GRADE 12 EXAMINATION
NOVEMBER 2018

ADVANCED PROGRAMME MATHEMATICS: PAPER I
MODULE 1: CALCULUS AND ALGEBRA

Time: 2 hours | 200 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 9 pages and an Information Booklet of 4 pages (i–iv). Please check that your question paper is complete.
2. Non-programmable and non-graphical calculators may be used, unless otherwise indicated.
3. All necessary calculations must be clearly shown and writing should be legible.
4. Diagrams have not been drawn to scale.
5. Round off your answers to two decimal digits, unless otherwise indicated.

QUESTION 1

1.1 Solve for $x \in \mathbb{R}$ without using a calculator and showing all working:

(a) $|x^2 + x| = -2x - 2$ (8)

(b) $\ln x^3 + 2\ln x^2 = 7$ (6)

1.2 The equation for radioactive decay of a radioactive element is:

$$y = y_0 e^{-kt}, \quad k > 0$$

where y_0 is the initial amount, y is the amount after time t (in years) and k is a constant.

The half-life of an element is the time taken for half of the quantity to decay.

(a) Make k the subject of the formula. (4)

(b) Determine the value of k for Carbon-14 if the half-life of Carbon-14 is 5 700 years. Give the answer correct to 6 decimal places. (2)

(c) How old is a sample in which 10% of the Carbon-14 nuclei originally present have decayed? In other words, 90% of the original quantity remain. (4)

[24]

QUESTION 2

2.1 Determine, in standard form $\left(ax^3 + bx^2 + cx + d = 0\right)$ with *a, b, c* and *d* real, a cubic equation which has roots -3 and $3+2i$. (8)

2.2 Explain why every cubic equation with real coefficients must have at least one real root. (4)

2.3 Thabo is practising his division of complex numbers of the form $a+bi$ where $a,b \in \mathbb{R}$.

He notices that $\dfrac{3+2i}{-2+3i} = -i \qquad \dfrac{5-7i}{7+5i} = -i \qquad$ *and* $\qquad \dfrac{4+5i}{-5+4i} = -i$.

Prove that $\dfrac{a+bi}{-b+ai} = -i \quad \textit{for} \quad \textit{all} \quad a,b \in \mathbb{R}$. (8)

[20]

QUESTION 3

Use Mathematical Induction to prove that $2^{3n} - 3^n$ is divisible by 5 for $n \in \mathbb{N}$.

[14]

QUESTION 4

4.1 Consider the function $f(x) = e^{|x|}$

(a) Draw a sketch graph of *f*, showing at least 2 points on the graph. (8)

(b) Write down the *x*-coordinate of a point at which *f* is not differentiable. (2)

4.2 Consider the function *f(x)*, where *a* and *b* are rational.

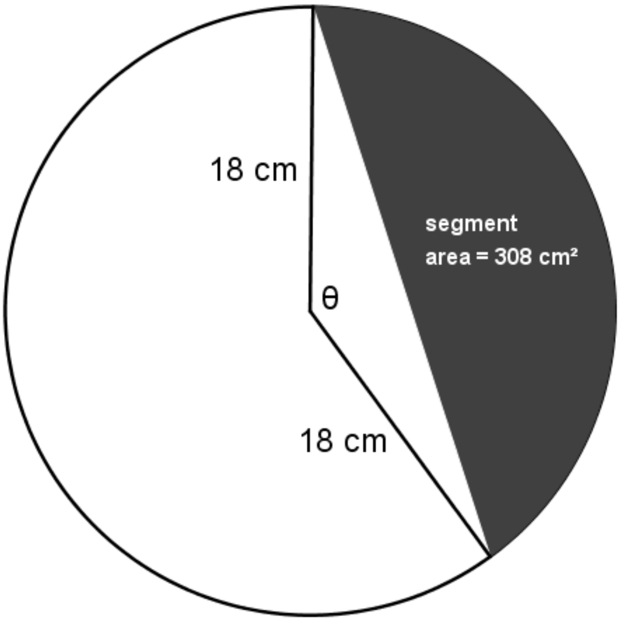
$$f(x) = \begin{cases} ax - b - 1, & x < 2 \\ bx^2 - ax + 5, & x \geq 2 \end{cases}$$

Determine a *and* b if *f* is differentiable at $x = 2$. (10)

[20]

QUESTION 5

Consider the diagram below. The shaded segment has an area of 308 cm^2.



5.1 By using the given area show that θ satisfies the equation:
$162\theta - 162\sin\theta - 308 = 0$. (6)

5.2 Use Newton-Raphson iteration to determine θ (in radians) to 5 decimal places using an initial approximation of 2 radians. You must state the iterative formula you have used to produce your answer. (8)

[14]

QUESTION 6

Consider the function $f(x) = \dfrac{2x^2 - 3x - 2}{x - 4}$.

6.1 Determine the intercepts with both axes. (6)

6.2 Determine the equations of any asymptotes. (8)

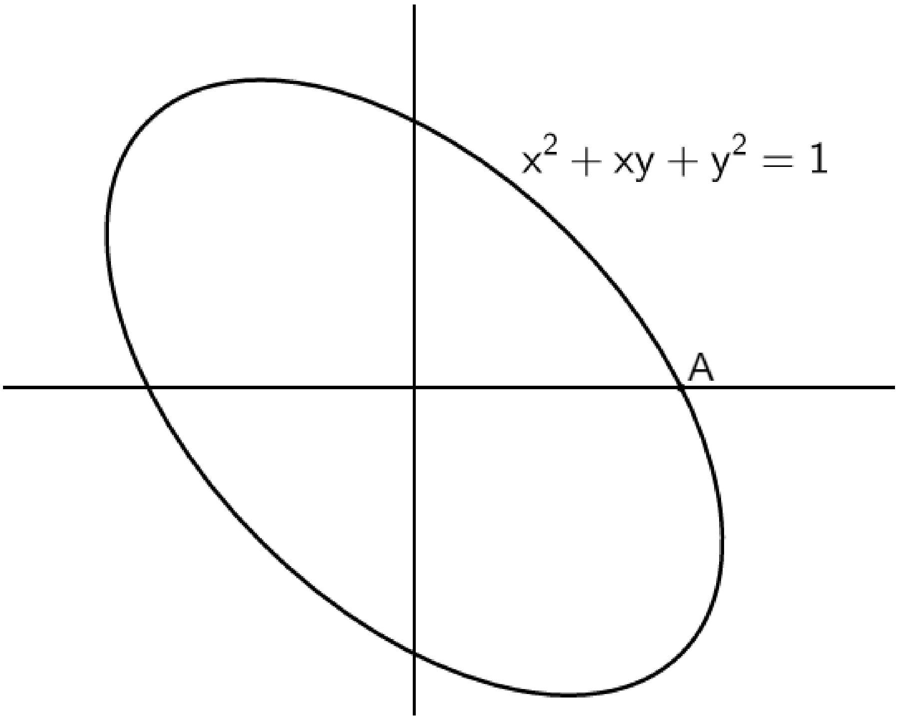
6.3 Determine the coordinates of any stationary points. (8)

6.4 Use the fact that $f''(x) = \dfrac{36}{(x-4)^3}$ to determine the nature of the stationary points you found in Question 6.3. (4)

[26]

QUESTION 7

The graph of an ellipse with equation $x^2 + xy + y^2 = 1$ is given below.



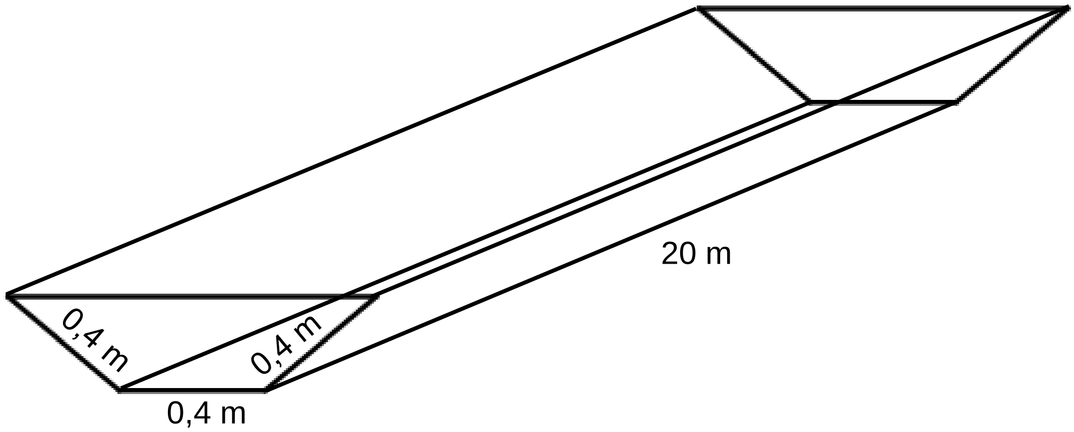
7.1 Use implicit differentiation to show that $\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$. (5)

7.2 Hence, or otherwise, determine the equation of the tangent to the ellipse at the point A, its positive *x*-intercept. (5)

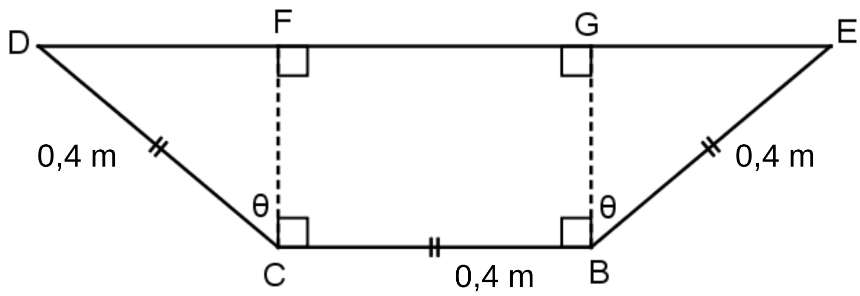
[10]

QUESTION 8

A farmer has a flat sheet of steel measuring 20 m × 1,2 m. He wishes to make a water trough by folding the sides up as shown in the diagram below. The ends of the trough will be constructed separately and will be at right-angles to the sides and base.



The cross section of the trough is as follows:



8.1 Show that the volume V is given by $V = 1{,}6\sin 2\theta + 3{,}2\cos\theta$. (8)

8.2 Hence, or otherwise, determine the size of θ which will give the trough maximum volume. (8)

[16]

QUESTION 9

9.1 (a) Prove that $\sin^3\theta = \sin\theta - \sin\theta\cos^2\theta$. (4)

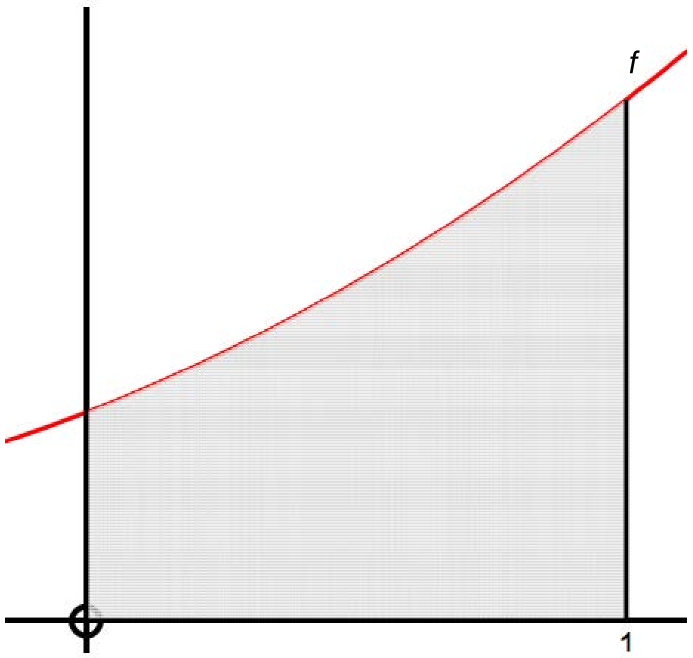
(b) Hence, or otherwise, determine $\int \sin^3\theta\, d\theta$. (8)

9.2 Determine $\int \frac{x}{\sqrt{2+x}}\,dx$. (8)

[20]

QUESTION 10

Jessica is attempting a Riemann sum to determine the area bounded by the curve *f*, the *x*-axis, the *y*-axis and the line x = 1 as shown below:



She has correctly determined that, if she uses *n* rectangles, then the area will be:

$$Area = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}$$

10.1 Determine the area when four rectangles are used. (3)

10.2 Will this be an under-approximation or an over-approximation? Explain your answer. (3)

10.3 Determine the exact area. (3)

10.4 Determine the value of $\int_{-1}^{0} f(-x)\,dx$, explaining your answer. (3)

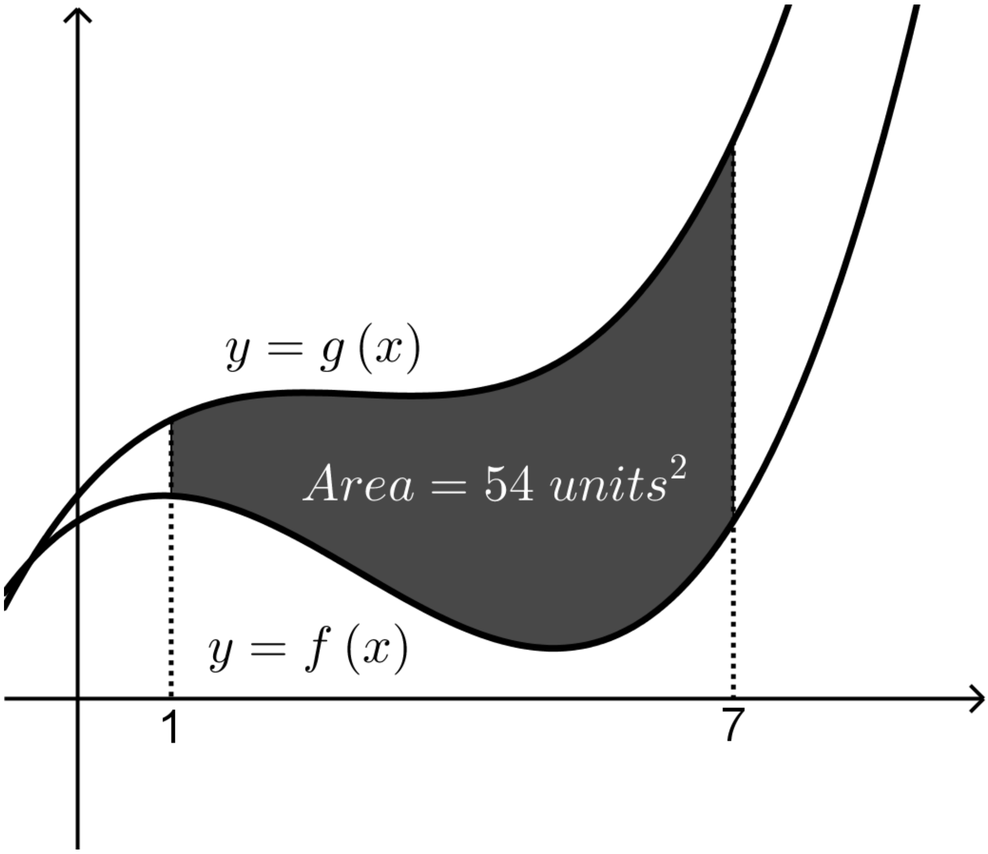
[12]

QUESTION 11

In the diagram below the function $f(x)$ has been drawn.

The function $g(x) = f(x) + kx + 1$, with k rational, has also been drawn.

If the area bounded by the two functions and the lines $x = 1$ *and* $x = 7$ is 54 units^2 then determine the value of k.



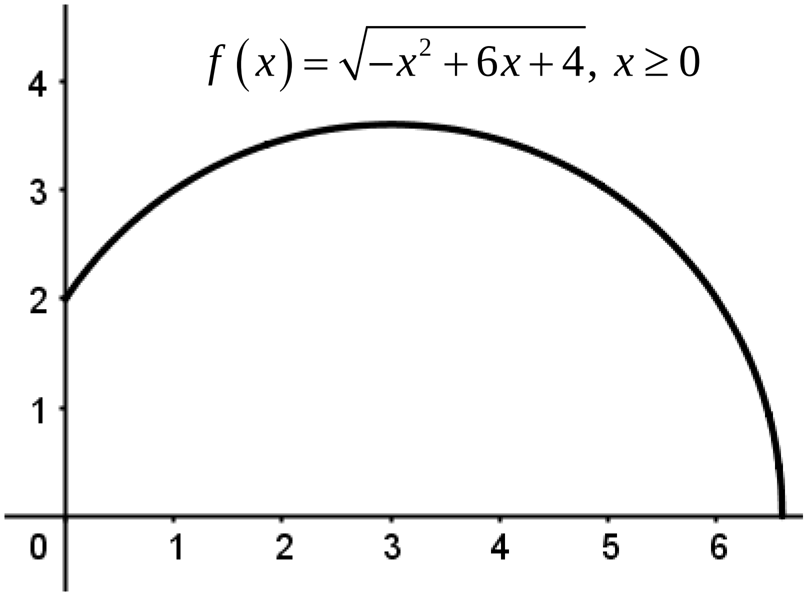
[12]

QUESTION 12

A perfume bottle is in the shape of a truncated sphere with a flat bottom as shown in the picture.

Image from: <https://www.newchic.com/fragrance-4359/p-1124134.html?rmmds=search&r_keywords=spray-perfume>

One half of the bottle can be modelled using the function *f*, illustrated below (with units in cm). Imagine the bottle has been turned on its side and the base of the bottle is on the *y*-axis.



To what height should the bottle be filled if it is to contain 175 ml of perfume? Remember that 1 ml = 1 cm^3.

[12]

Total: 200 marks