



Trigonometric Substitutions  
Math 121 Calculus II  
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Now that we have trig functions and their inverses, we can use trig subs. They're special kinds of substitution that involves these functions. For these, you start out with an integral that doesn't have any trig functions in them, but you introduce trig functions to evaluate the integrals. These depend on knowing

(1) the Pythagorean identities

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \sec^2 \theta = 1 + \tan^2 \theta$$

(2) the definitions

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \sec \theta = \frac{1}{\cos \theta}$$

(3) the derivatives

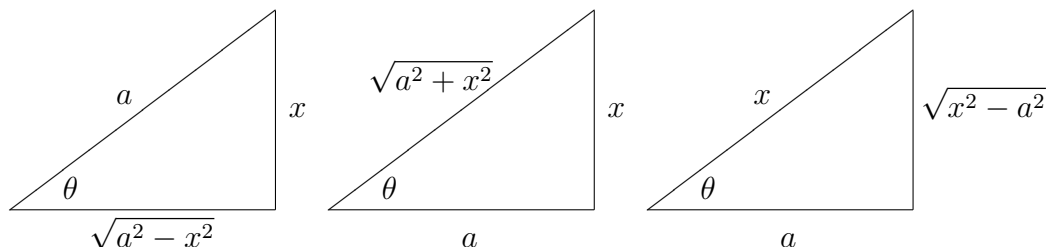
$$\begin{aligned} (\sin \theta)' &= \cos \theta & (\cos \theta)' &= -\sin \theta \\ (\tan \theta)' &= \sec^2 \theta & (\sec \theta)' &= \sec \theta \tan \theta \end{aligned}$$

There are three kinds of trig subs. You use them when you see as part of the integrand one of the expressions  $\sqrt{a^2 - x^2}$ ,  $\sqrt{a^2 + x^2}$ , or  $\sqrt{x^2 - a^2}$ , where  $a$  is some constant. In each kind you substitute for  $x$  a certain trig function of a new variable  $\theta$ . The substitution will simplify the integrand since it will eliminate the square root. Here's a table summarizing the substitution to make in each of the three kinds.

If use see	use the sub	so that	and
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$dx = a \cos \theta d\theta$	$\sqrt{a^2 - x^2} = a \cos \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$dx = a \sec^2 \theta d\theta$	$\sqrt{a^2 + x^2} = a \sec \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$dx = a \sec \theta \tan \theta d\theta$	$\sqrt{x^2 - a^2} = a \tan \theta$

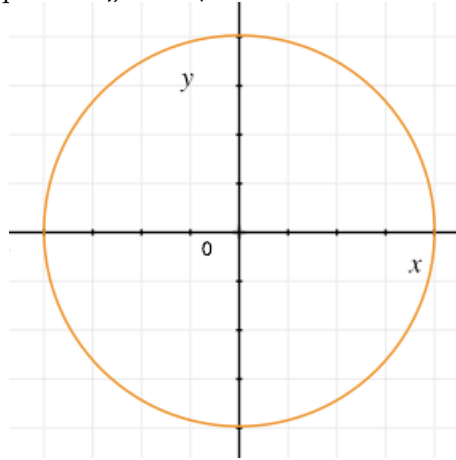
In each line, the last entry follows from the second entry by one of the Pythagorean identities.

There are also right triangles you can draw to make the connections between  $x$ ,  $a$ , and  $\theta$ . The three triangles below refer to the three trig subs, respectively.



We'll look at three examples to illustrate these trig subs.

**Example 1.** The area of a circle. We can finally show that the area of a circle of radius  $r$  is  $\pi r^2$ . Let's set our coordinates so that the center of the circle is at the origin,  $(0, 0)$ . Then the points on the circumference of the circle are at distance  $r$  from the origin, so the equation of the circle is  $x^2 + y^2 = r^2$ . The upper semicircle has the equation  $y = \sqrt{r^2 - x^2}$  and the lower semicircle has the equation  $y = -\sqrt{r^2 - x^2}$ .



Therefore, the area of the circle is given by the integral

$$\int_{-r}^r (\sqrt{r^2 - x^2} - (-\sqrt{r^2 - x^2})) dx = \int_{-r}^r 2\sqrt{r^2 - x^2} dx.$$

Because the expression  $\sqrt{r^2 - x^2}$  appears in the integral, we'll try the first trig sub with  $x = r \sin \theta$ ,  $dx = r \cos \theta d\theta$ , and  $\sqrt{r^2 - x^2} = r \cos \theta$ . Note that at the limits of integration  $x = \pm r$ , so  $\sin \theta = \pm 1$ , and  $\theta = \pm \frac{\pi}{2}$ . Therefore, the integral becomes

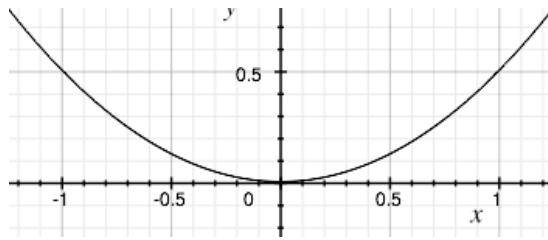
$$\int_{-\pi/2}^{\pi/2} 2r^2 \cos^2 \theta d\theta.$$

We recently found that an antiderivative of  $\cos^2 \theta$  is  $\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$ , so that last integral equals  $2r^2(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta) \Big|_{-\pi/2}^{\pi/2} = 2r^2(\frac{1}{2}\frac{\pi}{2} - 0) - 2r^2(\frac{1}{2}\frac{-\pi}{2} - 0) = \pi r^2$ . Thus, we've shown that the area of a circle of radius  $r$  is  $\pi r^2$ .

**Example 2.** The arclength of a parabola. We've seen how the length of a curve given by the equation  $y = f(x)$  for  $a \leq x \leq b$  is equal to the integral

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Let's apply that to find the length of the parabola  $y = \frac{1}{2}x^2$  for  $0 \leq x \leq 1$ .



Since the derivative of  $f(x) = \frac{1}{2}x^2$  is  $x$ , the length is

$$L = \int_0^1 \sqrt{1 + x^2} dx.$$

We'll use the trig sub of the second kind with  $x = \tan \theta$ ,  $dx = \sec^2 \theta d\theta$ , and  $\sqrt{1 + x^2} = \sec \theta$ . Then the integral becomes

$$L = \int_0^{\pi/4} \sec^3 \theta d\theta.$$

It takes an application of integration by parts to find that an antiderivative of  $\sec^3 \theta$  is  $\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta|$ . Given that, we find

$$\begin{aligned} L &= \left( \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) \Big|_{\theta=0}^{\pi/4} \\ &= \frac{1}{2} \sec \frac{\pi}{4} \tan \frac{\pi}{4} + \frac{1}{2} \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| \\ &= \frac{1}{2} \sqrt{2} + \frac{1}{2} \ln(\sqrt{2} + 1) \end{aligned}$$

Thus, the length of this arc of the parabola is about 1.1478.

**Example 3.** Consider the integral  $I = \int_2^3 \sqrt{1 - \frac{4}{x^2}} dx$ .



We can rewrite it as

$$I = \int_2^3 \frac{\sqrt{x^2 - 4}}{x} dx$$

and see that the third kind of trig sub applies. We'll let  $x = 2 \sec \theta$ ,  $dx = 2 \sec \theta \tan \theta d\theta$ , and  $\sqrt{x^2 - 4} = 2 \tan \theta$ . Then the integral becomes

$$I = \int_0^\beta \frac{2 \tan \theta}{2 \sec \theta} 2 \sec \theta \tan \theta d\theta = \int_0^\beta 2 \tan^2 \theta d\theta$$

where the upper limit of integration  $\beta$  is such that  $\cos \beta = \frac{2}{3}$ , that is,  $\beta = \arccos \frac{2}{3} = 0.84107$ . We can use the Pythagorean identity  $\sec^2 \theta = 1 + \tan^2 \theta$  to rewrite the integrand in a form we can integrate.

$$\begin{aligned} I &= \int_0^\beta 2(\sec^2 \theta - 1) d\theta \\ &= (2 \tan \theta - 2\theta) \Big|_{\theta=0}^\beta \\ &= 2 \tan \beta - 2\beta \end{aligned}$$

Note that  $\cos \beta = \frac{2}{3}$ , so  $\tan \beta = \frac{\sin \beta}{\cos \beta} = \frac{\sqrt{1 - 4/9}}{2/3} = \frac{\sqrt{5}}{2}$ . Thus,  $I = \sqrt{5} - 2 \arccos \frac{2}{3}$ .

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