# Math 113 HW #9 Solutions

## §4.1

**50.** Find the absolute maximum and absolute minimum values of

$$f(x) = x^3 - 6x^2 + 9x + 2$$

on the interval $[-1, 4]$.

**Answer:** First, we find the critical points of $f$. To do so, take the derivative:

$$f'(x) = 3x^2 - 12x + 9.$$

Then $f'(x) = 0$ when

$$0 = 3x^2 - 12x + 9 = (3x - 3)(x - 3),$$

i.e., when $x = 1$ or $3$. To find the absolute maximum and absolute minimum, then, we evaluate $f$ at the critical points and on the endpoints of the interval:

$$\begin{aligned} f(-1) &= (-1)^3 - 6(-1)^2 + 9(-1) + 2 = -14 \\ f(1) &= (1)^3 - 6(1)^2 + 9(1) + 2 = 6 \\ f(3) &= (3)^3 - 6(3)^2 + 9(3) + 2 = 2 \\ f(4) &= (4)^3 - 6(4)^2 + 9(4) + 2 = 6. \end{aligned}$$

Therefore, $f$ achieves its absolute minimum of $-14$ at $x = -1$ and its absolute maximum of 6 at both $x = 1$ and $x = 4$.

**54.** Find the absolute maximum and absolute minimum values of

$$f(x) = \frac{x^2 - 4}{x^2 + 4}$$

on the interval $[-4, 4]$.

**Answer:** First, find the critical points by finding where the derivative equals zero:

$$f'(x) = \frac{(x^2+4)(2x) - (x^2-4)(2x)}{(x^2+4)^2} = \frac{(2x^3+8x) - (2x^3-8x)}{(x^2+4)^2} = \frac{16x}{(x^2+4)^2}.$$

Therefore, $f'(x) = 0$ when $16x = 0$, meaning that $x = 0$ is the only critical point. Plugging in the endpoints and the critical point gives:

$$\begin{aligned} f(-4) &= \frac{(-4)^2 - 4}{(-4)^2 + 4} = \frac{12}{20} = \frac{3}{5} \\ f(0) &= \frac{0^2 - 4}{0^2 + 4} = \frac{-4}{4} = -1 \\ f(4) &= \frac{4^2 - 4}{4^2 + 4} = \frac{12}{20} = \frac{3}{5}. \end{aligned}$$

Therefore, $f$ achieves its absolute minimum of $-1$ at $x = 0$ and its absolute maximum of $\frac{3}{5}$ at both $x = -4$ and $x = 4$.

**58.** Find the absolute maximum and absolute minimum values of

$$f(t) = t + \cot(t/2)$$

on the interval $[\pi/4, 7\pi/4]$.

**Answer:** The derivative of $f$ is

$$f'(t) = 1 - \frac{1}{2}\csc^2(t/2).$$

Therefore, $f'(t) = 0$ when

$$1 = \frac{1}{2}\csc^2(t/2) = \frac{1}{2\sin^2(t/2)}$$

or, equivalently, when

$$\sin^2(t/2) = \frac{1}{2}.$$

Since $\sin\theta = \frac{1}{\sqrt{2}}$ for $\theta = \pi/4, 3\pi/4, 9\pi/4, 11\pi/4, \ldots$, $f'(t) = 0$ when

$$t = \pi/2, 3\pi/2$$

(of course, there are infinitely many such $t$, but these are the only two in the given interval).

Now, plugging in the endpoints and the critical points gives:

$$\begin{aligned} f(\pi/4) &= \frac{\pi}{4} + \cot(\pi/8) \approx 3.2 \\ f(\pi/2) &= \frac{\pi}{2} + \cot(\pi/4) = \frac{\pi}{2} + 1 \approx 2.6 \\ f(3\pi/2) &= \frac{3\pi}{2} + \cot(3\pi/4) = \frac{3\pi}{2} - 1 \approx 3.7 \\ f(7\pi/4) &= \frac{7\pi}{4} + \cot(7\pi/8) \approx 3.1 \end{aligned}$$

Therefore, $f$ achieves its absolute minimum of $\approx 2.6$ at $t = \pi/2$ and its absolute maximum of $\approx 3.7$ at $t = 3\pi/2$.

## §4.2

**16.** Let $f(x) = 2 - |2x - 1|$. Show that there is no value of $c$ such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

**Answer:** If there were a $c$ such that $f(3) - f(0) = f'(c)(3 - 0)$, then it would be the case that

$$f'(c) = \frac{f(3) - f(0)}{3 - 0} = \frac{-3 - 1}{3} = -\frac{4}{3}.$$

Now, we can write $f$ as the following piecewise function:

$$f(x) = \begin{cases} 2 - (1 - 2x) & \text{if } x < 1/2 \\ 2 - (2x - 1) & \text{if } x \geq 1/2. \end{cases}$$

Therefore, on the interval $(-\infty, 1/2)$, $f'(x) = 2$, whereas on the interval $(1/2, +\infty)$, $f'(x) = -2$. Since $f'(1/2)$ is undefined, this means that there is no $c$ such that $f'(c) = -4/3$, so there cannot be a $c$ satisfying the condition stated in the problem.

This does not violate the Mean Value Theorem because the function $f$ is not differentiable on $(0, 3)$: in particular, it is not differentiable at $x = 1/2$.

**18.** Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.

*Proof.* Let $f(x) = 2x - 1 - \sin x$. Then note that

$$f(0) = 2(0) - 1 - \sin 0 = -1 < 0$$
$$f(\pi) = 2\pi - 1 - \sin \pi = 2\pi - 1 - (-1) = 2\pi > 0$$

so, by the Intermediate Value Theorem, there exists $a$ between 0 and $\pi$ such that $f(a) = 0$. In other words, the given equation has at least one solution.

Suppose that the equation has more than one solution. In particular, this will mean there exist $x_1$ and $x_2$ such that $f(x_1) = 0$, $f(x_2) = 0$ and $x_1 \neq x_2$. If $x_1 < x_2$, then, by the Mean Value Theorem, there exists $c$ between $x_1$ and $x_2$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0.$$

However, we know that

$$f'(x) = 2 - \cos x$$

for any $x$, so, in particular,

$$f'(c) = 2 - \cos c.$$

Since $\cos c \leq 1$, $2 - \cos c \geq 1$, so we have that $f'(c) = 0$ and $f'(c) \geq 1$. Clearly both of these can't be true, so the assumption that there is more than one solution to the equation must have been false.

Therefore, we can conclude that there is exactly one solution to the equation. □

**28.** Suppose $f$ is an odd function and is differentiable everywhere. Prove that for every positive number $b$, there exists a number $c$ in $(-b, b)$ such that $f'(c) = f(b)/b$.

*Proof.* Let $b$ be a positive number. Then, since $f$ is odd, $f(-b) = -f(b)$. Since $f$ is differentiable everywhere, the Mean Value Theorem says that there exists $c$ between $-b$ and $b$ such that

$$f'(c) = \frac{f(b) - f(-b)}{b - (-b)} = \frac{f(b) - (-f(b))}{b + b} = \frac{2f(b)}{2b} = \frac{f(b)}{b},$$

as desired. □

## §4.3

**14.** Let

$$f(x) = \cos^2 x - 2\sin x, \quad 0 \le x \le 2\pi.$$

**(a)** Find the intervals on which $f$ is increasing or decreasing.

**Answer:** To find the intervals on which $f$ is increasing or decreasing, take the derivative of $f$:

$$f'(x) = 2\cos x(-\sin x) - 2\cos x = -2\cos x(\sin x + 1).$$

Since $\sin x + 1 \ge 0$ for all $x$, we see that the sign of $f'(x)$ is the opposite of that of $\cos x$. Thus, $f'(x) < 0$ (meaning $f$ is decreasing) on the intervals

$$[0, \pi/2), (3\pi/2, 2\pi]$$

and $f'(x) > 0$ (meaning $f$ is increasing) on the intervals

$$(\pi/2, 3\pi/2).$$

**(b)** Find the local maximum and minimum values of $f$.

**Answer:** $f'(x) = 0$ when either $\cos x = 0$ or $\sin x = -1$, which is to say when

$$x = \pi/2, 3\pi/2.$$

$f'$ changes from negative to positive at

$$x = \pi/2$$

so, by the first derivative test, the local minimum value of $f$ is

$$f(\pi/2) = \cos^2(\pi/2) - 2\sin(\pi/2) = 0^2 - 2(1) = -2.$$

$f'$ changes from positive to negative at

$$x = 3\pi/2$$

so, by the first derivative test, the local maximum value of $f$ is

$$f(3\pi/2) = \cos^2(3\pi/2) - 2\sin(3\pi/2) = 0^2 - 2(-1) = 2.$$

**(c)** Find the intervals of concavity and inflection points.

**Answer:** To do so, we need to use the second derivative

$$f''(x) = -2\left[(-\sin x)(\sin x + 1) + \cos x(\cos x)\right] = -2\left[-\sin^2 x - \sin x + \cos^2 x\right].$$

Then $f''(x) = 0$ when

$$x = \pi/6, 5\pi/6, 3\pi/2.$$

Moreover, $f''(x) \le 0$ on the intervals

$$[0, \pi/6), (5\pi/6, 2\pi]$$

so $f$ is concave down on these intervals, and $f''(x) > 0$ on the interval

$$x = (\pi/6, 5\pi/6)$$

so $f$ is concave up on this interval.
Thus, we see that $f$ switches concavity at the points

$$x = \pi/6, 5\pi/6$$

i.e., so these are the inflection points of $f$.

**20.** Find the local maximum and minimum values of

$$f(x) = \frac{x}{x^2 + 4}$$

using both the First and Second Derivative Tests. Which method do you prefer?

**Answer:** Taking the first derivative,

$$f'(x) = \frac{(x^2 + 4)(1) - x(2x)}{(x^2 + 4)^2} = \frac{-x^2 + 4}{(x^2 + 4)^2}.$$

Therefore, $f'(x) = 0$ when $x^2 = 4$, so the critical points of $f$ are at $x = \pm 2$.

First we see whether these critical points are local minima/maxima using the first derivative test, meaning we need to know when $f'(x) < 0$ and when $f'(x) > 0$. Notice that the denominator of $f'$ is always positive, so we can safely ignore it. Thus, $f'(x) < 0$ when $-x^2 + 4 < 0$, i.e., when

$$4 < x^2.$$

Hence, $f'(x) < 0$ when $x < -2$ or $x > 2$. On the other hand, $f'(x) > 0$ when $-x^2 + 4 > 0$, i.e., when

$$4 > x^2.$$

Hence, $f'(x) > 0$ when $-2 < x < 2$. Therefore, the sign of $f'$ switches from negative to positive at $x = -2$ and from positive to negative at $x = 2$, so $f$ has a local minimum at $x = -2$ and a local maximum at $x = 2$. In particular, the local minimum value of $f$ is

$$f(-2) = \frac{-2}{(-2)^2 + 4} = -\frac{2}{8} = -\frac{1}{4}$$

and the local maximum value of $f$ is

$$f(2) = \frac{2}{(2)^2 + 4} = \frac{2}{8} = \frac{1}{4}.$$

We could also have determined this by the second derivative test, which requires determining the second derivative of $f$:

$$\begin{aligned} f''(x) &= \frac{(x^2+4)^2(-2x) - (-x^2+4)(2(x^2+4)(2x))}{(x^2+4)^4} \\ &= \frac{(x^4+8x^2+16)(-2x) - (-x^2+4)(4x^3+16x)}{(x^2+4)^4} \\ &= \frac{-2x^5 - 16x^3 - 32x + 4x^5 - 64x}{(x^2+4)^4} \\ &= \frac{2x^5 - 16x^3 - 96x}{(x^2+4)^4}. \end{aligned}$$

Therefore,

$$\begin{aligned} f''(-2) &= \frac{2(-2)^5 - 16(-2)^3 - 96(-2)}{(-2)^2+4)^4} = \frac{-64+128+192}{8^4} = \frac{256}{4096} = \frac{1}{16} > 0 \\ f''(2) &= \frac{2(2)^5 - 16(2)^3 - 96(2)}{(2^2+4)^4} = \frac{64-128-192}{8^4} = \frac{-256}{4096} = -\frac{1}{16} < 0. \end{aligned}$$

Therefore, by the second derivative test, $f$ has a local minimum at $x = -2$ and a local maximum at $x = 2$, agreeing with the first derivative test.

**50.** Let

$$f(x) = \frac{e^x}{1+e^x}.$$

**(a)** Find the vertical and horizontal asymptotes.

**Answer:** Since $f(x)$ is defined for all $x$, there aren't any vertical asymptotes. To check for horizontal asymptotes, compute

$$\lim_{x\to\infty} \frac{e^x}{1+e^x} = \lim_{x\to\infty} \frac{1}{\frac{1}{e^x}+1} = 1$$

and

$$\lim_{x\to-\infty} \frac{e^x}{1+e^x} = \frac{0}{1+0} = 0,$$

so $f$ has a horizontal asymptote at $y = 0$ to the left and at $y = 1$ to the right.

**(b)** Find the intervals of increase or decrease.

**Answer:** To do so, find the first derivative:

$$f'(x) = \frac{(1+e^x)(e^x) - e^x(e^x)}{(1+e^x)^2} = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}.$$

Both numerator and denominator are always positive, so $f$ is increasing on $(-\infty, +\infty)$.

**(c)** Find the local maximum and minimum values.

**Answer:** Since $f'(x) > 0$ for all $x$, $f$ has no critical points and, hence, no local maxima or minima.

**(d)** Find the intervals of concavity and the inflection points.

**Answer:** First, compute the second derivative:

$$
\begin{aligned}
f''(x) &= \frac{(1+e^x)^2(e^x) - e^x(2(1+e^x)(e^x))}{(1+e^x)^4} \\
&= \frac{(1+2e^x+e^{2x})(e^x) - e^x(2e^x+2e^{2x})}{(1+e^x)^4} \\
&= \frac{e^x + 2e^{2x} + e^{3x} - 2e^{2x} - 2e^{3x}}{(1+e^x)^4} \\
&= \frac{e^x - e^{3x}}{(1+e^x)^4}.
\end{aligned}
$$

Since the denominator is always positive, the sign of $f''(x)$ depends only on the sign of the numerator. Hence, $f''(x) < 0$ when $e^x - e^{3x} < 0$, i.e., when

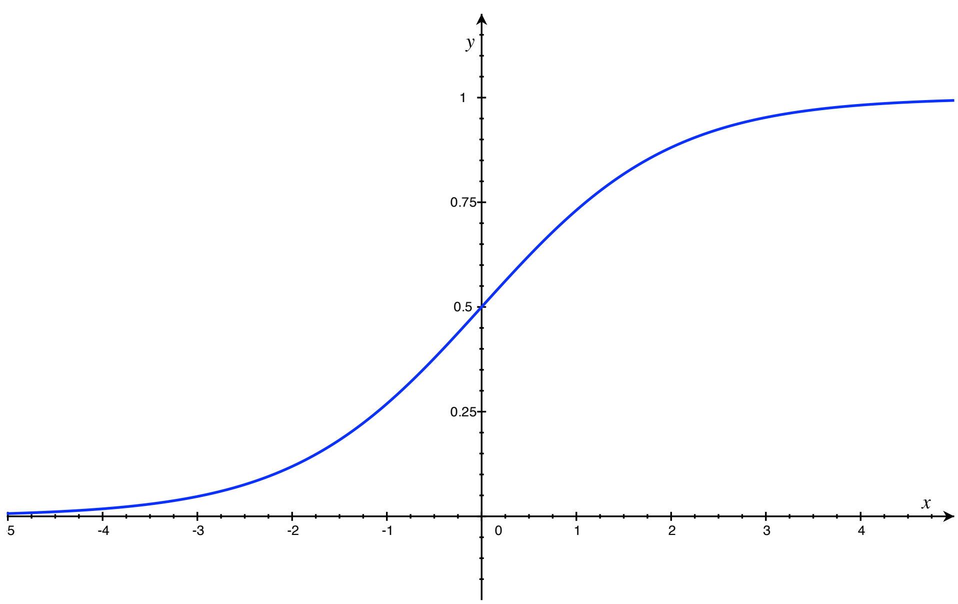
$$
\begin{aligned}
e^x &< e^{3x} \\
1 &< e^{2x} \quad \text{by dividing both sides by } e^x \\
0 &< 2x \quad \text{by taking the natural log of both sides.}
\end{aligned}
$$

Therefore, $f''(x) < 0$ (and, thus, $f$ is concave down) when $x > 0$. On the other hand, reversing the inequality in each of the above steps shows that $f''(x) > 0$ (and, thus, $f$ is concave up) when $x < 0$.

The sign of $f''$ changes at $x = 0$, so $(0, 1/2)$ is the single inflection point on the graph of $f$.

**(e)** Use the information from parts (a)-(d) to sketch the graph of $f$.

**Answer:** The graph of $f$ is pictured below.



## §4.4

For each problem, find the limit. Use l'Hôpital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hôpital's Rule doesn't apply, explain why.

**20.**

$$\lim_{x\to 1} \frac{\ln x}{\sin \pi x}.$$

**Answer:** Note that $\ln 1 = 0$ and $\sin(\pi \cdot 1) = \sin \pi = 0$, so l'Hôpital's Rule applies. Hence,

$$\lim_{x\to 1} \frac{\ln x}{\sin \pi x} = \lim_{x\to 1} \frac{\frac{1}{x}}{\pi \cos \pi x} = \lim_{x\to 1} \frac{1}{\pi x \cos \pi x} = \frac{1}{\pi \cos \pi} = -\frac{1}{\pi}.$$

**46.**

$$\lim_{x\to\infty} x \tan(1/x).$$

**Answer:** We can't immediately apply l'Hôpital's Rule, but the above limit is equal to

$$\lim_{x\to\infty} \frac{\tan(1/x)}{1/x}.$$

This is in $\frac{0}{0}$ form, so we can apply l'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\tan(1/x)}{1/x} = \lim_{x\to\infty} \frac{\sec^2(1/x)\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \sec^2(1/x) = \sec^2(0) = 1.$$

**50.**

$$\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right)$$

**Answer:** First, a bit of algebraic manipulation:

$$\cot x - \frac{1}{x} = \frac{\cos x}{\sin x} - \frac{1}{x} = \frac{x\cos x}{x \sin x} - \frac{\sin x}{x \sin x} = \frac{x\cos x - \sin x}{x\sin x}.$$

Therefore,

$$\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x\cos x - \sin x}{x\sin x}.$$

Both numerator and denominator are going to zero, so we can apply l'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{\cos x + x(-\sin x) - \cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{-x\sin x}{\sin x + x\cos x}.$$

Again, both numerator and denominator are going to zero, so we apply l'Hôpital's Rule again to get

$$\lim_{x\to 0} \frac{-\sin x - x\cos x}{\cos x + \cos x + x(-\sin x)} = \lim_{x\to 0} \frac{-\sin x - x\cos x}{2\cos x - x\sin x} = \frac{0}{2+0} = 0.$$