APPENDIX: FINANCIAL CALCULATIONS

Transactions in money and fixed-interest markets are based on financial calculations of the price of and return on securities. These calculations are often based on esoteric formulae and equations which require complex solution techniques. Understanding of these advanced techniques requires some mathematical background.

However, most of the market participants carrying out these calculations on a day-to-day basis do not possess such a background. They perform the calculations on one of the many inexpensive financial calculators currently available. These calculations are easy to master and any student hoping to use the financial markets will need to have the ability to use them. A financial calculator will also be useful for more advanced courses in finance.

Therefore, this Appendix shows how basic financial calculations can be done on three of the calculators most frequently used – the Casio FC100, the Sharp EL-735 and the Hewlett Packard 12C. The inputs for the Casio and the Sharp are similar and only the occasional differences will be noted.

THE TIME VALUE OF MONEY

$10,000 to be received two years from now is worth less than $10,000 to be received now because we can generate $10,000 in two years time by investing a smaller amount now. How much we need to invest is determined by the interest rate which is therefore a measure of the time value of money.

Assume that the interest rate is 10% p.a., then:

$$\text{The present value of \$10,000 to be received in two years} = \text{The amount of that has to be invested to generate \$10,000 in two years}$$

$$= \frac{\mathbf{10{,}000}}{\mathbf{(1.10)^2}}$$

$$= \frac{\mathbf{10{,}000}}{\mathbf{1.21}}$$

$$= \$8{,}264.46$$

This calculation is performed on the Casio FC100 and Sharp EL-735 as follows:

10,000	**FV**		
2	**n**		
10	**i%**		
	COMP	**PV**	**–8264.46**

The input for the HP12C is:

10,000	**FV**	
2	**n**	
10	**i**	
	PV	**–8264.46**

QUESTION 1:

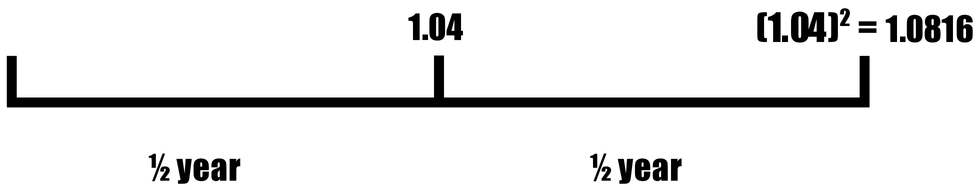
What is the present value of \$20,000 to be received in three years when the interest rate is 8% p.a.? (Solutions to the questions are given at the end of the Appendix).

COMPOUNDING

Compounding means getting interest on interest. Consider an investment which pays 8% p.a. but which compounds semi-annually.

Interest rate for half year = 4%

Consider a dollar invested for one year. In the second half of the year, we receive interest on $1.04 rather than the dollar initially invested.



The effective interest rate (i.e. the return we actually get) is 8.16% p.a.

Assume now that compounding is quarterly.

$$\textbf{Effective Interest Rate} \quad = (1.02)^4 - 1 = 1.0824 - 1 = 0.0824 \text{ or } 8.24\% \text{ p.a.}$$

The keystrokes on the Casio and Sharp calculators are:

1	+/–	PV	
2	i%		
4	n		
	COMP	FV	1.0824

On the HP12C they are:

1	CHS	PV
2	i	
4	n	
	FV	1.0824

The $\boxed{+/–}$ and $\boxed{\text{CHS}}$ keys change the sign of the preceding number, i.e. the

The $\boxed{+/-}$ and $\boxed{\text{CHS}}$ keys change the sign of the preceding number, i.e. the PV register contains –1.

Assume now that the compounding is daily. On the Casio and Sharp calculators:

1	**+/–**	**PV**			
8	**÷**	**365**	**=**	**i%**	
365	**n**				
	COMP	**FV**			**1.0833**

On the HP12C:

1	**CHS**	**PV**					
8	**E**	**365**	**÷**	**I**	**E**	**= Enter**	
365	**n**						
	FV						**1.0833**

i.e. the effective rate of interest is 8.33% p.a.

QUESTION 2:

We borrow money for one year at an interest rate of 11% p.a. What is the effective interest rate if compounding is:

- **half yearly?**
- **quarterly?**
- **daily?**

DIGRESSION ON THE USE OF THE CALCULATORS

Note:

- You should always clear your calculator before undertaking a new calculation:

SHIFT	AC	**in the case of the Casio FC 100 and**
CA	ENT	**for the Sharp EL-735**
F	CLX	**on the HP12C**

  These instructions will not be given in each example but it is assumed that registers have been properly cleared before each calculation.

- Calculations are done to as many as twelve decimal places regardless of the number of decimal places shown on the display. The decimal places displayed can be changed to d as follows:

Casio FC100:	MODE	**5 d**	
Sharp EL-735:	2nd F	TAB	**d**
HP12C:	f	**d**	

- Make sure BEG is not displayed. This means that the calculator is in 'begin mode', i.e. interest is assumed to be paid at the beginning rather than the end of the period.

In terms of the calculations done so far, we note that there are four registers of relevance:

PV	**present value**
FV	**future (or face) value**
i	**interest rate per period**
n	**number of interest periods**

You can input any three of these, in any order, and then calculate the fourth.

Note that, if both PV and FV are input, they must have opposite signs. This is a result of the ***cash flow convention*** followed by all financial calculators. Each calculation must represent an identifiable financial transaction.

PV	**positive**	**Loan**
FV	**negative**	

PV	**negative**	**Investment**
FV	**positive**	

There is no valid financial transaction that would have PV and FV with the same sign. If you input only one of PV and FV, and calculate the other, you need not worry about the sign. The one calculated will have the opposite sign to the one input.

Let us consider a case in which **n** must be calculated. We invest $100,000 at 10% p.a. and, at the end of the investment, it is worth $147,500. What is the term of the investment?

On the Casio and Sharp calculators:

100000	+/–	**PV**	
147500	**FV**		
10	**i%**		
	COMP	**n**	**4.08**

On the HP12C:

100000	**CHS**	**PV**	
147500	**FV**		
10	**i**		
	COMP	**n**	**5**

Note that the HP12C rounds up to the next full number.

SHORT-TERM SECURITIES

Securities with a term of less than one year are called Bills or Notes. The following are examples of such securities:

Treasury Notes – issued by the RBA at tender for terms of 91 days and 182 days

Bank Bills – issued by private borrowers but repayment is guaranteed by a bank through a process of endorsement or acceptance

– credit standing is determined by the guaranteeing bank's credit rating

– initial terms typically 90 or 180 days

– slightly higher yield than TNs on occasion

– very liquid market

Certificates of Deposit – issued directly by banks

– equivalent to bank bills, paying the same return

Promissory Notes – issued by borrowers without a bank guarantee (one name paper)

– return depends on the credit rating of the borrower

– illiquid market

These securities are sold at a discount. You buy them at a price below the face value and on maturity you receive the face value.

GAIN = FACE VALUE – PRICE

There is no cash flow over the life of the security.

Interest rates are stated on a per annum basis and must be adjusted to the term of the security.

Assume that a 180-day bill pays 7% p.a. Its return for the 180 days is:

$$7 \times \frac{180}{365} = 3.452055\%$$

This is based on a 365-day year convention which is used by former British Empire countries.

Other countries (such as the USA, Japan and Europe) use the 360-day year convention. In that case, the interest rate calculation would be:

$$7 \times \frac{180}{360} = 7 \times \frac{1}{2} = 3.50\%$$

Assume that we buy a 90-day bill with face value $500,000 at a market yield of 7.85% p.a. What is its price?

$$\begin{aligned} \textbf{Interest Rate} &= 7.85 \times \frac{90}{365} \\ &= 1.9356164\% \\ \textbf{Price} &= \textbf{Present Value of \$500,000 to be received} \\ &\quad \textbf{90 days from now} \\ &= \frac{\$500000}{1.019356164} = \$490{,}505.69 \end{aligned}$$

On the Casio and Sharp calculators:

500000 $\boxed{\text{FV}}$

7.85 $\boxed{\text{x}}$ 90 $\boxed{=}$ $\boxed{\div}$ 365 $\boxed{=}$ $\boxed{\text{i\%}}$

1 $\boxed{\text{n}}$

$\boxed{\text{COMP}}$ $\boxed{\text{PV}}$ –490505.69

On the HP12C:

500000 $\boxed{\text{FV}}$

7.85 $\boxed{\text{E}}$ 90 $\boxed{\text{X}}$ 365 $\boxed{\div}$ $\boxed{\text{i\%}}$

1 $\boxed{\text{n}}$

$\boxed{\text{PV}}$ –490505.69

We buy a 90-day bank bill with face value $100,000 for $97,750. At what yield are we buying the bill?

$$\textbf{Gain} = \$2{,}250$$

$$\textbf{Yield} = \frac{2250}{97750} \times \frac{365}{90} \times 100 = 9.34\%\ \text{p.a.}$$

On the Casio and Sharp calculators:

97750	**+/–**	**PV**						
100000	**FV**							
1	**n**							
	COMP	**I%**						**2.301779**
	x	**365**	**=**	**÷**	**90**	**=**		**9.34**

On the HP12C:

97750	**CHS**	**PV**		
100000	**FV**			
1	**n**			
	i			**2.301779**
365	**x**	**90**	**÷**	**9.34**

The return for 90 days is 9.34% p.a. The return is stated on a per annum basis although it only applies for 90 days and the final step in the calculation is necessary to convert the 90-day return to an annual return.

QUESTION 3:

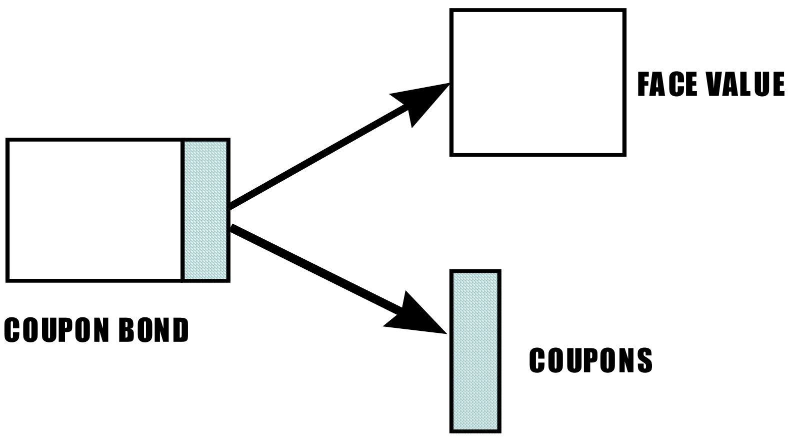
We buy a 180-day bank bill with face value $1,000,000 at 6.75% p.a. How much do we pay for it? We hold the bill for 28 days and then sell it at a yield of 6.35% p.a. What price do we sell at and what return did we earn over the holding period?

ZERO COUPON BONDS

In Australia, the term bond is used to describe a security with an initial term of more than one year. This usage does not prevail around the world. For example, US government securities are called "bills" and the standard US government security is the 30-year Treasury bill.

A zero coupon bond (ZCB) pays its face value at maturity but no cash flow over its life. It is similar to a discount security, but because it covers more than one year, interest accrues on it as time passes. By convention, interest on bonds is compounded half-yearly.

In the United States, zero coupon bonds are frequently created by "stripping" the coupons from a coupon bond (see next section) and trading them separately.



The face value is a single payment at maturity, i.e. a zero coupon bond. The coupon payments are regular (usually half-yearly) payments of the same amount. They are called a "strip".

Consider a ten-year ZCB with face value $500,000. The market yield is 6.75% p.a. What is its price?

On the Casio and Sharp calculators:

500000	FV				
6.75	÷	2	=	i%	
20	n				
	COMP	PV			–257430.24

On the HP12C calculator:

500000	FV				
6.75	E	2	÷	i	
20	n				
	PV				–257430.24

QUESTION 4:

Consider a five-year ZCB with face value $1,000,000. The market yield is 8.95% p.a. What is the price of the bond?

We hold the bond for one year and then sell it at a yield of 8.90% p.a. What is its price and what return have we earned over the holding period?

COUPON BONDS

A coupon bond pays:

(a) a half-yearly coupon payment

(b) its face value at maturity

The word "coupon" is used because in earlier times the payments were represented by coupons attached to the bond. This occurred when the bonds were "bearer" bonds which gave ownership and entitlement to face value and coupon payments to whoever held the bond. Most bonds are now "inscribed stock". No bond is actually issued but purchasers are recorded on a register as the owners of a certain amount of stock.

Commonwealth Government Securities:

- are also known as Treasury Bonds
- are sold at tender by the RBA
- smaller amounts can be purchased over-the-counter at the RBA (yield equal to the average established in the tender)
- have a liquid market and the RBA will buy small amounts back ($50,000 per day)

State Government Bonds:

- are similar to CGS
- pay higher returns, depending on the issuing State's credit rating.

Consider a ten-year bond with face value of $1,000,000 and paying a semi-annual coupon of $35,000. The market yield is 7.8% p.a. What is the price of the bond?

The coupon rate is 3.5% per half year or 7% p.a. In this case, we need a new key – the payment key:

$$\boxed{\textbf{PMT}}$$

This key will be used to enter the coupon payments.

On the Casio and Sharp calculators:

1000000	**FV**		
3.9	**i%**		
20	**n**		
35000	**PMT**		
	COMP	**PV**	**–945154.14**

On the HP12C calculator:

1000000	**FV**	
3.9	**i**	
20	**n**	
35000	**PMT**	
	PV	**–945154.14**

QUESTION 5:

Consider a 5-year bond with face value $100,000 and semi-annual coupon $3,200. The market yield is 6% p.a. What is the price of the bond?

We hold the bond for 1½ years and sell it at a yield of 6.6% p.a. What is its price and what have we earned over the holding period?

It should be noted that the preceding calculations indicate how to determine the price of a bond at a coupon period. If a bond is sold between coupon dates, the price must take the accrued coupon into account. Techniques for doing this are given in Valentine (1991, pp.113-117).

Consider a ten-year bond with face value $1,000,000, semi-annual coupons of $40,000 and selling at a price of $973,300. What is the yield on this security?

On the Casio and Sharp calculators:

1000000	**FV**				
973300	**+/–**	**PV**			
20	**n**				
40000	**PMT**				
	COMP	**i%**			**4.20**
	i	**x**	**2**	**=**	**8.40**

The yield is 8.40% p.a. This is the yield to maturity. Investors buying the bond at the price given will achieve this return only if:

- they hold the bond to maturity; and
- they can reinvest each coupon payment at the same yield

The second requirement is unlikely to be satisfied. Thus, the return actually earned is likely to be a little higher or lower than the purchase yield depending on whether the rates at which coupon payments are reinvested is higher or lower than that yield.

MORTGAGES

Lenders providing the funds for the purchase of property usually secure their loans on the properties themselves. That is, if the borrower defaults, the lender has the right to take over the property and to sell it to recover its funds. The instrument that provides this security is called a mortgage.

There are two types of mortgage loan. Interest-only loans require the borrower to make regular interest payments and to repay the principal of the loan at the end of its term. The cash flow pattern of this type of loan is the same as that of a coupon bond and it is most commonly used to finance the purchase of investment properties. The second type of mortgage is a “credit foncier” mortgage under which the loan is repaid in equal regular payments covering both principal and interest. This form of mortgage is standard in the purchase of a home for occupation by the owner. Since this is the most common form, it is the one that will be assumed here.

We take a 15-year mortgage for \$150,000 at 10.75% p.a. It is to be repaid fortnightly (that is, every two weeks). What is the repayment?

In solving this example, we will assume that there are **n = 15 x 26 = 390** payments. Actually, there are slightly more than 391 fortnights in 15 years. Also, the following calculation assumes that interest is compounded fortnightly. In practice, lenders compound interest daily, although payments are fortnightly or monthly.

On the Casio and Sharp calculators:

150000	**PV**				
0	**FV**				
26	**x**	**15**	**=**	**n**	**390**
10.75	**÷**	**26**	**=**	**i%**	
	COMP	**PMT**			**–775.29**

On the HP12C calculator:

150000	**PV**				
0	**FV**				
26	**E**	**15**	**x**	**n**	
10.75	**E**	**26**	**÷**	**i**	
	PMT				**–775.29**

We pay the mortgage for one year. How much do we owe then? We can continue from the previous calculations.

On the Casio and Sharp calculators:

26 $\boxed{\textbf{x}}$ **14** $\boxed{=}$ $\boxed{\textbf{n}}$ **364**

$\boxed{\textbf{COMP}}$ $\boxed{\textbf{PV}}$ **145751.95**

On the HP12C calculator:

26 $\boxed{\textbf{E}}$ **14** $\boxed{\textbf{x}}$ $\boxed{\textbf{n}}$

$\boxed{\textbf{PV}}$ **145751.95**

We have paid off:	150000 – 145751.95	= 4,248.05
But over that year we paid:	26 x 775.29	=20,157.54
This means that we have paid:	(20,157.45 – 4248.05)	=15,909.49 of interest

The early payments include a lot of interest and little principal. The final payments include mainly principal.

QUESTION 6 (a):

We take a ten-year mortgage for \$100,000 at 8.75% p.a. It is to be repaid in monthly repayments. What is the repayment amount? Assume that interest is compounded monthly.

After two years, what is the balance outstanding? How much principal and how much interest has been paid?

So far, we have assumed that the interest rate on the mortgage is fixed. Money mortgages are actually variable rate so that the lender can change the rate whenever market interest rates change.

QUESTION 6 (b):

In question 6 (a), assume that it is a variable-rate mortgage and that the

interest rate increases to 9.50% p.a. after two years. What is the new payment?

Many lenders providing fixed-rate mortgages include a penalty for the prepayment of the mortgage. The reason for this penalty is that mortgage holders have an incentive to prepay their loan (and refinance it) when interest rates fall. Lenders will have offset the fixed rate in some way (for example, by funding it from deposits which have a fixed rate for the same period) and, with an interest rate fall, they will have suffered a loss on that hedge. If the mortgage is prepaid, they lose the offsetting gain on their loss. That is, they have a net loss.

Prepayment penalties can take two basic forms. First, there can be a fixed dollar amount (sometimes defined as a number of monthly payments). The problem with this approach is that there will be some fall in interest rates that makes it profitable for the borrower to prepay the loan. Secondly, prepayment can be allowed only at the market value of the loan. In this case, the lender cannot lose from the prepayment. The problem with this approach is the difficulty in explaining it to customers, particularly when the new market value exceeds the amount originally borrowed.

QUESTION 6 (c):

Assume in question 6 (a) that, after two years, the interest rate falls to 8.25% p.a. What prepayment penalty would make it unattractive to prepay the loan?

DURATION

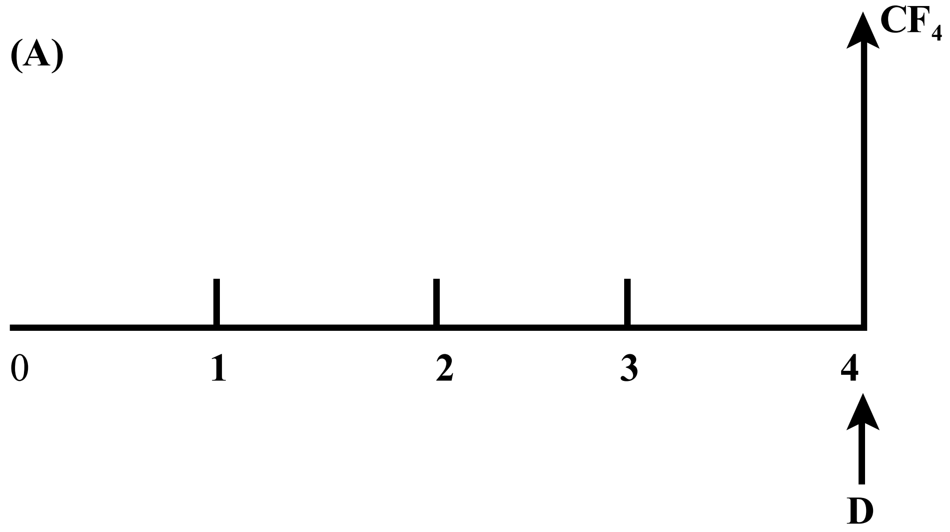
The earlier sections show that the prices of fixed-interest securities fall as interest rates increase. The reader will not be surprised to be told that the sensitivity of a security's price to a change in interest rates appears to increase with the term of the security. In general, longer-term securities have prices that are more sensitive to interest rates than shorter-term securities.

Consider a bond with face value $100,000 and semi-annual coupon $5,000. At 10% p.a., it will sell at par, (i.e. at its face value of $100,000), regardless of its term. Assume now that the yield drops to 9% p.a. The following Table indicates what happens to the price of the bond for different terms to maturity.

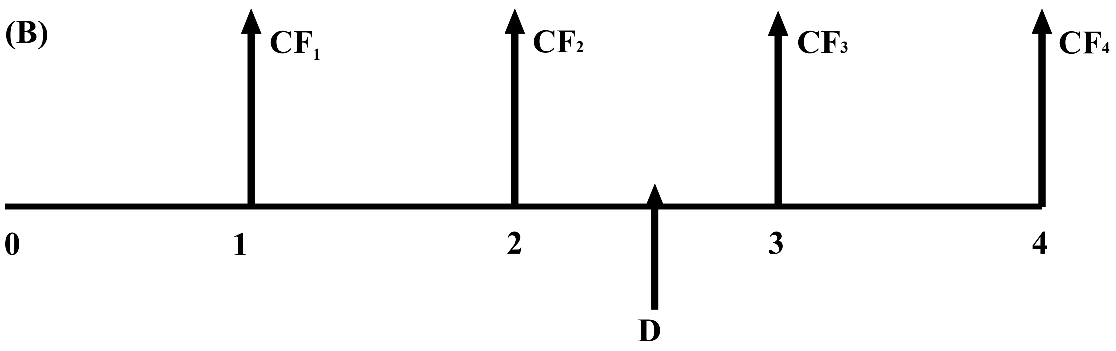
TERM (YEARS)	PRICE AT 9% P.A. ($)
2	101,793.76
5	103,956.36
10	106,503.97
20	109,200.79
30	110,319.01

It is clear that, as the term of the security increases, the sensitivity of its price to changes in interest rates increases. However, it is also clear that the sensitivity of the security price does not go up in proportion to its term. For example, in the Table, the price of the ten-year bond increases by 6.50%. If the price of the thirty-year bond reacted in a proportional fashion, it would have increased by 19.50%, whereas the actual increase is 10.32%.

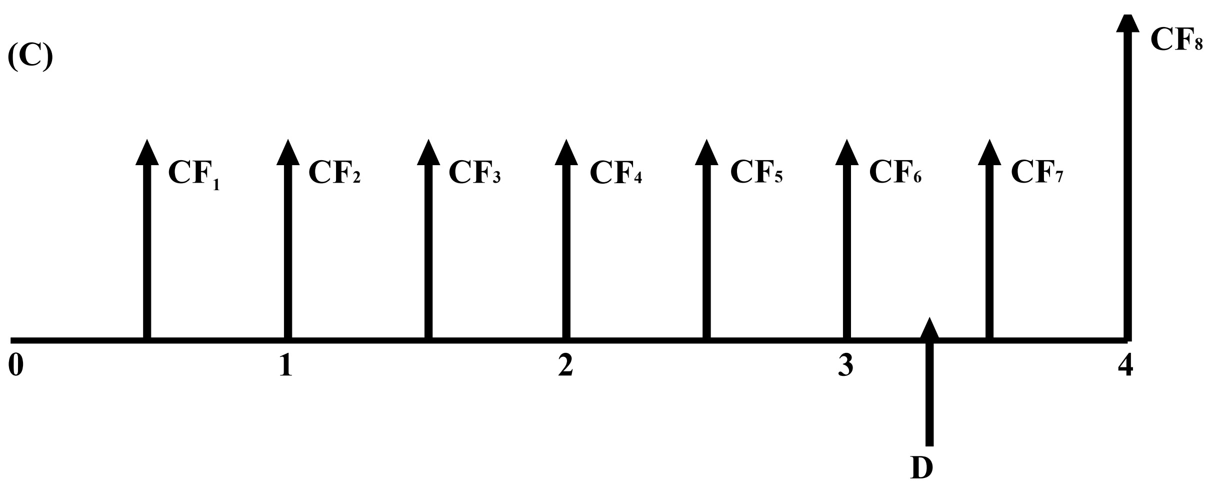
The reason for this reaction is that the sensitivity of security prices to interest rate changes depends on the patterns of their cash flows as well as their terms. This point can be illustrated by the following three four-year securities.



Security **(A)** is a zero-coupon bond in which all the cash flow occurs at maturity.



Security **(B)** is an **annuity** which generates equal cash flows in every period. A strip of coupon payments also has cash flows of this form. Also, credit-foncier loans (such as housing loans and personal loans) are repaid in equal payments, including both interest and principal.



Security **(C)** is a coupon bond which has a large payment, consisting of its face value and the final coupon payment at maturity and half-yearly coupon payments.

Security **(A)** is the most sensitive to interest rate changes. If the interest rate increases, a holder would have to wait four years before receiving any cash flow that could be reinvested at the higher rate. Security **(B)** is least sensitive to interest rates because three-quarters of the cash flow is received before maturity. Security **(C)** is between **(A)** and **(B)** in interest-rate sensitivity. It has some earlier cash flows, but there is a large concentration at the maturity date.

Clearly, the price sensitivity of a security depends on its pattern of cash flows. The **duration** of a security is a measure that takes this pattern into account. The duration of a security is the weighted average of the times to the receipt of various elements of a security's cash flow where the weights are the present values of the cash flows divided by the price of the security (the total of the present values of the cash flows). The duration is a weighted average of times and it is therefore stated in time units.

The approximate durations of securities **(A)**, **(B)** and **(C)** are shown on the diagrams. In the case of **(A)**, the duration is four because this is the only point in time when a cash flow occurs. **(B)** has a duration above two but less than 2.5 (the average of the four values 1, 2, 3, 4). The cash flows are equal, but the present values of those cash flows (which determine the weights used in the average) decline as the time to their receipt increases.

The coupon bond **(C)** has a duration below its term, but not very far below it. For a bond of this maturity, the weight arising from the large final payment dominates the average. However, as the term increases, this final cash flow is more heavily discounted and becomes increasingly less important. Therefore, the gap between the duration and the term increases as the term of the coupon bond increases.

The definition given above can be used to calculate the duration of a security. Consider the case of a three-year annuity which pays $100 per year at a market interest rate of 5% p.a. Its duration can be calculated as follows:

CASH FLOW	TIME TO RECEIPT OF CASH FLOW	PRESENT VALUE	TIME x PRESENT VALUE
100	1	95.238095	95.238095
100	2	90.702948	181.405896
100	3	86.383760	259.151280
	PRICE =	**272.324803**	**535.795271**

$$\textbf{Duration} = \frac{\mathbf{535.695271}}{\mathbf{272.324803}} = \mathbf{1.967\ years}$$

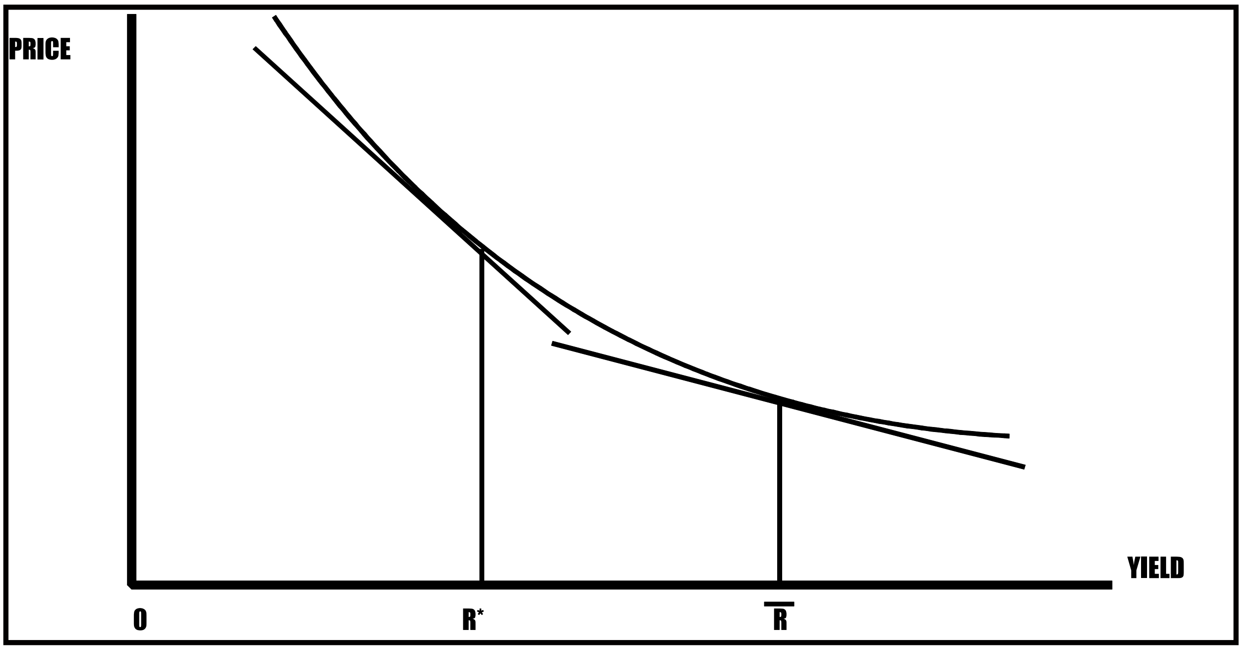
It is easier to make use of an approximation to the percentage change in price that results from an interest rate change. Consider a security that generates cash flows $\mathbf{CF_1, CF_2, \ldots., CF_K}$ over **K** periods. The price of this security **(P)** is given by:

$$\mathbf{P} = \frac{\mathbf{CF_1}}{\mathbf{1+r}} + \frac{\mathbf{CF_2}}{(\mathbf{1+r})^2} + \ldots\ldots + \frac{\mathbf{CF_K}}{(\mathbf{1+r})^K} \quad \mathbf{(*)}$$

where **r** is the yield to maturity.

This equation shows that the relationship between the price of a security and the market yield is a nonlinear one. A relationship of this type is illustrated by the curve in Figure 1.

FIGURE 1 : THE DURATION APPROXIMATION TO SECURITY PRICE CHANGES



We can approximate the change in the price of the security around a yield of **R*** by using the tangent to the curve at **R***. Clearly, this approximation becomes less accurate as we move further away from **R***. However, in practice, the accuracy of the approximation is good even for changes as large as 100 basis points (one percentage point). A basis point change is a change of one unit in the second decimal place of a market yield. It is the smallest change that can occur in a market yield.

The approximating tangent can be represented by the equation:

$$\frac{\Delta \mathbf{P}}{\mathbf{P}} = \frac{\mathbf{D} \times \textbf{Fall in the Interest Rate}}{\textbf{One plus the Interest Rate}} \qquad \textbf{(**)}$$

where **D** is the duration of the security and **ΔP** is the change in the price and the interest rate is expressed as a decimal. The change can be converted into a percentage by multiplying both sides of the equation by 100.

This equation can be derived by differentiating **(*)** with respect to **r** (see Advanced Topics 1 at the end of this appendix). It shows that **D** changes as the yield changes. Figure 1 shows that, at higher yields, the approximating tangent is flatter, i.e. it has a lower slope. In other words, duration falls as the yield increases. The prices of securities are more interest rate sensitive in low interest rate environments than when interest rates are high.

Applying the approximation **(**)** to the annuity discussed above, we will calculate the change in price that occurs when the market yield increases by 0.01 (one basis point). This value is the **price value of a basis point (PVBP)**. A basis point is the smallest change that can occur in a market interest rate because they are commonly stated to two decimal points. We have chosen this change because **(**)** applies strictly only for infinitesimally small changes in interest rates and the error involved is smaller the smaller the interest rate change we consider.

On the Casio and Sharp calculators:

100	$\boxed{\textbf{PMT}}$		
3	$\boxed{\textbf{n}}$		
5.0	$\boxed{\textbf{i}}$		
	$\boxed{\textbf{COMP}}$	$\boxed{\textbf{PV}}$	**–272.324803**
5.01	$\boxed{\textbf{i}}$		
	$\boxed{\textbf{COMP}}$	$\boxed{\textbf{PV}}$	**–272.273783**

The only difference with the HP12C is that we do not need the $\boxed{\textbf{COMP}}$ key. The PVBP is 0.05102.

As a percentage: $$\frac{(0.05102)100}{272.324803} = \frac{D \times 0.01}{1.05}$$

That is, D = 1.967 years

As a further example, consider a ten-year bond with a face value of \$100,000 and a coupon rate of 6% p.a. The market yield is 5.80% p.a. We calculate the price at this yield and then at a yield which is one basis point higher (i.e. at 5.81).

On the Casio and Sharp calculators:

100000	**FV**				
3000	**PMT**				
20	**N**				
5.80	**÷**	**2**	**=**	**i%**	
	COMP	**PV**			**–101501.60**
5.81	**÷**	**2**	**=**	**i%**	
	COMP	**PV**			**–101425.85**

On the HP12C:

100000	**FV**				
3000	**PMT**				
20	**N**				
5.80	**E**	**2**	**÷**	**i**	
	PV				**–101501.60**
5.81	E	**2**	÷	**i**	
	PV				**–101425.85**

The PVBP is $75.75.

As a percentage: $$\frac{(75.75)100}{101501.60} = 0.0746\%$$

From the equation above, the duration is calculated from:

$$0.0746 = \frac{D \times 0.005}{1.029}$$
$$D = 15.35 \text{ half years or } 7.68 \text{ years}$$

Note that, since the half-year is the basic period, interest rates and the basis point change are stated in half-yearly terms.

We can also calculate the modified duration **(DM)** as:

$$\mathbf{D_M = \frac{D}{One\ plus\ Interest\ Rate}}$$

In the example:

$$\mathbf{D_M = \frac{15.35}{1.029} = 14.92}$$

From the formula above:

Percentage Increase in Price	**=–DM x Change in Interest Rate**
	=– 14.92 x 0.005
	=– 0.0746%

Therefore, the modified duration of a security represents the percentage change in the price of a security when the interest rate changes by one percent.

QUESTION:

We take a 15-year mortgage for $200,000 which is to be paid off monthly. The interest rate is 7.75% p.a. What is the duration of this mortgage?

SOLUTION:

We will calculate the payment on this mortgage and then examine how its price changes when there is a basis point increase in the yield.

On the Casio and Sharp calculators:

200000	**PV**				
15	**x**	**12**	**=**	**n**	
7.75	**÷**	**12**	**=**	**i%**	
	COMP	**PMT**			**–1882.55**
7.76	**÷**	**12**	**=**	**i%**	
	COMP	**PV**			**199878.35**

On the HP12C:

200000	**PV**				
15	**E**	**12**	**x**	**n**	
7.75	**E**	**12**	**÷**	**i**	
	PMT				**–1882.55**
7.76	**E**	**12**	**÷**	**i**	
	PV				**199878.35**

The PVBP is \$121.65 and the percentage fall in price is 0.0608%.

Therefore, the duration is given by:

$$\mathbf{0.0608 = \frac{D}{1.006458} \times 0.000833}$$

$$\mathbf{D = 73.44 \text{ months or } 6.12 \text{ years}}$$

$$\mathbf{D_M = \frac{73.44}{1.006458} = 72.97 \text{ months or } 6.08 \text{ years}}$$

In this case, the duration is well below the term because the payments are evenly spread over the life of the security. Indeed, the duration of this mortgage is less than the duration of the ten-year bond discussed earlier.

USING DURATION FOR HEDGING

Assume that we want to invest in an annuity of the type discussed in the previous subsection. We want to fund this investment by issuing a security that will leave our overall position unaffected by movements in interest rates. Such a security would be a zero coupon bond with term (duration) equal to 1.967 years, current price equal to \$272.324803 and, therefore, face value of:

$$\begin{aligned} \mathbf{FV} &= \mathbf{\$272.324803(1.05)}^{1.967} \\ &= \mathbf{\$299.755078} \end{aligned}$$

A change in interest rates will affect the value of our asset (the annuity) and our liability (the zero coupon bond) by the same amount, leaving our net position unchanged.

This hedging rule can be formalised as follows:

Let: $\mathbf{D}_A$ be the duration of the asset, and

$\mathbf{D}_L$ be the duration of the liability.

We define the duration gap (DGAP) of our position as:

$$\mathbf{DGAP = D_A - D_L}$$

In the example, **DGAP = 0**. In such cases, a change in interest rates will have the same dollar value effect on the prices of the asset and liability. It is said that our position is "immunised" against interest rate movements.

If **DGAP >0**, $\mathbf{D}_A$ must be above $\mathbf{D}_L$. An interest rate increase will lead to a fall in the value of both the asset and the liability but, since it has the higher duration, the price of the asset will fall further. That is, we will lose. Consequently, a fall in the interest rate will increase the net value of our position.

If **DGAP <0**, $\mathbf{D}_A$ is below $\mathbf{D}_L$. An interest rate fall will cause the asset to increase in value by less than the liability. That is, our net position will deteriorate. Conversely, an interest rate increase would lead to a capital gain.

However, there are some problems with this hedge. First, we have already indicated that it is only completely accurate for extremely small changes in the market yield. The hedge will be very accurate for a basis point change in the interest rate. However, consider the case of a 100 basis point (one percentage point) increase in the interest rate. The price of the annuity goes to \$267.301. The price of the zero goes to:

$$\frac{\$299.755078}{(1.06)^{1.967}} = \$267.294$$

leaving a slight error.

Secondly, the position is only immunised against uniform shifts in the yield curve; that is, cases where the same change occurs in the yields for all maturities. In the example, we are protected against equal changes in two- and three-year yields. If these yields change by different amounts, the net value of the position will be affected.

Thirdly, the hedge is instantaneous. That is, it protects us against uniform changes in yields today, but its accuracy erodes as time passes. Consider the position described above after one year has passed. The duration of the zero coupon bond would now be 0.967 years. We can recalculate the duration of the annuity (only the Casio calculation is shown):

100	**PMT**		
2	**N**		
5	**i**		
	COMP	**PV**	**–185.941043**
5.01	**i**		
	COMP	**PV**	**–185.914699**

Therefore, the duration is:

$$\frac{(0.026344)100}{185.941043} = \frac{D}{1.05} \times 0.01$$

D = 1.488 years

The duration gap is now:

1.488 – 0.967 = 0.521 years

and we would lose from an interest rate increase. This hedge needs to be constantly rebalanced as time passes. The reason that this problem arises is that the price-yield curves for the annuity and zero coupon bonds have a different sensitivities to time so that duration changes by different amounts as time passes.

Fourthly, in the previous example, we assumed that the interest rate remained constant. As discussed above, duration changes as interest rates change. Moreover, the sensitivity of duration to interest rate changes differs amongst different securities because the curvature of their price-yield curves differ. Therefore, the hedge is strictly appropriate only for small changes in interest rates although, in practice, it is quite accurate for movements of one percentage point (100 basis points).

CONVEXITY

It can be seen from Figure 1 that the curve that represents the actual price change resulting from a given change in the yield moves further away from the approximating tangent as the size of the yield change increases. The error in the approximation increases as the size of the yield change increases because the price/yield curve is convex.

We can improve the accuracy of the approximation by taking account of the curvature of the price/yield curve. This is done by calculating the convexity (CX) of the security. It is:

> **The weighted average of the squares of the times to the receipt of cash flows where the weights are the present values of the corresponding elements of the cash flow**

The weighted average of the squares of the times to the receipt of cash flows where the weights are the present values of the corresponding elements of the cash flow

plus

the weighted average of the times to receipt of the cash flows.

The calculation for the annuity discussed above can be set as follows:

CASH FLOW	PRESENT VALUE	TIME x PRESENT VALUE	TIME2 x PRESENT VALUE
100	95.238095	95.238095	95.238095
100	90.702948	181.405896	362.811792
100	86.383760	259.151280	777.453840
PRICE =	**272.324803**	**535.795271**	**1235.503727**

$$\mathbf{CX} = \frac{\mathbf{1}}{\mathbf{272.324803}}\left[\mathbf{535.795271 + 1235.503727}\right]$$
$$= \mathbf{6.504362\ years}$$

For a zero coupon bond where the term is stated in years:

$$\mathbf{CX = Term^2 + Term}$$

In the case of the zero used to hedge the annuity:

$$\begin{aligned}\mathbf{CX} \quad &= \mathbf{(1.967486)^2 + 1.967486} \\ &= \mathbf{5.838488}\end{aligned}$$

Since these securities have different convexities, the hedge becomes less accurate the further we move away from the yield at which the durations were calculated.

Convexity increases as:

- the term of the security increases;
- the market yield falls; and
- the value of the coupon payments falls.

Convexity also has the additive property possessed by duration.

APPROXIMATING PRICE CHANGES USING CONVEXITY

A better approximation to the relative change in the price of the security can be obtained by adding a convexity term to **(**)**. This gives:

$$\frac{\Delta P}{P} = \frac{D.\Delta r}{1+r} + \frac{\frac{1}{2} \times CX(\Delta r)^2}{(1+r)^2} \qquad (***)$$

Convexity can also be used to construct a more accurate hedge for a security, although the hedge position must then include at least two securities. Consider hedging the annuity discussed above with **two** zero coupon bonds. Assume that the term (and duration) of the first is **X** and of the second is **Y**. Then, the convexity of the first is $(X^2 + X)$ and the convexity of the second is $(Y^2 + Y)$. We can set the average duration and average convexity of these two zero coupon bonds equal to the duration and convexity of the annuity by solving the following two equations:

$$\tfrac{1}{2}X + \tfrac{1}{2}Y = 1.967486$$

$$\tfrac{1}{2}X^2 + \tfrac{1}{2}X + \tfrac{1}{2}Y^2 + \tfrac{1}{2}Y = 6.504362$$

Solving these equations gives terms of 1.151476 years and 2.783496 years.

Since the price of each security is $½(272.324803) = $136.162402, their face values are:

$$\begin{aligned} FV &= \$136.162402(1.05)^{1.151476} \\ &= \$144.031066 \\ \text{and} \quad FV &= \$136.162402(1.05)^{2.783496} \\ &= \$155.968729 \end{aligned}$$

Assume now that the interest rate goes to 6% p.a. The new prices are:

$$\text{Price} = \frac{\$144.031066}{(1.06)^{1.151476}} = \$134.684375 \qquad \text{Price} = \frac{\$155.968729}{(1.06)^{2.783496}} = \$132.616885$$

The total value of the zeros is then $267.301 which is the value of the annuity at that yield.

ANOTHER COMVEXITY APPLICATION: BULLETS AND DUMBBELLS

The performance of funds managers is often judged against that of a benchmark portfolio. That benchmark portfolio will have a specific duration and one way for the funds manager to make certain of matching the performance of the benchmark is to hold zero coupon bonds with terms and duration equal to the duration of the benchmark – a **bullet** portfolio. Then, if interest rates change, the gains and losses on the funds manager's portfolio will approximately match those of the benchmark.

The discussion of the previous section suggests that an alternative is to adopt a **dumbbell** portfolio consisting of two zeros with average duration equal to the duration of the benchmark.

As an example, assume that the market yield is 10% p.a. and we have $100,000 to invest. The duration of the benchmark portfolio is two years.

Bullet Portfolio:	Two-Year Zero	Price	= $100,000
		FV	= $121,000
		Convexity	= 6 years
Dumbbell Portfolio:	One-Year Zero	Price	= $50,000
		FV	= $55,000
		Convexity	= 2 years
	Three-Year Zero	Price	= $50,000
		FV	= $66,550
		Convexity	= 12 years
		Average Convexity	= 7 years

Assume now that the market yield changes by one percentage point. The resulting portfolio values are:

	MARKET YIELD	
	9% p.a.	**11% p.a.**
Bullet Portfolio	$101,843.28	$98,206.31
Dumbbell Portfolio	$101,847.53	$98,210.34

In both cases, the dumbbell portfolio gives the better performance, a result that arises from its higher convexity (see **(***)**). The funds manager is able to bring in a slightly better result by adopting the dumbbell portfolio.

DURATION AND SECURITY PRICING WITH UNEQUAL CASH FLOWS

Consider a security that pays the following annual cash flows:

1	**300,000**
2	**300,000**
3	**300,000**
4	**500,000**
5	**500,000**

The interest rate is 8% p.a. What is the price and duration of this security?

In order to solve this problem, we need to use the cash flow registers.

On the Casio calculator:

	MODE	**4**	
0	**CFj**		
300000	**CFj**		
3	**Nj**		
500000	**CFj**		
2	**Nj**		
8	**i%**		
	COMP	**NPV**	**1480935.62**
8.01	**i%**		
	COMP	**NPV**	**1480502.56**

The price of the security is $1,480,935.62 and the PVBP is $433.06. Therefore, the duration is given by:

$$\frac{\mathbf{433.06}}{\mathbf{1,480,935.62}} = \frac{\mathbf{D}}{\mathbf{1.08}} \times \mathbf{0.0001}$$

$$\textbf{i.e. D = 3.16 years}$$

On the Sharp calculator:

0	**CFj**		
3	**2nd F**	**Nj**	
300000	**CFj**		
2	**2nd F**	**Nj**	
500000	**CFj**		
8	**i**		
	COMP	**NPV**	**1480935.62**
8.01	**i**		
	COMP	**NPV**	**1490502.56**

Note that the number of equal cash flows **(Nj)** must be entered before the cash flow **(i)** value.

On the HP12C:

0	g	CFO	
300000	g	CFj	
3	g	Nj	
500000	g	CFj	
2	g	Nj	
8	i		
	f	NPV	1480935.62
8.01	i		
	f	NPV	1480502.56

In the case of the Casio and Sharp calculators, the instructions for clearing the cash flow registers are as follows:

Casio	MODE	4	MODE	6	SHIFT	AC
Sharp	RCL	CFj	2nd F	CA	ENT	

For the HP12C, the instruction given earlier will also clear the cash flow registers.

QUESTION:

Consider a security with the following cash flows:

1	**–500**
2	**–500**
3	**–500**
4	**–500**
5	**6000**

The market yield is 10% p.a. What is its duration?

SOLUTION:

On the Casio FC100:

	MODE	4	
0	CFj		
500	+/–	CFj	
4	Nj		
6000	CFj		
10	i		
	COMP	NPV	**2140.60**
10.01	i%		
	COMP	NPV	**2139.25**

The PVBP is \$1.35. Therefore, to calculate the duration:

$$\frac{1.35(100)}{2140.60} = \frac{D}{1.10} \times 0.01$$

$$\text{Therefore, } D = 6.94 \text{ years}$$

The basic calculations on the other calculators are:

On the Sharp calculator:

0	CFj		
4	2nd F	Nj	
500	+/–	CFj	
6000	CFj		
10	i		
	COMP	NPV	**2140.60**
10.01	i		
	COMP	NPV	**2139.25**

On the HP12C:

0	**g**	**CFO**	
500	**CHS**		
4	**g**	**Nj**	
6000	**g**	**CFj**	
10	**i**		
	f	**NPV**	**2140.60**
10.01	**i**		
	f	**NPV**	**2139.25**

You will note that, in this case, the duration is above the term of the security because of the negative cash flows (outflows). It might seem that this is an unrealistic example, but one security that fits this pattern is an insurance policy. There is a small market in Australia for such policies. You can buy an insurance policy from its original holder and use it as an investment, but this means that you must pay the premia on it to maturity date.

REFERENCE

Valentine, T.J. (1991), *Interest Rates and Money Markets in Australia* (Financial Analysis and Training; Sydney)

ADVANCED TOPIC 1: APPROXIMATION TO THE PRICE OF A SECURITY AND CONVEXITY

Taylors' Theorem

Consider a function of a variable x,y = f(x). We can look at an approximation to this function at a particular value x0, namely:

$$\mathbf{f(x) = f(x^0) + \frac{dy}{dx}(x - x^0) + \frac{\frac{1}{2}d^2y}{dx^2}(x - x^0)^2 + \ldots\ldots}$$

The differentials are evaluated at x0 and the approximation can be improved by taking additional higher order terms.

The relationship of interest to us is the price/yield relationship

$$\mathbf{P = f(r)}$$

and we wish to obtain an approximation to it at the current market yield **(ro)**. A first-order approximation to the price/yield relationship is:

$$\mathbf{P = P^o + (r - r^o)\frac{dP}{dr}}$$

where P – Po is ΔP, r – ro is Δr and

$$\mathbf{\frac{dP}{dr} = \frac{-D.P}{1 + r}}$$

which is the approximation used in the text. This approximation is clearly a first-order one.

A second-order approximation is:

$$\mathbf{P = P^o + \frac{dP}{dr}(r - r^o) + \tfrac{1}{2}\frac{d^2P}{dr^2}(r - r^o)^2}$$

The convexity of the security is:

$$\mathbf{CX = \frac{(1 + r)^2}{P} . \frac{d^2P}{dr^2}}$$

Thus, the second-order approximation can be written:

$$\mathbf{DP} = \frac{-\mathbf{D}}{(1+\mathbf{r})}.\mathbf{P}.\mathbf{Dr} + \tfrac{1}{2}\frac{\mathbf{CX}.\mathbf{P}.(\mathbf{Dr})^2}{(1+\mathbf{r})^2}$$

Now,

$$\frac{\mathbf{dP}}{\mathbf{dr}} = \frac{-\mathbf{D}}{1+\mathbf{r}}.\mathbf{P}$$

Thus,

$$\frac{\mathbf{d^2P}}{\mathbf{dr^2}} = \frac{\mathbf{P}}{(1+\mathbf{r})^2}.\mathbf{D} - \frac{\mathbf{P}}{1+\mathbf{r}}\frac{\mathbf{dP}}{\mathbf{dr}} + \frac{\mathbf{D}^2}{(1+\mathbf{r})^2}.\mathbf{P}$$

and

$$\mathbf{CX} = \mathbf{D} + \mathbf{D}^2 - (1+\mathbf{r})\frac{\mathbf{dP}}{\mathbf{dr}}$$

Consider the duration of a series of cash flows CF_1, ...CF_M at an interest rate **r**.

$$\mathbf{D} = \frac{1}{\mathbf{P}}.\sum_{\mathbf{i}=1}^{\mathbf{M}} \mathbf{iCFi}$$

$$\frac{\mathbf{dD}}{\mathbf{dr}} = \frac{-\mathbf{dP}}{\mathbf{dr}}.\frac{1}{\mathbf{P}^2}\sum_{\mathbf{i}=1}^{\mathbf{M}}\frac{\mathbf{iCFi}}{(1+\mathbf{r}^{\mathbf{i}})} - \frac{1}{\mathbf{P}(1+\mathbf{r})}\sum_{\mathbf{i}=1}^{\mathbf{M}}\frac{\mathbf{i^2CFi}}{(1+\mathbf{r})^{\mathbf{i}}}$$

$$= \frac{\mathbf{D}}{1+\mathbf{r}}.\mathbf{P}.\frac{1}{\mathbf{P}^2}.\mathbf{D}.\mathbf{P} - \frac{1}{\mathbf{P}(1+\mathbf{r})}\sum_{\mathbf{i}=1}^{\mathbf{M}}\frac{\mathbf{i^2CFi}}{(1+\mathbf{r})}$$

$$= \frac{\mathbf{D}^2}{(1+\mathbf{r})} - \frac{1}{\mathbf{P}(1+\mathbf{r})}\sum_{\mathbf{i}=1}^{\mathbf{M}}\frac{\mathbf{i^2CFi}}{(1+\mathbf{r})^{\mathbf{i}}}$$

$$\textbf{Therefore,} \mathbf{CX} = \mathbf{D} + \mathbf{D}^2 - \mathbf{D}^2 + \frac{1}{\mathbf{P}}\sum_{\mathbf{i}=1}^{\mathbf{M}}\frac{\mathbf{i^2CFi}}{(1+\mathbf{r})^{\mathbf{i}}}$$

$$= \mathbf{D} + \frac{1}{\mathbf{P}}\sum_{\mathbf{i}=1}^{\mathbf{M}}\frac{\mathbf{i^2CFi}}{(1+\mathbf{r})^{\mathbf{i}}}$$

which is the computation formula given in the text.

SOLUTIONS

QUESTION 1:

On the Casio FC100 and Sharp EL-735:

20000	FV		
3	n		
8	i%		
	COMP	PV	–15876.64

On the HP12C:

20000	FV	
3	n	
8	i	
	PV	–15876.64

The present value is $15876.64.

QUESTION 2:

On the Casio FC100 and Sharp EL-735:

1	+/–	PV			
11	÷	2	=	i%	
2	n				
	COMP	FV			1.1130
11	÷	4	=	i%	
4	n				
	COMP	FV			1.1146
11	÷	365	=	i%	
365	n				
	COMP	FV			1.1163

On the HP12C:

1	CHS	PV		
11	E	2 ÷	i	
	FV			1.1130
11	E	4 ÷	i	
4	n			
	FV			1.1146
11	E	365 ÷	i	
365	n			
	FV			1.1163

The effective interest rates are 11.30% p.a. for half-yearly compounding, 11.46% p.a. for quarterly compounding and 11.63% p.a. for daily compounding.

QUESTION 3:

On the Casio and Sharp calculators:

1000000	FV					
1	n					
6.75	x	180 =	÷	365 i%		
	COMP	PV				–967784.70
6.35	x	152 =	÷	365 i%		
	COMP	PV				–974237.43

When the bill is sold, it is a 152-day bill. The holding period return can be calculated as follows:

967784.70	+/–	PV

974237.43	**FV**						
1	**n**						
	COMP	**i%**					**0.6668**
	x	**365**	**=**	**÷**	**28**	**=**	**8.69**

The return earned over the 28-day holding period is 8.69% p.a. It is higher than the purchase yield because the yield at which the bill was sold was lower than the purchasing yield.

On the HP12C:

100000	**FV**					
1	**n**					
6.75	**E**	**180**	**x**	**365**	**÷**	**i**
	PV					**–967784.70**
6.35	**E**	**152**	**x**	**365**	**÷**	**i**
	PV					**–974327.43**

To calculate the holding period yield:

967784.70	**CHS**	**PV**		
974237.43	**FV**			
1	**n**			
	i			**0.6668**
365	**x**	**28**	**÷**	**8.69**

QUESTION 4:

On the Casio and Sharp calculators:

1000000	**FV**				
10	**n**				
8.95	**÷**	**2**	**=**	**i%**	
	COMP	**PV**			**–645470.21**
8	**n**				
8.90	**÷**	**2**	**=**	**i**	
	COMP	**PV**			**–705882.55**

To calculate the holding period return:

645470.21	**+/–**	**PV**		
705882.55	**FV**			
2	**n**			
	COMP	**i%**		**4.575**
	x	**2**	**=**	**9.15**

The holding period yield is 9.15% p.a.

On the HP12C:

1000000	**FV**				
10	**n**				
8.95	**E**	**2**	**÷**	**i**	
	PV				**–645470.21**
8	**n**				
8.90	**E**	**2**	**÷**	**i**	
	PV				**–705882.55**

Calculating the holding period return:

645470.21	CHS	PV	
705882.55	FV		
2	n		
	i		4.575
2	x		9.15

QUESTION 5:

On the Sharp and Casio calculators:

100000	FV		
10	n		
3200	PMT		
3	i%		
	COMP	PV	–101706.04
7	n		
3.3	i%		
	COMP	PV	–99383.96

To calculate the holding period return:

101706.04	+/–	PV	
99383.96	FV		
3200	PMT		
3	n		
	COMP	i%	2.403
	x	2 =	4.81

The holding period return is 4.81% p.a. It is lower than the purchase yield because the yield at sale is higher than it.

On the HP12C:

100000	**FV**	
10	**n**	
3200	**PMT**	
3	**i**	
	PV	**–101706.04**
7	**n**	
3.3	**i**	
	PV	**–99383.96**

To calculate the holding period return:

101706.04	**CHS**	**PV**	
99383.96	**FV**		
3200	**PMTR**		
3	**n**		
	i		**2.403**
2	**x**		**4.81**

QUESTION 6 (a):

On the Casio calculator:

100000	PV				
0	FV				
120	n				
8.75	÷	12	=	i%	
	COMP	PMT			–1253.27
8	x	12	=	n	
	COMP	PV			86308.20

The amount of principal paid off is: **$100,000 – $86,308.20 = $13,691.80**

The total amount paid is: **24 x $1,253.27 = $30,078.48**

Therefore, the amount of interest paid is: **$16,386.68**.

On the Sharp EL-735, we can take advantage of the **x 12** and **÷ 12** keys.

Thus:

100000	PV		
0	FV		
10	2nd F	x 12	
8.75	2nd F	÷ 12	
	COMP	PMT	–1253.27
8	2nd F	x 12	
	COMP	PV	86308.20

On the HP12C:

100000	**PV**				
0	**FV**				
120	**n**				
8.75	**E**	**12**	**÷**	**i**	
	PMT				**–1253.27**
8	**E**	**12**	**x**	**n**	
	PV				**86308.20**

QUESTION 6 (b):

On the Casio calculator:

8	**x**	**12**	**=**	**n**	
86308.20	**PV**				
9.50	**÷**	**12**	**=**	**i%**	
	COMP	**PMT**			**–1286.93**

The payment increases to $1286.93.

QUESTION 6 (c):

In order to break even, the amount to be refinanced should produce the same payment of $1253.27.

On the Casio calculator:

8	**x**	**12**	**=**	**n**	
1253.27	**+/–**	**PMT**			
8.25	**÷**	**12**	**=**	**i%**	
	COMP	**PV**			**87861.99**

Since the amount of the loan outstanding is \$86,308.20, the penalty needs to be \$1,553.79 to create a break-even situation.