DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

2. Curves in Space

2.1. Curvature, Torsion, and the Frenet Frame. Curves in space are the natural generalization of the curves in the plane which were discussed in Chapter 1 of the notes. Namely, a *parametrized curve in the space* is a differentiable function $\alpha : (a, b) \to \mathbb{R}^3$. It has the form

$$\alpha(t) = (x(t), y(t), z(t)),$$

where $a < t < b$. The velocity of α is

$$\alpha'(t) = (x'(t), y'(t), z'(t)),$$

for $a < t < b$. The entire Section 1.3 of the notes can be immediately reformulated for curves in the space: Definition 1.3.2 (of the length of a curve over a closed interval), Definition 1.3.3 and Theorem 1.3.4 (concerning reparametrization of curves), Definition 1.3.4 (of a regular curve), Theorem 1.3.6 and Proposition 1.3.7 (concerning parametrization by arc length).

As about Section 1.4 (that is, the curvature and the fundamental theorem of curves), things are different. First of all, a purely formal observation is that Definition 1.4.1. (of the curvature) will not be satisfactory: it is "obvious" that only $x(t)$ and $y(t)$ (and their derivatives) are not enough to determine the curvature, one needs to bring also $z(t)$ into this game. But nothing prevents us from requesting Theorem 1.4.3., the first equation:

$$T'(s) = \kappa(s)N(s). \tag{1}$$

Here the curve α is parametrized by arc length, $T(s)$ is the velocity vector, that is $T(s) = \alpha'(s)$, and $N(s)$ is a vector of length 1 parallel to $T'(s) = \alpha''(s)$.

Throughout this section the following assumption will be always in force.

$$\boxed{\textbf{Assumption. } \text{The curve } \alpha : (a, b) \to \mathbb{R}^3 \text{ is parametrized by arc length and } \alpha''(s) \neq 0 \text{ for all } s.}$$

Definition 2.1.1. If α is like in the assumption, then the *curvature* of α at s is

$$\kappa(s) = \|\alpha''(s)\|.$$

We also define the vectors

$$T(s) = \alpha'(s),$$

called the *tangent vector* at s,

$$N(s) = \frac{1}{\kappa(s)}\alpha''(s)$$

called the *normal vector* at s, and[1]

$$B(s) = T(s) \times N(s),$$

called the *binormal* vector.

Let us note that N is perpendicular to T:

$$\|\alpha'(s)\|^2 = 1 \Rightarrow \alpha'(s) \cdot \alpha'(s) = 1 \Rightarrow 2\alpha''(s) \cdot \alpha'(s) = 0 \Rightarrow T(s) \cdot N(s) = 0.$$

[1]Here $\times$ denotes the cross product. Let us recall that if a and b are two vectors which are different from 0 and not collinear, then the vector $a \times b$ has the direction perpendicular to both a and b, the sense determined by the right hand rule, and the length

$$\|a \times b\| = \|a\|\|b\| \sin\theta$$

where θ is the angle between a and b.

Let us also note that $\|N(s)\| = 1$. Because also $T(s)$ has length 1, from the definition of the cross product we deduce that $B(s)$ is perpendicular to both $T(s)$ and $N(s)$, and

$$\|B(s)\| = 1.$$

Let's summarize: the vectors in the triple $(T(s), N(s), B(s))$ are all of length 1 and are any two perpendicular. This triple is called the Frenet frame. It is important to note that we can bring this frame onto (e_1, e_2, e_3) by a rotation, where (e_1, e_2, e_3) are the vectors of length 1 pointing in the positive direction of the coordinate axes (this fact is a direct consequence of the right hand rule).

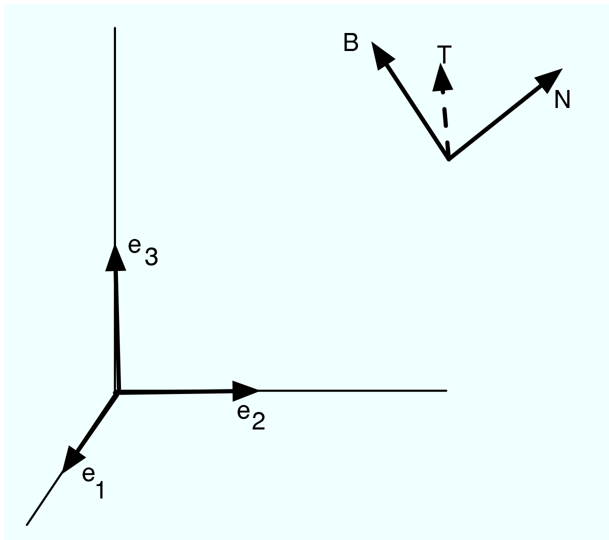

FIGURE 1. The vectors T, N, B in the figure (T is behind the plane determined by B and N) have length 1 and are any two orthogonal. But the triple (T, N, B) cannot be brought to (e_1, e_2, e_3) by a rotation. The reason is simple: $T \times N$ is $-B$, not B.

Only the curvature is not sufficient to determine the curve, like in the Fundamental Theorem of Plane Curves (Theorem 1.4.6.) For example, let us consider the helix, which is the trajectory of a point on the propeller of a helicopter which ascends (vertically); we assume that both the angular velocity of the propeller and the linear velocity of the helicopter are constant (see also Figure 2).

Exercise. Show that the helix is the trace of the curve

$$\alpha(t) = (a \cos t, a \sin t, bt),$$

with t in $\mathbb{R}$. **Hint.** Use Figure 2. to find the coordinates of $\alpha(t)$. Denote by τ the time, by ω the angular velocity of the propeller, and by v the velocity of the helicopter along the z axis, then find the coordinates at the moment τ.

The trace of the helix can be seen in Figure 3. To find its curvature, we will parametrize it by arc length. We determine

$$s(t) = \int_0^t \sqrt{a^2 \sin^2 u + a^2 \cos^2 u + b^2} du = \sqrt{a^2 + b^2} t.$$

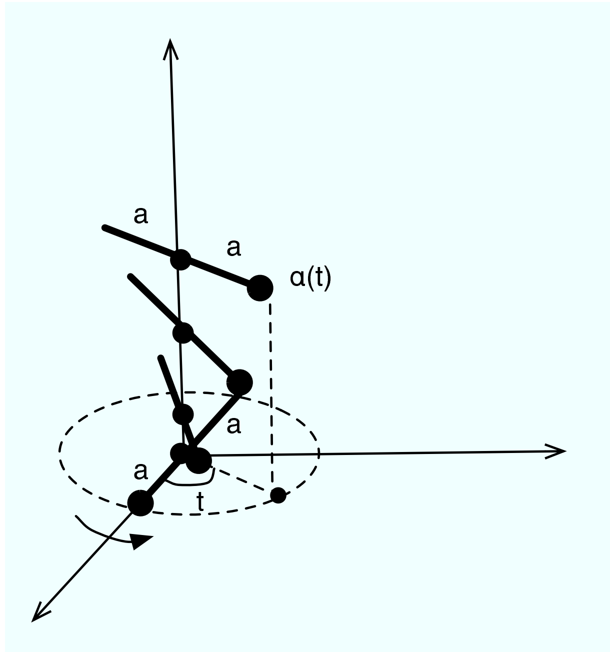

FIGURE 2. This is how the curve called helix arises.

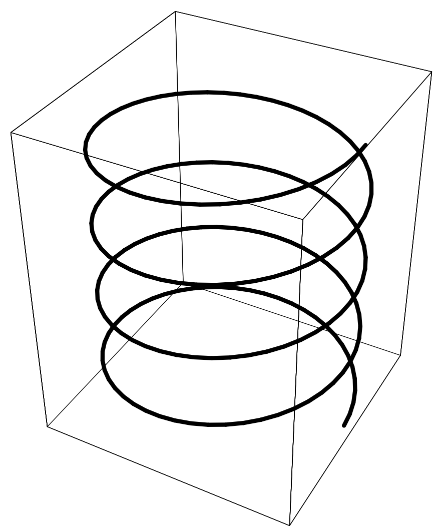

FIGURE 3. The helix.

We solve $s = \sqrt{a^2 + b^2} t$ with respect to t, and find

$$t = \frac{1}{\sqrt{a^2 + b^2}} s.$$

Denote

$$c = \sqrt{a^2 + b^2}$$

and obtain the following parametrization by arc length of the helix:

$$\bar{\alpha}(s) = (a \cos(\frac{s}{c}), a \sin(\frac{s}{c}), \frac{bs}{c}).$$

The curvature is

$$\kappa(s) = \|\alpha''(s)\| = \frac{a}{a^2 + b^2}.$$

Note that $\kappa(s)$ does not depend on s, so the helix is identically curved at any of its points.

Now if we choose $r := \frac{a^2+b^2}{a^2}$, then a circle of radius r has curvature

$$\frac{1}{r} = \frac{a^2}{a^2+b^2},$$

which is the same as for the helix. But it's obviously *not* possible to obtain the (trace of the) helix from (trace of the) the circle by rotations and translations (compare to Theorem 1.4.6.)

So if we want to have a Fundamental Theorem for curves in the space, we need to associate to a curve something more than just the curvature. Let us analyze the case when the trace of α is contained in a plane (see Figure 4).

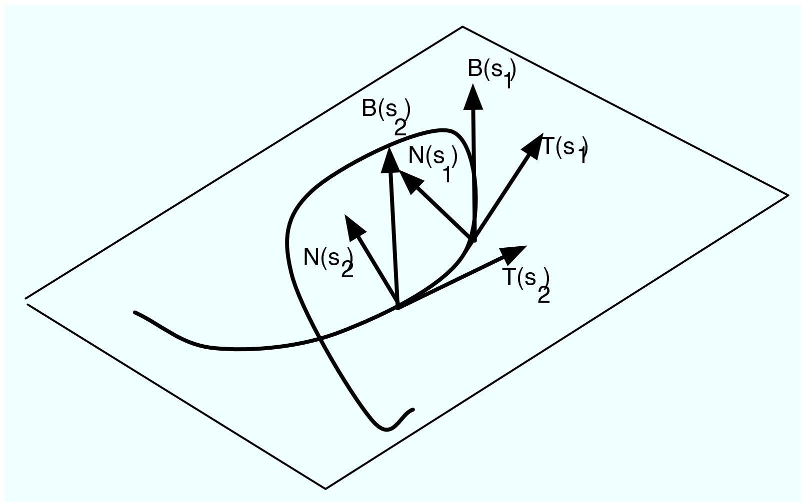

FIGURE 4. The trace of the curve α is contained in a plane: one can see the vectors T, N, and B at two different values of s.

Exercise. If the trace of α is contained in a plane, then the vectors $T(s)$ and $N(s)$ are parallel to that plane. **Hint.** Consider a vector v perpendicular to the plane and a point P in the plane. For any point $\alpha(s)$ on the curve, the vector $\alpha(s)-P$ is parallel to the plane, so it's perpendicular to v. You only need to show that both $\alpha'(s)$ and $\alpha''(s)$ are perpendicular to v (keyword: dot product).

Consequently, if the trace of α is contained in a plane, then the vector $B(s)$ is the same for all s: it is one of the two vectors of length 1 which are perpendicular to the plane. For an arbitrary curve, it seems natural to try to measure the extent to which it is not a plane curve. The discussion above suggests that one could do that by looking at the (rate of) change of $B(s)$, that is, $B'(s)$. Indeed, the number $\|B'(s)\|$ is, up to a possible negative sign, the torsion of a curve at a given point, which is defined in the following.

First, let us note that $B'(s)$ is parallel to $N(s)$. This is because on the one hand

$$\|B(s)\|^2 = 1 \Rightarrow B(s)\cdot B(s) = 1 \Rightarrow B'(s)\cdot B(s) = 0,$$

which means that $B'(s)$ is perpendicular to $B(s)$. On the other hand

$$\begin{aligned} B'(s) &= \frac{d}{ds}(T(s)\times N(s)) = T'(s)\times N(s) + T(s)\times N'(s) \\ &= T(s)\times N'(s) \text{ (as } T'(s) \text{ is parallel to } N(s)) \end{aligned}$$

which implies that $B'(s)$ is perpendicular to $T(s)$. Now a vector perpendicular to both $B(s)$ and $T(s)$ must be collinear to $N(s)$, so we must have[2]

$$B'(s) = -\tau(s)N(s), \tag{2}$$

for some number $\tau(s)$.

Definition 2.1.2. The number $\tau(s)$ determined by (2) is called the <u>*torsion*</u> of α at s.

Exercise. Show that a curve α has its trace contained in a plane if and only if $\tau(s) = 0$ for all s (see also the previous exercise).

Let us now look at Equations (1) and (2). They describe the derivatives of two of the components of the Frenet frame $(T(s), N(s), B(s))$ in terms of the original vectors. What about $N'(s)$? The answer can be deduced as a direct consequence of (1), (2), and the product rule, as follows:

$$\begin{aligned} N'(s) &= \frac{d}{ds}N(s) = \frac{d}{ds}(B(s) \times T(s)) = B'(s) \times T(s) + B(s) \times T'(s) \\ &= -\tau(s)N(s) \times T(s) + \kappa(s)B(s) \times N(s) = -\kappa(s)T(s) + \tau(s)B(s). \end{aligned}$$

We record the formulas for the derivatives of T, N, and B in the following theorem.

Theorem 2.1.3. (The Frenet formulas). *We have*

$$\begin{aligned} T' &= \qquad \kappa N \\ N' &= -\kappa T \qquad + \tau B \\ B' &= \qquad -\tau N \end{aligned}$$

These formulas have several applications. In the following we will mention only a few of them. The first one are explicit formulas for the curvature, and especially the torsion, of a curve, which is not necessarily parametrized by arc length.

Theorem 2.1.4. *Let $\alpha : (a, b) \to \mathbb{R}^3$ be a regular curve, not necessarily parametrized by arc length.*

(a) The parametrization by arc length of α satisfies the assumption above if and only if $\alpha'(t) \times \alpha''(t) \neq 0$ for all t.

(b) If $\alpha'(t) \times \alpha''(t) \neq 0$, for all t, then the curvature and the torsion of α at t are given by

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}$$

$$\tau(t) = \frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{\|\alpha'(t) \times \alpha''(t)\|^2}$$

Proof. We reparametrize the curve by the arc length. As explained in these notes, Chapter 1, Section 1.3., we obtain a new curve β given by

$$\beta(s) = \alpha(t(s)),$$

where s is the arc length. The curvature and torsion of α at t are the curvature, respectively torsion, of β at $s(t)$: denote it by κ. Also denote by T, N, B the Frenet frame of β at $s(t)$.

[2] The negative sign chosen here has a reason, which will be explained later on. Some authors, like do Carmo, don't use the negative sign, so they write $B' = \tau N$.

Exercise. Determine $\alpha'(t) = \frac{d}{dt}\alpha$, $\alpha''(t) = (\frac{d}{dt})^2\alpha$ and $\alpha'''(t) = (\frac{d}{dt})^3\alpha$ as linear combinations of T, N, B. For example, we can write

$$\alpha'(t) = \frac{d}{dt}\beta(s(t)) = \frac{d}{ds}\beta(s)\frac{d}{dt}s(t) = \frac{ds}{dt}T,$$

$$\begin{aligned}\alpha''(t) &= \frac{d}{dt}(\frac{ds}{dt}T) = \frac{d^2s}{dt^2}T + \frac{ds}{dt}\frac{dT}{dt} = \frac{d^2s}{dt^2}T + \frac{ds}{dt}\frac{dT}{ds}\frac{ds}{dt}\\ &= \frac{d^2s}{dt^2}T + \left(\frac{ds}{dt}\right)^2\frac{dT}{ds} = \frac{d^2s}{dt^2}T + \left(\frac{ds}{dt}\right)^2\kappa N.\end{aligned}$$

Use similar computations for $\alpha'''(t)$, calculate the cross product $\alpha' \times \alpha''$, then $(\alpha' \times \alpha'') \cdot \alpha'''$ and prove the two formulas.

Corollary 2.1.5. *If $\alpha : (a, b) \to \mathbb{R}^3$ is parametrized by arc length, then the following relation involving the curvature κ and the torsion τ of α at s holds:*

$$\tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\kappa^2}.$$

The next corollary is also meant to explain the choice of the negative sign in equation (2). First, we mention that the plane through $\alpha(s)$ determined by the vectors $T(s)$ and $N(s)$ is called the *osculating plane* at s.

Corollary 2.1.6. *Fix s and assume that $\alpha'''(s) \neq 0$. If $\tau(s) > 0$, then the curve crosses the osculating plane in such a way that for all sufficiently small $h > 0$, the points $\alpha(s + h)$ are on the same side of the osculating plane as $B(s)$ (informally speaking, $B(s)$ indicates the sense of the motion along the curve).*

One should also note that if we hadn't chosen the negative sign in (2), we would have the opposite situation in the corollary: $B(s)$ would be opposite to the sense of motion along the curve.

Proof. One uses the Taylor expansion

$$\alpha(s + h) = \alpha(s) + h\alpha'(s) + \frac{h^2}{2}\alpha''(s) + \frac{h^3}{6}\alpha'''(s) + o(h^3),$$

where $o(h^3)$ are terms of degree at least 4. The first three terms in the right hand side represent a point in the osculating plane (since $\alpha'(s) = T(s)$ and $\alpha''(s) = \kappa(s)N(s)$). This implies that the left hand side of the equation

$$\alpha(s + h) - \left(\alpha(s) + h\alpha'(s) + \frac{h^2}{2}\alpha''(s)\right) = \frac{h^3}{6}\alpha'''(s) + o(h^3),$$

is a vector with tail on the osculating plane and tip at $\alpha(s + h)$. But then this vector must point to the same side of the osculating plane as $\alpha'''(s)$. The main point of the proof is that the vectors $\alpha'''(s)$ and $B(s)$ point to the same side of the osculating plane. This is because the numbers $(T(s) \times N(s)) \cdot \alpha'''(s)$ and $(T(s) \times N(s)) \cdot B(s)$ have the same sign. In turn, this follows immediately from Corollary 2.1.5 and the equations $\alpha'(s) = T(s)$ and $\alpha''(s) = \kappa(s)N(s)$. See also Figure 5.

Another application of the Frenet formulas is a proof of the following theorem.

Theorem 2.1.7. (The fundamental theorem of space curves) *If $\kappa, \tau : (a, b) \to \mathbb{R}$ are two arbitrary functions, where $\kappa(s) > 0$ for all s, then there exists a curve $\alpha : (a, b) \to \mathbb{R}^3$ whose curvature and torsion at s are $\kappa(s)$, respectively $\tau(s)$. If $\beta : (a, b) \to \mathbb{R}^3$ is another such*

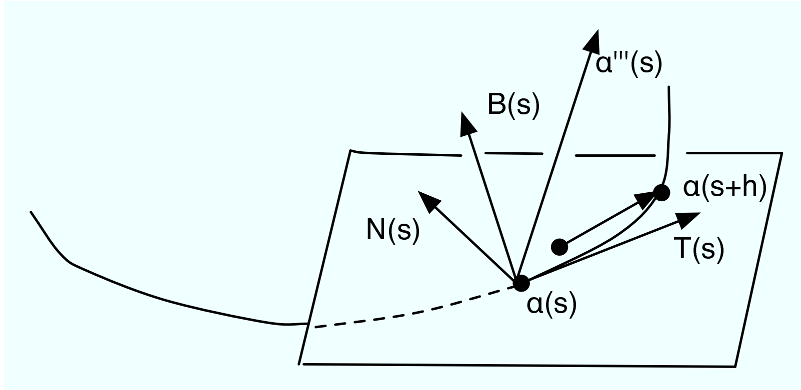

FIGURE 5. The parallelogram represents the osculating plane (the vectors $T(s)$ and $N(s)$ are contained there). The dot with no label on it in the plane is $\alpha(s)+h\alpha'(s)+\frac{h^2}{2}\alpha''(s)$.

curve, then β differs from α by a translation followed by an orthogonal linear transformation with positive determinant. By this we mean that

$$\beta(s) = A(\alpha(s)) + X$$

for all s in (a, b), where A is a linear transformation of $\mathbb{R}^3$ of the type mentioned above.

Proof (sketch). We prove first the existence part. Denote

$$(r_{ij}(s))_{1\le i,j\le 3} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}$$

From the general theory of linear systems of differential equations, there exist $v_1, v_2, v_3 : (a, b) \to \mathbb{R}^3$ which satisfy the system

$$v_i' = \sum_{j=1}^{3} r_{ij} v_j, \tag{3}$$

where $i = 1, 2, 3$. Moreover, the initial condition can be arbitrary. That is, we can pick c with $a < c < b$ and $v_1(c), v_2(c), v_3(c)$ arbitrary vectors of length 1, any two orthogonal and such that

$$v_3(c) = v_1(c) \times v_2(c).$$

They determine the solution uniquely. We prove that for any s, the vectors $v_1(s), v_2(s), v_3(s)$ are of length 1 and any two perpendicular (we prefer to skip this detail, see [Sp, Ch.1, Theorem 11]). We set

$$\alpha(s) := \int_c^s v_1(u)du.$$

This curve has

$$T(s) = \alpha'(s) = v_1(s).$$

Because $\|\alpha'(s)\| = 1$, for all s, the curve α is parametrized by arc length. We also have

$$\alpha''(s) = v_1'(s) = r_{11}v_1 + r_{12}v_2 + r_{13}v_3 = \kappa(s)v_2(s)$$

which implies that the curvature of α is κ and

$$v_2(s) = N(s).$$

We still need to show that the torsion of α is τ. By Corollary 2.1.5, the torsion of α is given by

$$\frac{(\alpha'(s) \times \alpha''(s)) \cdot \alpha'''(s)}{\kappa(s)^2}.$$

Because $\alpha'' = \kappa v_2$, we deduce

$$\alpha''' = \kappa' v_2 + \kappa v_2' = \kappa' v_2 + \kappa(-\kappa v_1 + \tau v_3) = \kappa' v_2 - \kappa^2 v_1 + \kappa\tau v_3.$$

Consequently, the torsion of α is

$$\frac{(v_1 \times \kappa v_2) \cdot (\kappa' v_2 - \kappa^2 v_1 + \kappa\tau v_3)}{\kappa(s)^2} = \tau,$$

as desired.

To prove the uniqueness part, we choose the linear orthogonal transformation A which maps the Frenet fame of α at c to the Frenet frame of β at c. Note that both frames are orthonormal systems of vectors with the last vector equal to the vector product of the previous two: thus the transformation A mentioned above exists and moreover, has $\det(A) = 1$. We have that

$$\beta'(s) = A\alpha'(s),$$

for all s, by using a uniqueness property of the solutions of the system (3). This in turn implies

$$\beta(s) = A\alpha(s) + X$$

where X is a constant. □

At the end of the section we mention one more example of a space curve (a few more will be given in the next sections).

Example. The *Viviani curve* is the intersection of the cylinder of diameter a with a sphere whose centre is on one of the generating lines of the cylinder and radius is a (see[3] Figure 6).

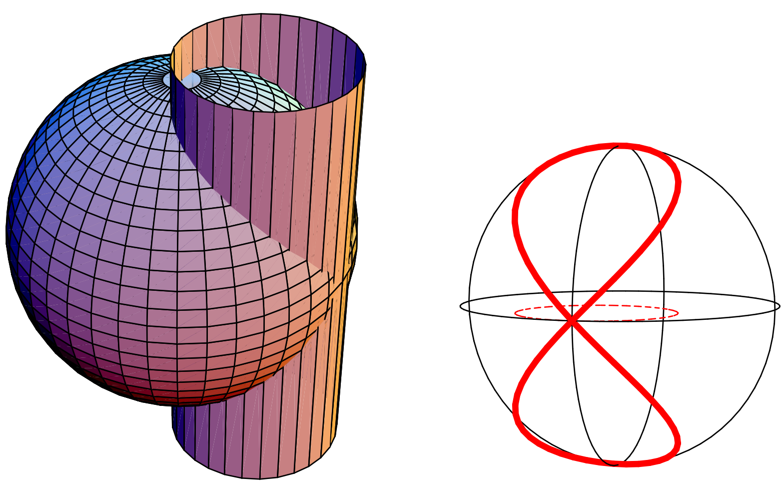

FIGURE 6. The Viviani curve.

An obvious choice of coordinates gives the implicit description of the curve as

$$x^2 + y^2 + z^2 = 4a^2$$
$$(x - a)^2 + y^2 = a^2$$

[3]I took the figure from `http://mathworld.wolfram.com/VivianisCurve.html`

It is not difficult to check that a parametrization of this curve is given by

$$\alpha(t) = (a(1+\cos t), a\sin t, 2a\sin\frac{t}{2}),$$

where t is in $\mathbb{R}$.

2.2. Space Curves That Lie on a Sphere

In this section we will make the following assumption:

Assumption. The curve $\alpha : (a,b) \to \mathbb{R}^3$ is parametrized by arc length and satisfies

$$\alpha''(s) \neq 0, \quad \tau(s) \neq 0, \quad \kappa'(s) \neq 0,$$

for all s.

Denote

$$\rho(s) := \frac{1}{\kappa(s)}.$$

The main goal of the section is to give a proof of the following theorem.

Theorem 2.2.1. *The trace of α is contained in a sphere of radius r if and only if we have*

$$\rho^2(s) + \frac{(\rho'(s))^2}{\tau(s)^2} = r^2, \tag{4}$$

for all s in (a,b).

Proof of "$\Rightarrow$". The hypothesis is that $\alpha(s)$ is contained in a sphere of radius r. Let A be the centre of the sphere. Consider the function

$$g(s) := \|\alpha(s) - A\| = (\alpha(s) - A)\cdot(\alpha(s) - A),$$

where s is in (a,b). We know that $g(s)$ is constant, equal to r^2. Its first derivative is

$$g'(s) = 2(\alpha(s) - A)\cdot T(s) = 0. \tag{5}$$

The second derivative is (we don't write "(s)" any more)

$$\frac{1}{2}g'' = (\alpha - A)\kappa N + 1 = (\alpha - A)\frac{1}{\rho}N + 1 = 0. \tag{6}$$

The third derivative is

$$\frac{1}{2}g''' = -(\alpha - A)\frac{\rho'}{\rho^2}N + (\alpha - A)\frac{1}{\rho}(-\kappa T + \tau B) + T\frac{1}{\rho}N = 0$$

which implies

$$(\alpha - A)\frac{\rho'}{\rho^2}N + (\alpha - A)\frac{1}{\rho}(\kappa T - \tau B) = 0. \tag{7}$$

From (5) and (6) we deduce that

$$\alpha - A = -\rho N + \lambda B$$

where λ is a number. We can determine it by using (7), as follows:

$$(-\rho N + \lambda B)\cdot\frac{\rho'}{\rho^2}N + (-\rho N + \lambda B)\frac{1}{\rho}(\kappa T - \tau B) = 0$$

which implies

$$-\frac{\rho'}{\rho} - \frac{\lambda\tau}{\rho} = 0 \Rightarrow \lambda = -\frac{\rho'}{\tau}.$$

We conclude that

$$\alpha(s) - A = -\rho N - \frac{\rho'}{\tau}B,$$

which implies (by using Pythagora's theorem) that

$$\|\alpha(s) - A\|^2 = \rho^2(s) + \frac{(\rho'(s))^2}{\tau(s)^2}.$$

On the other hand, $\|\alpha(s) - A\|^2$ is equal to r^2, so the conclusion follows. $\square$

Now we want to prove that if (4) holds, then the race of α is contained in a sphere of radius a. To this end we will need the following lemma.

Lemma 2.2.2. We have

$$\frac{d}{ds}\left(\alpha + \rho N + \frac{\rho'}{\tau}B\right) = \left(\rho\tau + \left(\frac{\rho'}{\tau}\right)'\right)B$$

$$\frac{d}{ds}\left(\rho^2 + \frac{(\rho')^2}{\tau^2}\right) = 2\frac{\rho'}{\tau}\left(\rho\tau + \left(\frac{\rho'}{\tau}\right)'\right)$$

Exercise. Prove the two equations in Lemma 2.2.2. Then use them to prove the implication "$\Leftarrow$" in Theorem 2.2.1.

An example of a curve whose trace is on a sphere is the Viviani curve. We will give more examples in the next section.

2.3. Curves of constant slope. Loxodromes on the sphere.

From [Gr-Abb-Sa], Section 8.5 (page 246) take Definition 8.17, Lemma 8.18, and Lemma 8.19 (with proofs).

We say that a curve $\alpha : (a, b) \to \mathbb{R}^3$ is of *constant slope* if there exists a unit vector u such that the angle between $T(t)$ (the unit tangent vector to α) and u is independent of t.

An example of a curve of constant slope is the helix (see Figure 3, and also Homework no. 2, Question no. 1 (d)). Indeed, we have seen that for this curve we have

$$T(s) = \bar{\alpha}'(s) = \left(\frac{a}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{a}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{b}{\sqrt{a^2+b^2}}\right).$$

The last component of $T(s)$ is just $T \cdot e_3 = \frac{b}{\sqrt{a^2+b^2}}$, which is constant (here $e_3 := (0, 0, 1)$).

In the following we will construct an example of a curve of constant slope whose trace is contained in a sphere. We start with the cardioid in the xy plane; we actually consider its realization given in Figure 7. The corresponding parametrization is

$$\alpha(t) = (1 + 2\cos t(1 - \cos t), 2\sin t(1 - \cos t)).$$

Then we consider the intersection of the cylinder over the trace of this curve with the sphere of radius 3 and center at 0. The latter curve can be parametrized as

$$\beta(t) = (1 + 2\cos t(1 - \cos t), 2\sin t(1 - \cos t), 2\sqrt{2}\cos\frac{t}{2}).$$

Exercise. It is obvious that the trace of β is on the the cylinder over the trace of α. Show that the trace of β is contained in the sphere of radius 3 and center at 0.

We want to show that β has constant slope with respect to $e_3 = (0, 0, 1)$. To this end we determine the unit tangent vector at an arbitrary point, that is $\beta'(t)/\|\beta'(t)\|$. By using also the solution to Question no. 5, Assignment no. 1 we deduce that

$$\beta'(t) = (-2\sin t + 2\sin 2t, 2\cos t - 2\cos 2t, -\sqrt{2}\sin\frac{t}{2}),$$

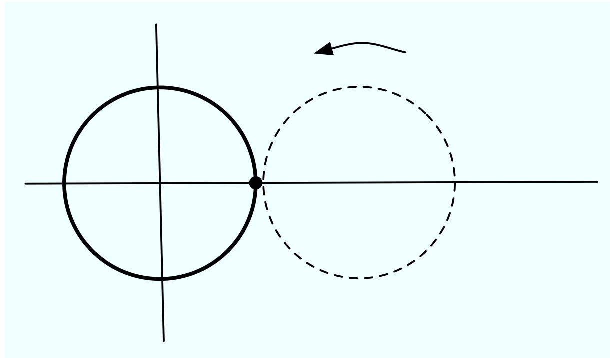

FIGURE 7. The dotted circle rotates around the continuous one and generates the cardioid (the coordinate system is slightly different from the one chosen in Section 1.6, and this gives a different parametrization).

Exercise. Show that

$$\|\beta'(t)\| = 3\sqrt{2}\sin\frac{t}{2}.$$

You may want to note that t is between 0 and 2π (from the geometric construction of the cardioid), so $t/2$ is between 0 and π, hence $\sin t/2 \geq 0$.

We are interested in

$$\beta'(t)/\|\beta'(t)\| \cdot (0,0,1) = -\frac{1}{3},$$

which is indeed constant (independent of t). So β is a curve of constant slope (see also Figure 8).

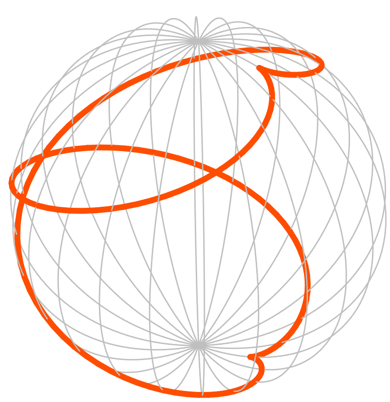

FIGURE 8. The spherical cardioid (the sphere has radius equal to 3).

We note that the construction above can be done by starting with two circles of *different* radii. We let one of them rotate around the other one: the resulting curve is a generalization of the cardioid and it's called a *epicycloid* (see [Gr-Abb-Sa], page 144). The cone over such a curve intersects a certain sphere with centre at O: the resulting curve is again of constant slope. These curves with trace on the sphere are the *spherical helices* (for more details, see [Gr-Abb-Sa], Section 8.5).

In the last part of this section we will discuss about loxodromes on the sphere.

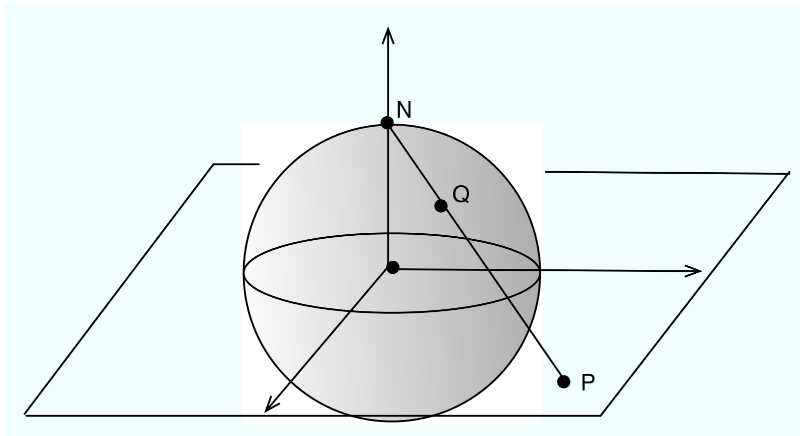

FIGURE 9. The stereographic projection of P is Q.

Definition 2.3.1. A *spherical loxodrome* is a curve with trace on the sphere which makes a constant angle[4] with any meridian that it crosses.

We want to determine all spherical loxodromes on the sphere S with centre at O and radius 1. An important instrument is the *stereographic projection*, which is a map, call it Φ, from the (horizontal) xy plane to the sphere. By definition, if $P = (x, y, 0)$ is in the horizontal plane, then $\Phi(P)$ is the intersection of the line segment NP with the sphere (here $N = (0, 0, 1)$ is the North pole).

Exercise. Prove that if $P = (x, y, 0)$ then

$$\Phi(P) = \frac{1}{x^2 + y^2 + 1}(2x, 2y, x^2 + y^2 - 1).$$

Hint. The straight line determined by P and N is the trace of $\alpha(t) = (tx, ty, 1 - t)$; the latter point is on the sphere if and only if $(tx)^2 + (ty)^2 + (1 - t)^2 = 1 \rightarrow t = \frac{2}{x^2+y^2+1}$ (we took $t \neq 0$).

The following lemma is intuitively clear, and will not be proved here.

Lemma 2.3.2.

(a) *A circle with center at O in the xy plane is mapped under Φ on a parallel on the sphere.*
(b) *A straight line through the origin in the xy plane is mapped under Φ on a meridian on the sphere (minus N, which is not in the image of Φ).*
(c) *The map Φ preserves the angles between curves (by this we mean that if $t \mapsto \alpha(t)$ and $u \mapsto \beta(u)$ are two curves in the plane which intersect at $\alpha(t) = \beta(u)$, then the*

[4]The *angle between two arbitrary curves* $t \mapsto \alpha(t)$ and $u \mapsto \beta(u)$ at a point $\alpha(t) = \beta(u)$ is actually the angle between the tangent vectors $\alpha'(t)$ and $\beta'(u)$ (for instance, the angle between a meridian and a parallel on a sphere is $\pi/2$).

angle between α *and* β *at the latter point is equal to the angle between* $t \mapsto \Phi(\alpha(t))$ *and* $u \mapsto \Phi(\beta(u))$ *at* $\Phi(\alpha(t)) = \Phi(\beta(u))$*).*

Consequently, loxodromes on the sphere are images under Φ of curves in the plane which meet straight lines through the origin under a constant angle. The latter curves are just the logarithmic spirals (see Section 1.3), parametrized as

$$\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$$

where t is in $\mathbb{R}$.

In conclusion, the general equation of the loxodrome is

$$\beta(t) = \frac{1}{1 + a^2e^{2bt}}(2ae^{bt}\cos t, 2ae^{bt}\sin t, a^2e^{2bt} - 1).$$

An example of a loxodrome can be visualized in Figure 10.

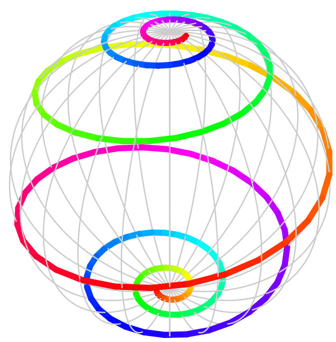

Figure 10. A loxodrome on the sphere.

References

[Gr-Abb-Sa] A. Gray, E. Abbena, and S. Salamon, *Modern Differential Geometry with MATHEMATICA*, 3rd edition

[Sp] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, volume II