Answers to review questions from Chapter 1 and section 3.1

(1) Use the definition $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ of the derivative to find $f'(x)$ when $f(x) = x^{-1/2}$.

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{1}{\sqrt{x}\sqrt{x+h}} \cdot \frac{\left(\sqrt{x}-\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}{\sqrt{x}+\sqrt{x+h}} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{1}{\sqrt{x}\sqrt{x+h}} \cdot \frac{x-(x+h)}{\sqrt{x}+\sqrt{x+h}} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{1}{\sqrt{x}\sqrt{x+h}} \cdot \frac{-h}{\sqrt{x}+\sqrt{x+h}} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x}\sqrt{x+h}} \cdot \frac{-1}{\sqrt{x}+\sqrt{x+h}} \\
&= -\frac{1}{\sqrt{x}\sqrt{x}(2\sqrt{x})} = -\frac{1}{2x^{3/2}} = -\frac{1}{2}x^{-3/2}
\end{aligned}
$$

(2) Use the definition $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ of the derivative to find $f'(x)$ when $f(x) = x^{-2}$.

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{x^2-(x+h)^2}{x^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{x^2-(x^2+2hx+h^2)}{x^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-2hx-h^2}{x^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2} \\
&= \frac{-2x}{x^2x^2} = -2x^{-3}
\end{aligned}
$$

(3) Assume x is a number such that $\tan x = 7$ and $\sin x < 0$. Find $\sec x$.
We know $\sec^2 x = 1 + \tan^2 x = 1 + 7^2 = 50$. This implies $\sec x = \pm\sqrt{50} = \pm 5\sqrt{2}$. Since $\frac{\sin x}{\cos x} = \tan x = 7 > 0$ and $\sin x < 0$, we conclude $\cos x < 0$, hence $\sec x = \frac{1}{\cos x} < 0$. This fact and $\sec x = \pm 5\sqrt{2}$ imply $\sec x = -5\sqrt{2}$.

(4) Simplify $\sin(\sin^{-1} x)$, $\cos(\sin^{-1} x)$, $\sec(\sin^{-1} x)$, $\tan(\sin^{-1} x)$.

The definition of inverse function implies $\sin(\sin^{-1} x) = x$. Since

$$\cos^2(\sin^{-1} x) = 1 - \sin^2(\sin^{-1} x) = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2,$$

we conclude $\cos(\sin^{-1} x) = \pm\sqrt{1-x^2}$. But $\sin^{-1} x$ is in the interval $[-\pi/2, \pi/2]$ and cos is positive or zero on that interval. This implies $\cos(\sin^{-1} x) = \sqrt{1-x^2}$. Now we know

$$\sec(\sin^{-1} x) = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}}.$$

Finally,

$$\tan(\sin^{-1} x) = \frac{\sin(\sin^{-1} x)}{\cos(\sin^{-1} x)} = \frac{x}{\sqrt{1-x^2}} \ .$$

(5) Consider the function $f(x) = \dfrac{x-8}{1+7x}$. Find a formula for the inverse function $f^{-1}(x)$.

The notation $t = f^{-1}(x)$ gives $x = f(t) = \dfrac{t-8}{1+7t}$. The notation $t = f^{-1}(x)$ gives $x = f(t) = \dfrac{t-8}{1+7t}$. When we solve for t, we get $t-8 = x(1+7t)$, which is $t-8 = x+7xt$, which is $(1-7x)t = x+8$, which is $t = \dfrac{x+8}{1-7x}$. We conclude $f^{-1}(x) = t = \dfrac{x+8}{1-7x}$.

(6) Solve for x in the equation $e^{4x+3} = 2e^{3-x}$.

Dividing both sides of the equation $e^{4x+3} = 2e^{3-x}$ by e^{3-x}, we get $e^{5x} = 2$. This is $5x = \ln 2$, which is $x = \dfrac{\ln 2}{5}$.

(7) Solve for x in the equation $\sqrt{e^{8x-6}} = e^{x^2}$.

When we solve for t, we get $t-8 = x(1+7t)$, which is $t-8 = x+7xt$, which is $(1-7x)t = x+8$, which is $t = \dfrac{x+8}{1-7x}$. We conclude $f^{-1}(x) = t = \dfrac{x+8}{1-7x}$.

(8) Solve for x in the equation $e^{4x+3} = 2e^{3-x}$.

Dividing both sides of the equation $e^{4x+3} = 2e^{3-x}$ by e^{3-x}, we get $e^{5x} = 2$. This is $5x = \ln 2$, which is $x = \dfrac{\ln 2}{5}$.

(9) Which of the given functions is even, which of the given functions is odd, and which of the given functions is neither? Explain carefully.

$$f(x) = x^4\sqrt{1+x^2} \qquad g(x) = x^3 + 1 \qquad h(x) = x\sqrt{1+x^2}$$

The function $f(x)$ is even because

$$f(-x) = (-x)^4\sqrt{1+(-x)^2} = x^4\sqrt{1+x^2} = f(x)$$

The function $h(x)$ is odd because

$$h(-x) = (-x)\sqrt{1+(-x)^2} = -\left(x\sqrt{1+x^2}\right) = -h(x)$$

The function $g(x)$ is neither because

$$g(-x) = (-x)^3 + 1 = -x^3 + 1 \neq x^3 + 1 = g(x)$$

and

$$g(-x) = (-x)^3 + 1 = -x^3 + 1 \neq -(x^3 + 1) = -g(x)$$

(10) Find functions $f(x)$ and $g(x)$ such that $f(x)$ is even, $g(x)$ is odd and $f(x) + g(x) = 5x^5 - 7x^4 - 5x^3 + 8x^2 - x + 10$.

We have $f(x) = -7x^4 + 8x^2 + 10$ and $g(x) = 5x^5 - 5x^3 - x$. The function $f(x)$ is even because

$$f(-x) = -7(-x)^4 + 8(-x)^2 + 10 = -7x^4 + 8x^2 + 10 = f(x)$$

The function $g(x)$ is odd because

$$g(-x) = 5(-x)^5 - 5(-x)^3 - (-x) = -(5x^5 - 5x^3 - x) = -g(x)$$

(11) Express the function $f(x) = \sqrt{1+\cos^2 x}$ as the composition of three simpler functions.

If $f_1(x) = \cos x,\ f_2(x) = 1 + x^2$ and $f_3(x) = \sqrt{x}$ then $f(x) = f_3(f_2(f_1(x)))$.