Does it converge or diverge? If it converges, find its value (if possible).

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$

   The terms of the sum go to zero. It looks similar to $\sum \frac{1}{n}$, which diverges. We also note that the terms of the sum are positive. We compare them:

$$\lim_{n\to\infty} \frac{\frac{1}{n-\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n-\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{\sqrt{n}}} = 1$$

   The series diverges by the limit comparison test, with $\sum(1/n)$.

2. $\left\{\frac{n}{1+\sqrt{n}}\right\}$

   In this case, we simply take the limit:

$$\lim_{n\to\infty} \frac{n}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\frac{1}{\sqrt{n}}+1} = \infty$$

   The sequence diverges.

3. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-1}$

   The terms of the sum go to zero, since there is an n^2 in the numerator, and n^3 in the denominator. In fact, it looks like $\sum \frac{1}{n}$, so we compare it to that:

$$\lim_{n\to\infty} \frac{\frac{n^2-1}{n^3-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-n}{n^3-1} = 1$$

   Therefore, the series diverges by the limit comparison test, with $\sum \frac{1}{n}$.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3}$

   We can temporarily break this apart to see if the pieces converge:

$$\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \frac{5}{n^3} - 2\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3}$$

   Both of these are $p-$series, the first with $p=3$, the second with $p=\frac{5}{2}$, therefore they converge separately, and so the sum also converges.

5. $\displaystyle\sum_{n=1}^{\infty} (-6)^{n-1}5^{1-n}$

   First, let's rewrite the terms of the sum:

$$(-6)^{n-1}5^{1-n} = \frac{(-6)^{n-1}}{5^{n-1}} = \left(\frac{-6}{5}\right)^{n-1}$$

so that this is a geometric series with $r = \frac{-6}{5}$. Since $|r| > 1$, this series diverges.

6. $\left\{\frac{n!}{(n+2)!}\right\}$
We first simplify:
$$\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}$$
so the limit as $n \to \infty$ is 0.

7. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$
A little tricky... First, note that this can be written as $\ln(n) - \ln(n+_1)$. Now, let's write out the n^{th} partial sum:
$$S_n = \ln(1) - \ln(2) + \ln(2) - \ln(3) + \ln(3) - \ln(4) + \ldots + \ln(n) - \ln(n+1)$$
with cancellations,
$$S_n = 0 - \ln(n+1)$$
Now, the limit of S_n as $n \to \infty$ is $-\infty$, so the sum diverges.

8. $\displaystyle\sum_{n=2}^{\infty} \frac{3^n + 2^n}{6^n}$
A sum of geometric series:
$$\sum_{n=2}^{\infty} \frac{3^n + 2^n}{6^n} = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n = \frac{(1/2)^2}{1-(1/2)} + \frac{(1/3)^2}{1-(1/3)} = \frac{2}{3}$$

9. $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$
Write out the first few terms of the sequence:
$$1, 0, -1, 0, 1, 0, -1, \ldots$$
so the sequence diverges.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$
First, we see the terms go to zero like $\frac{1}{n^3}$.
$$\lim_{n\to\infty} \frac{n^3}{n(n+1)(n+2)} = 1$$
so the series converges by the limit comparison test.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n\sqrt{n}}$

First, do the terms go to zero? The maximum value of the sine function is 1, and all terms of the sum are positive, so:

$$\frac{\sin^2(n)}{n^{3/2}} \le \frac{1}{n^{3/2}}$$

so the terms do go to zero. Actually, we've also done a direct comparison with the $p-$series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges.

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)2^n}$

It looks like the terms are going to zero like $\frac{1}{2^n}$, so let's compare it to $\sum(1/2)^n$, which is a convergent geometric series.

$$\frac{n}{n+1} \cdot \frac{1}{2^n} \le \frac{1}{2^n}$$

So the series converges by a direct comparison.

Evaluate, if possible.

1. $\displaystyle\int \frac{x^3}{x^3+1}\, dx$

Do long division first!

$$\frac{x^3}{x^3+1} = 1 - \frac{1}{x^3+1}$$

Can we factor x^3+1? We see $x=-1$ gives 0, so $x+1$ can be factored out. Using long division,

$$\frac{x^3+1}{x+1} = x^2 - x + 1$$

so that $x^3+1 = (x+1)(x^2-x+1)$ (NOTE: On the exam, you will be able to factor the polynomial easier than this!)

By Partial Fractions,

$$\frac{x^3}{x^3+1} = 1 - \frac{1}{(x+1)(x^2-x+1)} = 1 + \frac{1}{3} \cdot \frac{1}{x+1} + \frac{1}{3} \cdot \frac{2-x}{x^2-x+1}$$

Now you have to complete the square to finish things off, and after some long algebra,

$$x + \frac{1}{6}\ln(x^2-x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{(2x-1)}{\sqrt{3}}\right) - \frac{1}{3}\ln(x+1)$$

NOTE: This was a complicated exercise! If you made it through this one, you could probably stop now- You're ready! There won't be anything this complex on the exam...

2. $\displaystyle\int_0^1 \frac{1}{2-3x}\,dx$

There is a vertical asymptote at $x = \frac{2}{3}$, so we need to split the integral there:

$$\int_0^1 \frac{1}{2-3x}\,dx = \lim_{T\to 2/3^-} \int_0^T \frac{1}{2-3x}\,dx + \lim_{T\to 2/3^+} \frac{1}{2-3x}\,dx$$

Integrate by taking $u = 2 - 3x$, and we get that the antiderivative is $-\frac{1}{3}\ln|2-3x|$. Now, take the limits- we'll do one here:

$$\lim_{T\to 2/3^-} -\frac{1}{3}\ln\left(\frac{1}{2-3T}\right) + \frac{1}{3}\cdot\frac{1}{2} = \lim_{T\to 2/3^-} +\frac{1}{3}\ln(2-3T) + \frac{1}{3}\cdot\frac{1}{2}$$

which diverges, since 0 is a vertical asymptote for $\ln(x)$.

3. $\displaystyle\int \frac{1}{x^4 - x^2}\,dx$

Factor and use partial fractions:

$$\int \frac{1}{x^2(x-1)}\,dx = \frac{1}{x} + \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$

4. $\displaystyle\int_1^\infty \frac{1}{1+\mathrm{e}^x}\,dx$

Use $u = 1 + \mathrm{e}^x$, so $du = \mathrm{e}^x\,dx$, so that $du = (u-1)dx$.

Substitution gives:

$$\int \frac{1}{u(u-1)}\,du = \int \frac{-1}{u} + \frac{1}{u-1}\,du$$

Antidifferentiate, and we get:

$$-\ln(1+\mathrm{e}^x) + \ln(\mathrm{e}^x) = \ln\left(\frac{e^x}{1+e^x}\right)$$

Take the appropriate limit to get an answer of $\ln(2)$

5. $\displaystyle\int_0^\infty \frac{dx}{(x+1)^2(x+2)}\,dx$

Use partial fractions:

$$\int \frac{dx}{(x+1)^2(x+2)}\,dx = \int \frac{1}{(x+1)^2} + \frac{1}{x+2} - \frac{1}{x-1}\,dx$$

Antidifferentiate to get:

$$-\frac{1}{x+1} + \ln|x+2| - \ln|x+1| = -\frac{1}{x+1} + \ln\left|\frac{x+2}{x+1}\right|$$

And, take the limit to get $-1 + \ln(2)$

6. $\displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$

Use partial fractions to get:

$$\int \frac{-1}{x^2} + \frac{2}{x} + \frac{3}{x+2}\,dx$$

And integrate to get:

$$\frac{1}{x} + 2\ln(x) + 3\ln(x+2)$$