^{e_1}\overline{b})(\overline{b}a^{e_2})(a^{e_2}a^{e_2})^{m_2-1}(a^{e_2}b)(ba)$, giving $(ab)_w = 2$, $(a\overline{b})_w = 2$, $(aa)_w = n+m_1+m_2-3$, and $(bb)_w = 0$. So $(\{\overline{b}\}, a)$ and $(\{\overline{b}\}, \overline{a})$ are the only one-letter level automorphisms on these words by Lemma 4.1. Since $(\{\overline{b}\}, \overline{a})(\{\overline{b}\}, a)$ is the identity, the sequence of words is closed under all level automorphisms. Therefore there can be no more than $n+1$ distinct members because the final word in the chain has only one level one-letter automorphism. All these words are distinct because the arrangements of b and $\overline{b}$ in w are such that cyclic permutations of w can be identified by the parity of the exponent of each b-string.

The same argument holds for v. ■

We now give another as yet unproven observation.

Conjecture 6.9 *After sufficiently many generations, all equivalence classes increase in size by one word every generation except the classes a^n, $a^{n-m}b\overline{a}^mb$, and $a^{n-m}ba^e\overline{b}\overline{a}^m\overline{b}a^eb$, $e \in \{1, -1\}$.*

The first of these exceptions is simply the string of as, which is always isolated. The second is discussed by Virnig [4], who proves that there are $\lfloor \frac{n}{2}+1 \rfloor$ RCP elements in the equivalence class of $a^{n-m}b\overline{a}^mb$. $a^{n-m}b\overline{a}^mb$ appears clearly in Table 2 in the appendix, slightly altering the sequence $5, 12, 17, 24, 67, 196, \ldots$ of stable equivalence classes of a given size for each word length. $a^{n-m}b\overline{a}^mb$ appears in those table in positions satisfying $n = \left\lfloor \frac{|w|}{2} \right\rfloor$, where n is the equivalence class size.

$a^{n-m}ba^e\overline{b}\overline{a}^m\overline{b}a^eb$ are more subtle in their perturbations, appearing in positions satisfying $n = \left\lfloor \frac{|w|}{2} \right\rfloor - 2$. Their effects can be observed by noticing patterns beginning in positions satisfying $n = \frac{|w|-5}{2}$ (for odd word lengths). Letting $Q(|w|, n)$ be the entry of Table 2 for words of odd length $|w| > 8$ and equivalence class size n, we observe that $Q\left(|w|, \frac{|w|-5}{2}\right) = Q\left(|w|+1, \frac{|w|-3}{2}\right) + 4 = Q\left(|w|+2, \frac{|w|-1}{2}\right) + 8 = Q\left(|w|+2, \frac{|w|+1}{2}\right) + 11$; i.e. after the first generation the number of classes decreases by 4, then by another 4 in the next generation, then by another 3, after which it becomes stable (with the exception of Virnig's class increasing the count by 1 for two generations). By these numbers we would expect to find equivalence classes predictably dropping off at each generation, and there is presumably more occurring in these patterns than can be accounted for simply by these exceptions, but the situation has not been sufficiently investigated.

Another pattern found in Table 2 concerns equivalence classes of size 5. For words of length 8 there are two such classes, but these are root word classes and they do not follow conventional rules; class 8 : 42 has as descendant classes 9 : 97 and 9 : 100, and class 8 : 43 has as descendants 9 : 98 and 9 : 99 (while class 9 : 101 arises from 8 : 36, which only has size 2). Thus their behavior precludes the existence of any classes of size 6 of words of length 9. Likewise, for words of length 12 there are 48 classes of size 5, but since 31 are root word classes, only 17 (plus Virnig's exception) have size 6 on words of length 13. However, this reasoning does not seem to hold generally: There are 859 classes of size 5 on words of length 16 and 380 of these are root word classes, but we have 448 rather than the expected 479 classes of size 6 of 17-letter words. It is not known why this is the case.

Interestingly, it appears that the same exceptions discussed above are also exceptions when looking at words with the longest x-strings among other classes of similar size and word length. Let us therefore turn to properties of $\lambda(w)$.

The following table gives the upper bound on $\lambda(w)$ (ignoring the aforementioned exceptions) in terms of $|w| > 8$ and the size of the equivalence class containing w. (a^n is isolated, so it appears in the first column.) Upon examination we notice that there is a Great Unconformity running southeast through the table of slope -2, to the right of which we have columns of numbers that are simply one fewer than the equivalence class size and to the left of which are sequences of numbers for each word length that decrease by 1 each term after the initial drop of 2.

Equivalence Class Size

$\lvert w \rvert$	1	2	3	4	5	6	7	8	9	10	11	12
9	6	4	3	3	0	0	0	0	0	0	0	0
10	7	5	4	3	4	0	0	0	0	0	0	0
11	8	6	5	4	4	5	0	0	0	0	0	0
12	9	7	6	5	4	5	6	0	0	0	0	0
13	10	8	7	6	5	5	6	7	0	0	0	0
14	11	9	8	7	6	5	6	7	8	0	0	0
15	12	10	9	8	7	6	6	7	8	9	0	0
16	13	11	10	9	8	7	6	7	8	9	10	0
17	14	12	11	10	9	8	7	7	8	9	10	11

$\max \lambda(w)$

The following conjecture numerically describes the Unconformity.

Conjecture 6.10 *Let w not be a member of one of the above exceptional classes, and let the size of the equivalence class of w be n. (Thus $n \leq w - 5$.) If $|w| > 3$ and $n = 1$, then $\lambda(w) \leq |w| - 3$. If $|w| > 8$ and $1 < n < \left\lfloor \frac{|w|}{2} \right\rfloor$, then $\lambda(w) \leq |w| - n - 3$. If $|w| > 8$ and $n \geq \left\lfloor \frac{|w|}{2} \right\rfloor$, then $n - 5 \leq \lambda(w) \leq n - 1$ and furthermore every equivalence class of size n contains at least one member w' with $\lambda(w') = n - 1$ and one member w'' with $\lambda(w'') = n - 5$.*

If this conjecture holds, we easily obtain the following.

Corollary 6.11 *If $n \geq \left\lfloor \frac{w}{2} \right\rfloor$, then $n - \lambda(w) \leq 5$.*

A similar table for $\min \lambda(w)$ shows features reminiscent of those seen above for $\max \lambda(w)$; on the right side of the Unconformity are uniform columns increasing with equivalence class size. The numbers to the left of the Unconformity, however, do not show the same patterns as those for $\max \lambda(w)$.

The next conjecture, bounding the number of automorphisms between two equivalent words, is suggested by words in F_2 of length ≤ 16.

Conjecture 6.12 *Let w and v be minimal words such that $w \sim v$ with $|w| > 8$, and let m be the minimal number of one-letter automorphisms S_i such that $TS_m \cdots S_2 S_1(w) = v$, where T is a cycle composed with a permutation. Then $m \leq |w| - 6$. Furthermore, the words $a^n bba\overline{b}ab$ and $b\overline{a}^n ba\overline{b}ab$ achieve this bound.*

Finally we give a conjecture that could be useful in counting the number of words equivalent to a given word.

Conjecture 6.13 *Let a word v have the same syllable representation as a word w with possible permutations of the sets $\{(aa), (\overline{aa})\}$ and $\left\{(bb), \left(\overline{bb}\right)\right\}$. Then $w \sim v$.*

7 Miscellaneous Results

In this section we give some results that emphasize specific words and may find applications in studies of words not concentrating on equivalency and the topics addressed in this paper.

Remark 7.1 *$a^m b^n$ is RCP for $m \geq n \geq 2$.*

Proof. $w = a^2 b^2$ is minimal by Theorem 2.12. We can increase the length of the a-string or b-string by prepending a^k to a cyclic permutation of w. $a^m b^n$ is minimal by Corollary 2.13, and if $m \geq n$ then it is RCP as well. ■

Remark 7.2 *$a^m ba^n \overline{b}$ is RCP for $m \geq n \geq 1$.*

Proof. There are only two automorphisms that introduce cancellations in $a^m ba^n \overline{b}$, namely $(\{b\}, \overline{a})$ and $\left(\left\{\overline{b}\right\}, \overline{a}\right)$. Each of these will cancel one a and introduce one new a, so $a^m ba^n \overline{b}$ is minimal. $a^m ba^n \overline{b}$ is RCP because $m \geq n$. ■

Remark 7.3 *If w is RCP, then w^n is RCP.*

Proof. By Corollary 2.14, w^n is minimal. Let S be a nontrivial cyclic permutation for which $S(w^n)$ precedes w^n in a lexicographic ordering. It follows that $S(w)$ precedes w in such an ordering, and this is a contradiction. ■

Remark 7.4 *$w \sim v$ if and only if $w^n \sim v^n$ for minimal words v and w.*

Proof. Let $S(w) = v$. Then $S(w^n) = S(w)^n = v^n$. Similarly $S(w^n) = v^n$ implies $S(w) = v$. ■

The following gives a sufficient condition for minimality drawing from the lengths of the x-strings of a word.

Theorem 7.5 *If $(xx)_w \geq \frac{|w|}{4}$ and $(yy)_w \geq \frac{|w|}{4}$ for a word w, then w is minimal.*

Proof. We have $(xx)_w + (yy)_w \geq \frac{|w|}{2}$, so there are at most $\frac{|w|}{2}$ syllables in the word that are not counted by either $(xx)_w$ or $(yy)_w$. Since the other syllables come in pairs (by Theorem 2.5) we have $(xy)_w + (x\overline{y})_w < \frac{|w|}{4}$. Therefore $|(xy)_w - (x\overline{y})_w| \leq (xy)_w + (x\overline{y})_w < \frac{|w|}{4} \leq \min((xx)_w, (yy)_w)$, which shows the minimality of w. ■

The next theorem can be helpful in estimating the size of a generation given the previous one.

Theorem 7.6 *Let $\delta(w)$ be the number of cycles of $w \in F_n$ that can be permuted to obtain w and $\chi(w)$ be the number of x-strings in w. Then there are exactly $\frac{\chi(w)}{\delta(w)}$ distinct RCP descendants of w.*

Proof. List $\chi(w)$ cycles of w such that each x-string in w appears as the initial x-string of some word in the list. Prepend a to each; we obtain distinct words when the cycled word is distinct from both w and those cycled words preceding that in question. Since $\delta(w) = \delta([w])$ for any cyclic permutation $[w]$, we simply divide to obtain $\frac{\chi(w)}{\delta(w)}$ words. ■

Example 7.7 *$w = aab\overline{a}\overline{a}\overline{b}$. $\chi(w) = 4$. w can be cycled into $aab\overline{a}\overline{a}\overline{b}$, $\overline{b}aab\overline{a}\overline{a}$, $\overline{a}\overline{b}aab\overline{a}$, $\overline{a}\overline{a}\overline{b}aab$, $b\overline{a}\overline{a}\overline{b}aa$, and $ab\overline{a}\overline{a}\overline{b}a$, of which the first and fourth can be permuted back into w, so $\delta(w) = 2$. The two descendants guaranteed by the theorem are thus $aaab\overline{a}\overline{a}\overline{b}$ and $aabb\overline{a}\overline{a}\overline{b}$.*

$\delta(w)$ counts the symmetries of w; the prime factors of $\delta(w)$ must also be prime factors of some $m \leq n$ in F_n. Therefore there is no word in F_2 with three-fold symmetry, but there are words with two-fold symmetry, four-fold symmetry, etc.

Theorem 7.8 *The number of minimal alternating words w of length $4n$ is equal to $4N$, where N is the number of permutations v of $p^{2n}q^{2n}$.*

Proof. First we prove that the number of minimal alternating words w with initial letter a is equal to the number of permutations of the letters $p^{2n}q^{2n}$. We define the map $\phi : v \to w$. $\phi(p) \in \left\{(ab), (ba), (\overline{a}\overline{b}), (\overline{b}\overline{a})\right\}$ and $\phi(q) \in \left\{(a\overline{b}), (b\overline{a}), (\overline{a}b), (\overline{b}a)\right\}$, and let the first letter of w be a (so that if the first letter of v is p then the first syllable of w is (ab), and if the first letter of v is q then the first syllable of w is $\left(a\overline{b}\right)$). In mapping $v \to w$ we find the first syllable and then take successive syllables such that each syllable pair in the syllable representation of w is of the form $(xy)(yz)$ (i.e. they agree on the shared letter). q appears an even number of times in v, so the last syllable in w ends with a (not $\overline{a}$). Thus w is a cyclic word because the first and last syllables agree. Letting $\{ab\}_w$ count the number of occurrences of the syllable ab in w, we have $(p)_v = \{ab\}_w + \{ba\}_w + \left\{\overline{a}\overline{b}\right\}_w + \left\{\overline{b}\overline{a}\right\}_w = 2n$. This implies $(ab)_w + \left(\overline{a}\overline{b}\right)_w = 2n$, and by Corollary 2.5 we have $(ab)_w = \left(\overline{a}\overline{b}\right)_w$, so $(ab)_w = n$. Similarly we have $\left(a\overline{b}\right)_w = n$. $\left|(ab)_w - \left(a\overline{b}\right)_w\right| = 0 = (aa)_w = (bb)_w$, so w is minimal. Therefore ϕ is well-defined. ϕ is bijective because for any w we can obtain its unique preimage v by replacing each syllable in its syllable representation with the corresponding letter p or q. Letting N be the number of permutations v, we multiply N by 4 to account for the cases of $\overline{a}$, b, and $\overline{b}$ as initial letters of w. ■

We have thus counted the number of alternating words.

Corollary 7.9 *The number of minimal alternating words w of length $4n$ is $4\binom{4n}{2n}$.*

Conjecture 7.10 *The number of RCP alternating words w of length $4n$ is equal to the number of RCP permutations v of $p^{2n}q^{2n}$.*

This result would greatly simplify the counting of RCP alternating words.

Another endeavor to explore is the counting of root words. Using the characterization of root words one can set up a system of equations, the solutions to which correspond to distinct root words. Given an arrangement of syllables in the syllable form of w, excluding the m syllables of the form xx, Conjecture 6.13 can be used to count distinct root words, for the number of such permutations is the number of order-dependent partitions of the integer m (this number being 2^{m-1}).

8 Programs

Among the code used in the research presented in this paper is Lau's collection of *Maple V* functions [1]. It was this code that provided the concepts underlying our own *Mathematica 4.1* code, and indeed some of Lau's functions were ported directly. The functions described below are available at ftp://ftp.math.orst.edu/publications/garity/REU along with the data they have produced.

8.1 MakeWords

It is the belief of the authors that the reason some of the results appearing in this paper were not articulated before is that not enough data had been created. In 1998 [4], for example, only words of length ≤ 8 had been tabled (and these not without omissions), while the results have now been extended past words of twice that length. (Ironically it is exactly at words of length 9 that many of the trends begin, providing not only counterexamples to past conjectures but also definite clues as to the global behavior of equivalence classes.) Aside from the obvious advances in the computational ability of common machines achieved in the interim, the primary reason for the inadequate volume of data appears to be the lack of a good method for initially producing a list of candidate RCP words. Once this is done it is not terribly time-consuming to determine the set of equivalence classes of the words for lengths up to about 14, but generating the first list has been a problem. The MakeWords function, then, is possibly our most important contribution to the programs involved in this project and is certainly responsible for generating the large word tables that have been achieved.

The implementation of MakeWords is such that the words output for a given length are, if minimal, fairly good candidates for RCP words. Beginning with an initial a-string, MakeWords recursively calls itself until the words have reached a specified length, adding to each existing word up to three additional letters depending on the properties of that word and its final letter at each particular stage. For example, both b and $\overline{b}$ are added to aba to obtain $abab$ and $aba\overline{b}$ as initial substrings of viable RCP words, but not $\overline{a}$ because of the imminent cancellation and not a because the longest x-string would then not appear at the beginning of the word.

8.2 EquivalentWords

Most of the functions involved in testing the equivalence of two words were adapted from Lau's implementation [1]. The function EquivalentWH2List returns a list of all minimal words that are equivalent to a given word under a single Whitehead Type II automorphism. These words are obtained by applying cycles to the word and also by applying the four Whitehead Type II automorphisms given by Lemma 2.11 to the word. EquivalentWords takes this list of words and repeatedly applies EquivalentWH2List to each new word from the previous iteration until no new words are created, at that point returning a list of plausibly RCP words equivalent to the given RCP word.

8.3 DetermineEquivalenceClasses

After a list of RCP words is produced, it is necessary to partition the list into equivalence classes. DetermineEquivalenceClasses does just this in a sequence of steps. First it removes words that it can determine are isolated, immediately creating equivalence classes for them. As most words of any length are isolated, this process greatly reduces the computation time. DetermineEquivalenceClasses then partitions the remaining words by the weights of the generators, using Rapaport's result [2] that if two minimal words are equivalent then the weights of the generators are equal up to permutations. Once this is done, each set of words is sent to SortByEquivalenceClasses, which performs the actual comparisons needed to construct the equivalence classes. Specifically, all words equivalent to the first word in the set are removed and stored as a class until none remain in the original list. Finally each equivalence class in the list of classes is sorted lexicographically, and the list of equivalence classes is sorted by size.

8.4 Offspring

Offspring is a function written to assist in the study of equivalence classes over several generations of words. It prepends a to viable cyclic permutations of a word, effectively obtaining descendants. These descendants can be found in a table (such as Table 3 in the appendix) to chart the relationships between an equivalence class and its progeny.

9 Conclusion

We have given several results for cyclic words in the free group on two generators and have characterized minimality by using syllable representations. We have also given a characterization of root words, the fundamental words out of which others are constructed, and have proven that they occur only with lengths of multiples of four in minimal forms. Equivalence classes of minimal words have been studied with results regarding cross-generational relationships and various stable classes, and several unproven observations have been given with the hope that they are addressed.

Applications of the theory discussed in this paper most notably include the classification of curves on a once-punctured torus. In this setting, equivalent words as given in Table 3 in the appendix correspond to homotopic curves on the torus with curve generators a and b. Additionally Table 3 can be treated as a table for homotopic curves on the non-punctured torus by rewriting instances of the commutator $ab\bar{a}\bar{b} = 1$. The significance of "root curves" on the torus has not been examined extensively, but intuitively root words represent curves on the punctured torus out of which additional curves can be generated by "extra" repetitions of a^e or b^e loops. A topological reason why root curves are composed of $4m$ generators may lead to further insight.

A number of open questions still remain. Aside from determining the validity of the numerous conjectures given in Section 6, future research might generalize some of the results obtained here for free groups on more than two generators, specifically Theorems 2.5 and 2.7 and the characterization of root words. On F_n it is necessary to consider a larger class of automorphisms than the one-letter automorphisms that were sufficient here; indeed a study of the automorphisms will be interesting in itself. Initial data in these pursuits could conceivably be obtained by modifying MakeWords to provide the initial list of words for any number of generators.

10 Appendix

1. This table details the quantities of various types of words and equivalence classes for words of length ≤ 17. The column labeled "Original Words" is the number of words output by MakeWords, the second column is the number of these that are RCP, and the third column is the number of these that are non-equivalent. The remaining columns list the number of root words of length n and the number of these that are non-equivalent.

Word Length	Original Words	RCP Words	Equivalence Classes	Root Words	Root Word Classes
0	1	1	1	1	1
1	1	1	1	0	0
2	1	1	1	0	0
3	2	1	1	0	0
4	7	4	3	3	2
5	7	5	4	0	0
6	20	12	10	0	0
7	48	18	16	0	0
8	209	67	43	34	13
9	393	177	101	0	0
10	1118	489	340	0	0
11	3154	1164	911	0	0
12	9959	3588	2544	945	304
13	25283	10539	7224	0	0
14	71884	29898	22616	0	0
15	204128	79884	65376	0	0
16	597813	237981	187754	34832	11395
17	1657201	700161	545743	0	0

2. This table lists for root words of length $\leq$ 16 and for all words of length $\leq$ 17 the number of equivalence classes of each size. (The rightmost column in each table is reproduced from Table 1.)

Equivalence Class Size

$\|w\|$	1	2	3	4	5	6	7	8	9	10	11	12	Total
0	1												1
1	1												1
2	1												1
3	1												1
4	2	1											3
5	3	1											4
6	9	0	1										10
7	15	0	1										16
8	31	5	4	1	2								43
9	52	28	15	6									101
10	257	41	24	12	6								340
11	792	46	35	20	13	5							911
12	2076	78	293	31	48	13	5						2544
13	4711	1970	403	78	27	18	12	5					7224
14	17387	3796	1062	238	74	24	18	12	5				22616
15	55675	6445	2285	635	207	70	25	17	12	5			65376
16	159686	10303	15129	1448	859	203	67	25	17	12	5		187754
17	417137	110815	12926	3047	1045	448	199	68	24	17	12	5	545743

Root Word Equivalence Class Size

$\|w\|$	1	2	3	4	5	Total
0	1					1
4	1	1				2
8	2	5	4	0	2	13
12	5	19	249	0	31	304
16	12	89	10914	0	380	11395

3. This table contains all RCP words on two generators of length ≤ 9 sorted by equivalence class. Root words are designated by stars.

$0:1$	1 ★
$1:1$	a
$2:1$	aa
$3:1$	aaa
$4:1$	$aaaa$
$4:2$	$ab\bar{a}\bar{b}$ ★
$4:3$	$aabb$ ★ $aba\bar{b}$ ★
$5:1$	$aaaaa$
$5:2$	$aaba\bar{b}$
$5:3$	$aab\bar{a}\bar{b}$
$5:4$	$aaabb$ $aab\bar{a}b$
$6:1$	$aaaaaa$
$6:2$	$aaaba\bar{b}$
$6:3$	$aaabbb$
$6:4$	$aaab\bar{a}\bar{b}$
$6:5$	$aabaa\bar{b}$
$6:6$	$aaba\overline{bb}$
$6:7$	$aabba\bar{b}$
$6:8$	$aabb\bar{a}b$
$6:9$	$aab\overline{aa}\bar{b}$
$6:10$	$aaaabb$ $aaab\bar{a}b$ $aab\overline{aa}b$
$7:1$	$aaaaaaa$
$7:2$	$aaaaba\bar{b}$
$7:3$	$aaaabbb$
$7:4$	$aaaab\bar{a}\bar{b}$
$7:5$	$aaabaa\bar{b}$
$7:6$	$aaaba\overline{bb}$
$7:7$	$aaabba\bar{b}$
$7:8$	$aaabb\bar{a}b$
$7:9$	$aaabb\bar{a}\bar{b}$
$7:10$	$aaab\bar{a}bb$
$7:11$	$aaab\overline{aa}\bar{b}$
$7:12$	$aaab\bar{a}\overline{bb}$
$7:13$	$aabaa\overline{bb}$
$7:14$	$aabba\overline{bb}$
$7:15$	$aabb\overline{aa}\bar{b}$
$7:16$	$aaaaabb$ $aaaab\bar{a}b$ $aaab\overline{aa}b$
$8:1$	$aaaaaaaa$
$8:2$	$aaaaaba\bar{b}$
$8:3$	$aaaaabbb$
$8:4$	$aaaaab\bar{a}\bar{b}$
$8:5$	$aaaabaa\bar{b}$
$8:6$	$aaaaba\overline{bb}$
$8:7$	$aaaabba\bar{b}$
$8:8$	$aaaabbbb$
$8:9$	$aaaabb\bar{a}b$
$8:10$	$aaaabb\bar{a}\bar{b}$
$8:11$	$aaaab\bar{a}bb$
$8:12$	$aaaab\overline{aa}\bar{b}$
$8:13$	$aaaab\bar{a}\overline{bb}$
$8:14$	$aaabaaa\bar{b}$
$8:15$	$aaabaa\overline{bb}$
$8:16$	$aaaba\overline{bbb}$
$8:17$	$aaabbaa\bar{b}$
$8:18$	$aaabba\overline{bb}$
$8:19$	$aaabbba\bar{b}$
$8:20$	$aaabbb\bar{a}\bar{b}$
$8:21$	$aaabb\bar{a}bb$
$8:22$	$aaabb\overline{aa}b$
$8:23$	$aaabb\overline{aa}\bar{b}$
$8:24$	$aaabb\bar{a}\overline{bb}$
$8:25$	$aaab\overline{aa}bb$
$8:26$	$aaab\overline{aaa}\bar{b}$
$8:27$	$aaab\overline{aa}\overline{bb}$
$8:28$	$aabbaa\overline{bb}$
$8:29$	$aabb\overline{aa}\overline{bb}$
$8:30$	$aabb\bar{a}\bar{b}\bar{a}\bar{b}$ ★
$8:31$	$ab\bar{a}\bar{b}ab\bar{a}\bar{b}$ ★
$8:32$	$aabab\bar{a}bb$ ★ $aab\bar{a}bba\bar{b}$ ★
$8:33$	$aaba\bar{b}a\overline{bb}$ ★ $aaba\overline{bb}\bar{a}\bar{b}$ ★
$8:34$	$aabbaabb$ ★ $aba\bar{b}aba\bar{b}$ ★
$8:35$	$aab\bar{a}\overline{bb}\bar{a}b$ ★ $aba\bar{b}\bar{a}b\bar{a}\bar{b}$ ★
$8:36$	$aabba\bar{b}\bar{a}b$ ★ $aba\bar{b}ab\bar{a}\bar{b}$ ★
$8:37$	$aaababbb$ ★ $aababa\overline{bb}$ ★ $aabba\bar{b}a\bar{b}$ ★
$8:38$	$aaabbabb$ ★ $aaba\overline{bb}ab$ ★ $ababa\bar{b}a\bar{b}$ ★
$8:39$	$aababb\bar{a}\bar{b}$ ★ $aabab\bar{a}\overline{bb}$ ★ $aabb\bar{a}\bar{b}a\bar{b}$ ★

8 : 40	$aababba\bar{b}$ ★ $aababb\overline{a}b$ ★ $aaba\bar{b}\overline{a}\overline{bb}$ ★
8 : 41	$aaaaaabb$ $aaaaab\overline{a}b$ $aaaab\overline{aa}b$ $aaab\overline{aaa}b$
8 : 42	$aaba\bar{b}abb$ ★ $aabba\bar{b}ab$ ★ $aabba\bar{b}\overline{a}\bar{b}$ ★ $aabb\overline{a}\bar{b}\overline{a}b$ ★ $abab\overline{a}ba\bar{b}$ ★
8 : 43	$aaba\bar{b}\overline{a}bb$ ★ $aaba\overline{bb}\overline{a}b$ ★ $aab\overline{a}bb\overline{a}\bar{b}$ ★ $aabb\overline{a}\bar{b}ab$ ★ $abab\overline{a}b\overline{a}\bar{b}$ ★
9 : 1	$aaaaaaaaa$
9 : 2	$aaaaaaba\bar{b}$
9 : 3	$aaaaaabbb$
9 : 4	$aaaaaab\overline{a}\bar{b}$
9 : 5	$aaaaabaa\bar{b}$
9 : 6	$aaaaaba\overline{bb}$
9 : 7	$aaaaabba\bar{b}$
9 : 8	$aaaaabbbb$
9 : 9	$aaaaabb\overline{a}b$
9 : 10	$aaaaabb\overline{a}\bar{b}$
9 : 11	$aaaaab\overline{a}bb$
9 : 12	$aaaaab\overline{aa}\bar{b}$
9 : 13	$aaaaab\overline{a}\overline{bb}$
9 : 14	$aaaabaaa\bar{b}$
9 : 15	$aaaabaa\overline{bb}$
9 : 16	$aaaaba\overline{bbb}$

9 : 17	$aaaabbaa\bar{b}$
9 : 18	$aaaabba\overline{bb}$
9 : 19	$aaaabbba\bar{b}$
9 : 20	$aaaabbb\overline{a}b$
9 : 21	$aaaabbb\overline{a}\bar{b}$
9 : 22	$aaaabb\overline{a}bb$
9 : 23	$aaaabb\overline{aa}b$
9 : 24	$aaaabb\overline{aa}\bar{b}$
9 : 25	$aaaabb\overline{a}\overline{bb}$
9 : 26	$aaaab\overline{a}bbb$
9 : 27	$aaaab\overline{aa}bb$
9 : 28	$aaaab\overline{aaa}\bar{b}$
9 : 29	$aaaab\overline{aa}\overline{bb}$
9 : 30	$aaaab\overline{a}\overline{bbb}$
9 : 31	$aaabaaa\overline{bb}$
9 : 32	$aaabaa\overline{bbb}$
9 : 33	$aaabbaa\overline{bb}$
9 : 34	$aaabba\overline{bbb}$
9 : 35	$aaabbbaa\bar{b}$
9 : 36	$aaabbba\overline{bb}$
9 : 37	$aaabbb\overline{aa}\bar{b}$
9 : 38	$aaabbb\overline{a}\overline{bb}$
9 : 39	$aaabb\overline{aa}bb$
9 : 40	$aaabb\overline{aaa}b$
9 : 41	$aaabb\overline{aaa}\bar{b}$
9 : 42	$aaabb\overline{aa}\overline{bb}$
9 : 43	$aababaa\overline{bb}$
9 : 44	$aaba\bar{b}aba\bar{b}$
9 : 45	$aaba\bar{b}ab\overline{a}\bar{b}$
9 : 46	$aaba\bar{b}\overline{a}ba\bar{b}$
9 : 47	$aaba\bar{b}\overline{a}b\overline{a}\bar{b}$
9 : 48	$aabbaa\bar{b}\overline{a}\bar{b}$

9 : 49	$aab\overline{a}\bar{b}aba\bar{b}$
9 : 50	$aab\overline{a}\bar{b}ab\overline{a}\bar{b}$
9 : 51	$aab\overline{a}\bar{b}\overline{a}ba\bar{b}$
9 : 52	$aab\overline{a}\bar{b}\overline{a}b\overline{a}\bar{b}$
9 : 53	$aaaababbb$ $aaabb\overline{a}b\overline{a}b$
9 : 54	$aaaabbabb$ $aaab\overline{a}bb\overline{a}b$
9 : 55	$aaaabbbab$ $aaab\overline{a}b\overline{a}bb$
9 : 56	$aaababa\overline{bb}$ $aaabba\bar{b}a\bar{b}$
9 : 57	$aaababba\bar{b}$ $aaabb\overline{a}ba\bar{b}$
9 : 58	$aaababb\overline{a}b$ $aababba\overline{bb}$
9 : 59	$aaababb\overline{a}\bar{b}$ $aaabb\overline{a}b\overline{a}\bar{b}$
9 : 60	$aaabab\overline{a}bb$ $aabab\overline{aa}bb$
9 : 61	$aaabab\overline{a}\overline{bb}$ $aaabb\overline{a}\bar{b}a\bar{b}$
9 : 62	$aaaba\bar{b}a\overline{bb}$ $aaaba\overline{bb}\overline{a}\bar{b}$
9 : 63	$aaaba\bar{b}\overline{a}\overline{bb}$ $aaaba\overline{bb}a\bar{b}$
9 : 64	$aaaba\overline{bb}ab$ $aaba\bar{b}a\bar{b}ab$
9 : 65	$aaabbaba\bar{b}$ $aaab\overline{a}bba\bar{b}$
9 : 66	$aaabbab\overline{a}b$ $aabba\overline{bb}ab$

9 : 67	$aaabbab\overline{a}\overline{b}$ $aaab\overline{a}bb\overline{a}\overline{b}$
9 : 68	$aaabba\overline{b}\overline{a}\overline{b}$ $aaab\overline{a}ba\overline{bb}$
9 : 69	$aaabb\overline{a}bab$ $aabba\overline{b}a\overline{bb}$
9 : 70	$aaabb\overline{a}\overline{b}\overline{a}\overline{b}$ $aaab\overline{a}b\overline{a}\overline{bb}$
9 : 71	$aaab\overline{a}babb$ $aaba\overline{bb}abb$
9 : 72	$aaab\overline{a}bbab$ $aababb\overline{aa}b$
9 : 73	$aaab\overline{a}\overline{b}a\overline{bb}$ $aaab\overline{a}\overline{bb}\overline{a}\overline{b}$
9 : 74	$aaab\overline{a}\overline{b}\overline{a}\overline{bb}$ $aaab\overline{a}\overline{bb}a\overline{b}$
9 : 75	$aaab\overline{a}\overline{bb}\overline{a}b$ $aab\overline{a}\overline{b}a\overline{b}\overline{a}b$
9 : 76	$aabaa\overline{b}a\overline{bb}$ $aabaa\overline{bb}\overline{a}\overline{b}$
9 : 77	$aabaa\overline{b}\overline{a}\overline{bb}$ $aabaa\overline{bb}a\overline{b}$
9 : 78	$aababb\overline{aa}\overline{b}$ $aababb\overline{a}\overline{bb}$
9 : 79	$aabab\overline{aa}\overline{bb}$ $aabb\overline{aa}\overline{b}a\overline{b}$
9 : 80	$aaba\overline{bb}\overline{a}bb$ $aabb\overline{a}\overline{bb}ab$
9 : 81	$aaabaabbb$ $aabaab\overline{a}bb$ $aabab\overline{a}b\overline{a}b$
9 : 82	$aaabbaabb$ $aabbaab\overline{a}b$ $aab\overline{a}bab\overline{a}b$
9 : 83	$aaabbabbb$ $aabaabb\overline{a}b$ $aab\overline{a}b\overline{a}bab$
9 : 84	$aabaaba\overline{bb}$ $aabaa\overline{bb}ab$ $aababa\overline{b}a\overline{b}$
9 : 85	$aabaabba\overline{b}$ $aabab\overline{a}ba\overline{b}$ $aabba\overline{b}\overline{a}\overline{bb}$
9 : 86	$aabaabb\overline{a}\overline{b}$ $aabab\overline{a}b\overline{a}\overline{b}$ $aaba\overline{b}\overline{aa}bb$
9 : 87	$aabaab\overline{a}\overline{bb}$ $aabaa\overline{bb}\overline{a}b$ $aabab\overline{a}\overline{b}a\overline{b}$
9 : 88	$aabaa\overline{b}abb$ $aaba\overline{b}\overline{aa}\overline{bb}$ $aaba\overline{b}\overline{a}\overline{b}a\overline{b}$
9 : 89	$aabaa\overline{b}\overline{a}bb$ $aabb\overline{aa}\overline{b}ab$ $aab\overline{a}\overline{b}\overline{a}\overline{b}a\overline{b}$
9 : 90	$aaba\overline{b}aabb$ $aabb\overline{a}\overline{b}\overline{aa}b$ $aab\overline{a}baba\overline{b}$
9 : 91	$aaba\overline{b}aa\overline{bb}$ $aaba\overline{b}a\overline{b}\overline{a}\overline{b}$ $aaba\overline{bb}\overline{aa}\overline{b}$
9 : 92	$aaba\overline{bb}\overline{aa}b$ $aabb\overline{a}\overline{b}\overline{aa}\overline{b}$ $aab\overline{a}b\overline{a}\overline{b}\overline{a}\overline{b}$
9 : 93	$aabbaab\overline{a}\overline{b}$ $aabba\overline{b}\overline{aa}b$ $aab\overline{a}bab\overline{a}\overline{b}$
9 : 94	$aabbaa\overline{b}\overline{a}b$ $aabba\overline{bb}\overline{a}b$ $aab\overline{a}\overline{b}a\overline{b}\overline{a}\overline{b}$
9 : 95	$aabba\overline{b}\overline{aa}\overline{b}$ $aabb\overline{a}\overline{bb}\overline{a}b$ $aab\overline{a}ba\overline{b}\overline{a}\overline{b}$
9 : 96	$aaaaaaabb$ $aaaaaab\overline{a}b$ $aaaaab\overline{aa}b$ $aaaab\overline{aaa}b$
9 : 97	$aaaba\overline{b}abb$ $aaabb\overline{a}\overline{b}\overline{a}b$ $aaba\overline{b}ab\overline{a}b$ $aab\overline{a}b\overline{a}\overline{b}\overline{a}b$
9 : 98	$aaaba\overline{b}\overline{a}bb$ $aaabb\overline{a}\overline{b}ab$ $aaba\overline{b}\overline{a}b\overline{a}b$ $aab\overline{a}b\overline{a}\overline{b}ab$
9 : 99	$aaaba\overline{bb}\overline{a}b$ $aaab\overline{a}\overline{bb}ab$ $aaba\overline{b}a\overline{b}\overline{a}b$ $aab\overline{a}\overline{b}a\overline{b}ab$
9 : 100	$aaabba\overline{b}ab$ $aaab\overline{a}\overline{b}\overline{a}bb$ $aab\overline{a}ba\overline{b}ab$ $aab\overline{a}\overline{b}\overline{a}b\overline{a}b$
9 : 101	$aaabba\overline{b}\overline{a}b$ $aaab\overline{a}\overline{b}abb$ $aab\overline{a}ba\overline{b}\overline{a}b$ $aab\overline{a}\overline{b}ab\overline{a}b$

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