Solving Polynomial Inequalities

Note: This material is to supplement Section 3.6.

Unlike linear inequalities, polynomial inequalities cannot always be solved with just algebra, and other techniques will need to be used. We will learn to solve inequalities once they are of the form where one side of the inequality is a polynomial and the other is 0. If one side of the inequality is not 0, we need to do some algebra to change it into that form.

Example. Solve $x^2 < x + 2$

First we will need to use algebra to change the inequality so that one side is a polynomial and the other is 0. We can do this by subtracting x and adding 2 to both sides yielding, $x^2 - x - 2 < 0$.

In this example we are asking for what x-values is the function $f(x) = x^2 - x - 2$ negative, or on what interval(s) is the graph below the x-axis

In Section 3.4, we have learned how to graph polynomial functions using end-behaviors, x-intercepts, multiplicities, and sign charts. Our goal is to create a sign chart, a chart that shows where the function is positive or negative.

In order to create the sign chart for the function we need to identify the x-intercepts, and thus we must factor the function.

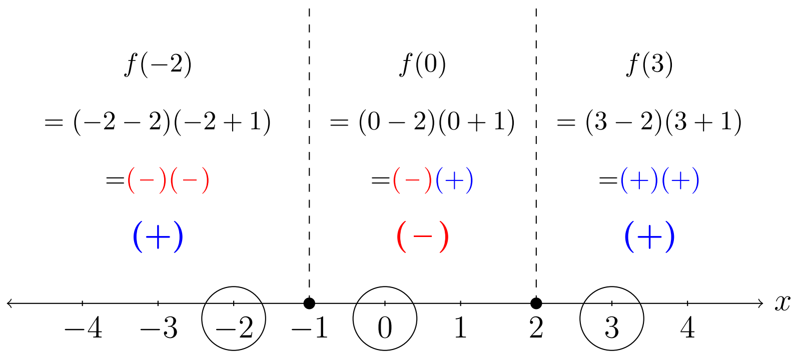
$f(x) = x^2 - x - 2 = (x - 2)(x + 1)$ and thus the x-intercepts are at $x = 2$ and $x = -1$.

The intermediate value theorem tells us that the function will be always positive or always negative on the intervals between and around the x-intercepts. There are then 3 intervals that we are concerned with, $(-\infty, -1)$, $(-1, 2)$, and $(2, \infty)$.

We will look at three different ways to finish this problem by creating a sign chart, all of which can be used for any problem. It is your choice on which technique works best for you.

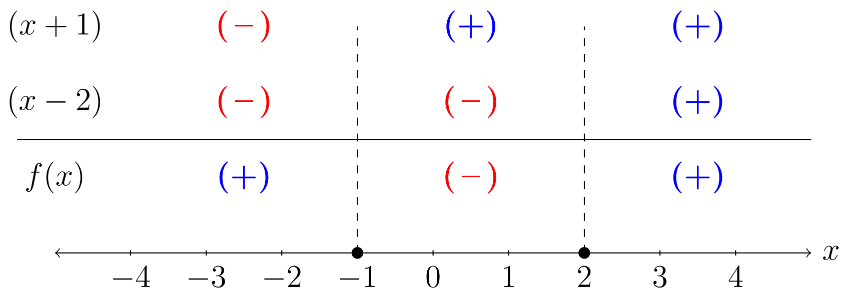
Technique 1: Test Point Method

One way to determine the sign of the function on each interval is to plug in a test point on the interval and determine the sign, we will call this the Test Point Method. It doesn't matter what test point you use as long as it is not the x-intercepts themselves (which will yield an answer of 0). Lets choose easy values $x = -2$ for the first interval, $x = 0$ for the second, and $x = 3$ for the third. Also note that we are not concerned about the value of the function at these points but only the sign of the function. It is much easier to determine the sign by plugging in the point into the factored form of the function.



Technique 2: Stacking Factors

The Stacking Factors method is very similar to the Test Point Method, and is merely just a different way of visualizing and organizing the information. It can be especially useful if there are a large number of factors (if the degree of the polynomial is large). For each factor (in this example $(x+1)$ and $(x-2)$) we will determine whether it contributes a positive or negative for each x-value. This is easy for linear factors as it will always be negative to the left of the associated x-intercept and positive to the right. Then we will count the number of negatives to determine if the function is positive or negative on that interval.



Technique 3: Using Properties of Polynomial Functions

Another way to determine the sign chart is to look at properties of polynomials discussed in Section 3.4. For instance since the degree of $f(x)$ is even and the leading coefficient is positive, then both the right and left end behaviors tend to infinity and thus the first and the last intervals must both be positive. Furthermore since the multiplicities of both zeros are 1, the function will cross the x-axis at these points, giving that the sign on the middle interval is negative. Note that only some of this information is needed in order to obtain the sign chart.

Then given the sign chart we now have that $f(x)$ is negative on the interval $(-1, 2)$ and is thus our answer.

ANSWER: $\boxed{\text{(-1,2)}}$

.

Example. Solve $x^2 - x - 2 \geq 0$.

To complete this problem we will use the same sign chart as in Example 1 since we are dealing with the same function. This time however we are looking for when the function is positive OR zero. This means that we must include the x-intercepts in our interval(s). The function is positive or zero on the intervals $(-\infty, -1)$ and $(2, \infty)$, and we will use a union ($\cup$) to combine these intervals.

$$\textbf{ANSWER: } \boxed{(\text{-}\infty, -1) \cup (2, \infty)}$$

.

Example. Solve $x^3 + x^2 - 5x + 9 \geq 6$

To begin this problem we must first put it in the correct form, that is a polynomial on one side and 0 on the other. After subtracting 6 from both sides we obtain

$$f(x) = x^3 + x^2 - 5x + 3 \geq 0.$$

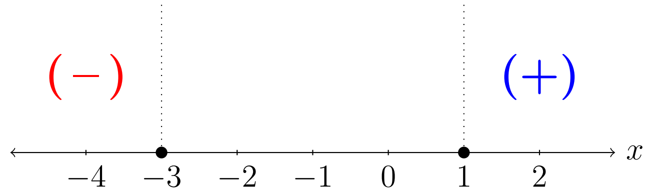
Then we must factor the function using techniques learned in Section 3.5. to obtain

$$f(x) = (x-1)^2(x+3) \geq 0.$$

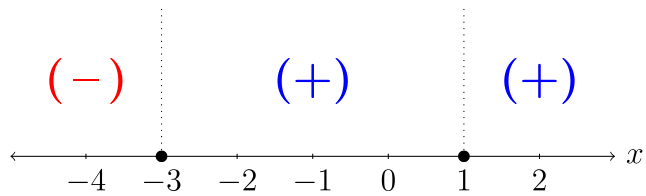
There are then three intervals we are concerned with, $(-\infty, -3)$, $(-3, 1)$, and $(1, \infty)$.

Using Properties of Polynomial Functions

In this example we will first create the sign chart using end-behaviors and multiplicities of zeros. Since the degree of $f(x)$ is odd and the leading coefficient is positive, the right end behavior tends to ∞ and the left end behavior tends to $-\infty$, and thus the sign on $(-\infty, -3)$ is negative and the sign on $(1, \infty)$ is positive.

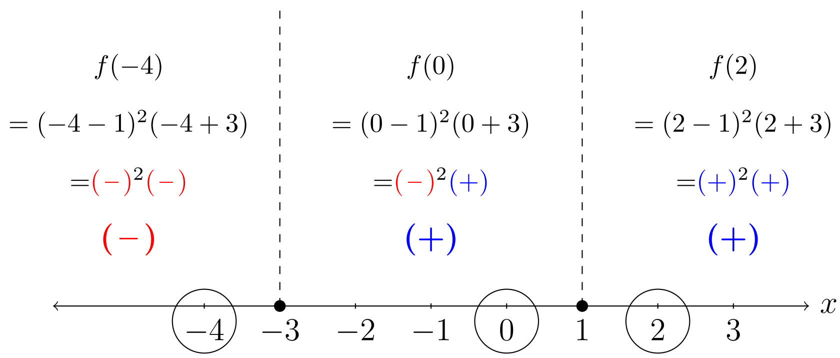


Then since the multiplicity of the zero $x = 1$ is even, the function will touch the x-axis, making the middle interval $(-3, 1)$ positive, or alternatively since the multiplicity of the zero $x = -3$ is odd, the function will cross the x-axis at this point giving the same result.



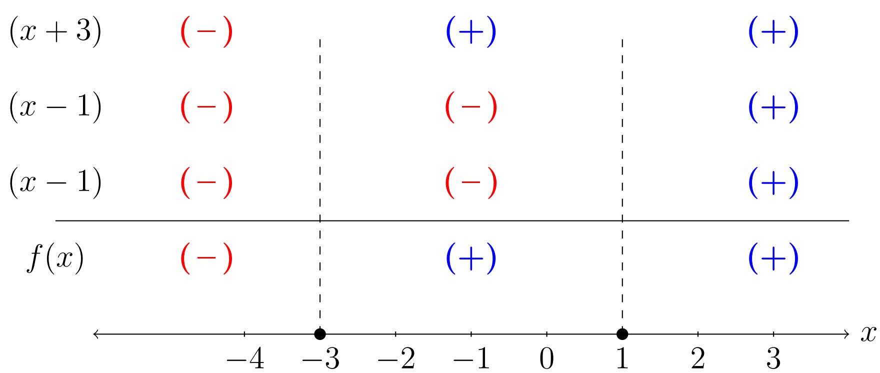
Test Point Method

We can also find the sign chart using the test point method.

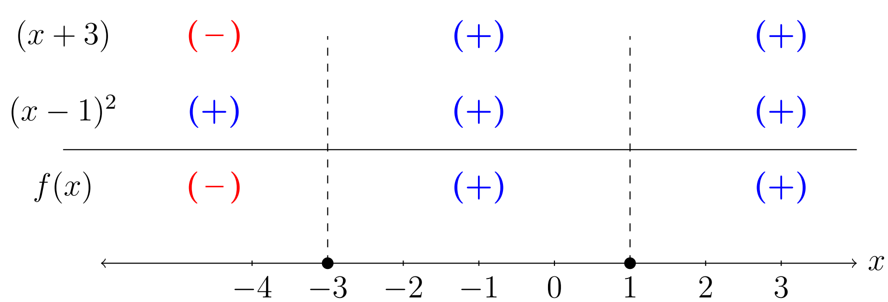


Stacking Factors

Or we can use the Stacking Factors method to create the sign chart. Note that we will consider $(x-1)^2$ as two factors, i.e. $(x-1)(x-1)$ although they will contribute the same signs.



You can use the Stacking Factors method by thinking of $(x-1)^2$ as one object. This is especially useful since the multiplicity is even and thus it will always be positive (where an odd multiplicity wouldn't change the sign).



Regardless of technique we arrive at the same sign chart and can then find our answer. Finally, note that we must also include the x intercepts since the problem calls for ≥ 0, so our answer will be $[-3, 1] \cup [1, \infty)$, which simplifies.

ANSWER: $\boxed{[-3,\infty)}$

Things to Remember

- We must first use algebra to modify the inequality so that there is a polynomial on one side and 0 on the other side. **This is a common mistake in problems like these**.
- The goal is to create a sign chart, each of the three techniques used generate the same sign chart. We only need to use **ONE** of these techniques, the choice is up to you. We have a lot of information (much of which we do not need), all of this information works together and can be used to check an answer quickly.
- Be careful of $<$ versus $\leq$. If we include zero, we must include the x-intercepts as well in our answer often leading to using closed brackets instead of open parentheses.
- The final answer must be in interval notation and simplified. As in the last example, directly reading the sign chart does not give the simplified answer.
- In the Test Point Method, we do NOT care what the function value is, only what the sign is (this will save a lot of time and anguish).
- Also in the Test Point Method, it is much easier to plug the test point into the factored form of the polynomial.
- In the Stacked Factors method, if a linear factor as even multiplicity, it will always contribute a positive value, and if odd multiplicity it will contribute the same sign as if the multiplicity were one.

Solving Rational Inequalities

Note: This material is to supplement Section 3.7.

In this section we will solve inequalities where one side of the inequality is a rational function and the other is zero. As before with polynomial inequalities, we may first need to use algebra to manipulate an inequality into this form.

Example. Manipulate the inequality $\dfrac{6-x}{(x+2)(x-3)} > -1$ so that there is a zero on one side of the inequality and a rational function on the other.

First we will add 1 to both sides, and then rewrite the left hand side as a rational function using a common denominator of $(x+2)(x-3)$.

$$\begin{aligned}
\frac{6-x}{(x+2)(x-3)} + 1 > -1 + 1 &\Rightarrow \frac{6-x}{(x+2)(x-3)} + \frac{(x+2)(x-3)}{(x+2)(x-3)} > 0 \\
&\Rightarrow \frac{6-x+(x+2)(x-3)}{(x+2)(x-3)} > 0 \\
&\Rightarrow \frac{6-x+(x^2-x-6)}{(x+2)(x-3)} > 0 \\
&\Rightarrow \frac{x^2-2x}{(x+2)(x-3)} > 0 \\
&\Rightarrow \frac{x(x-2)}{(x+2)(x-3)} > 0
\end{aligned}$$

ANSWER: $\boxed{\dfrac{x(x-2)}{(x+2)(x-3)} > 0}$

Example. Solve $\dfrac{6-x}{(x+2)(x-3)} > -1$.

The first step of this problem is to transform the inequality so that there is a rational function on the left and zero on the right (as done in the previous example). So we have the equivalent problem:

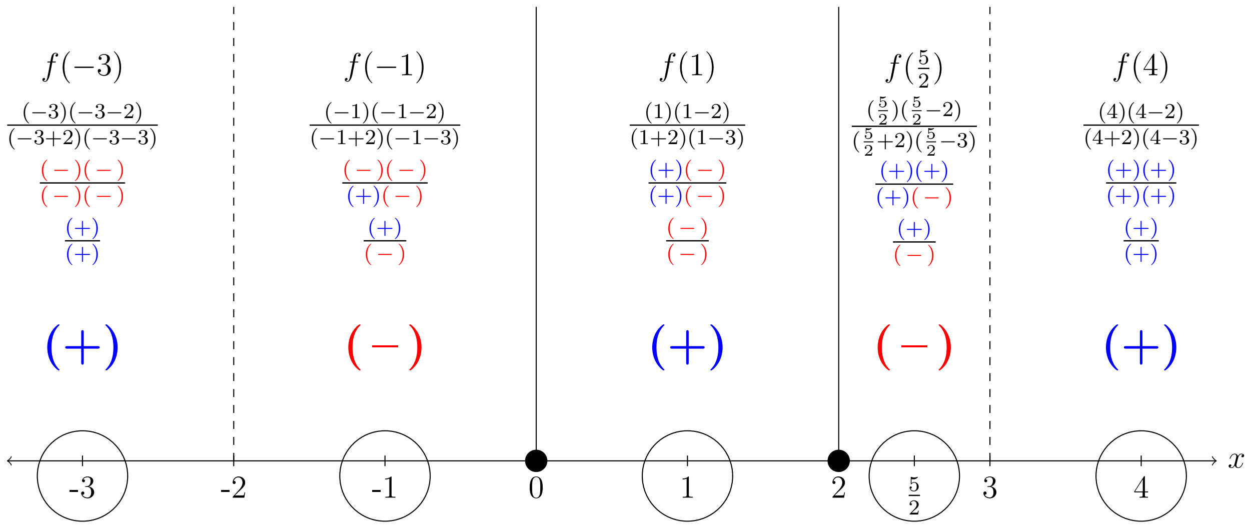
$$\text{Solve } \frac{x(x-2)}{(x+2)(x-3)} > 0.$$

Just as we did for polynomial inequalities, we will solve this by finding a sign chart using either the Test Point method or the Stacking Factors method. One difference from polynomial inequalities, though, is that we must look at both x-intercepts and vertical asymptotes, as the function can change sign at both.

The function $f(x) = \dfrac{x(x-2)}{(x+2)(x-3)}$ has x-intercepts at $x = 0$ and $x = 2$ (by setting the numerator equal to zero), and vertical asymptotes $x = -2$ and $x = 3$ (by setting the denominator equal to zero). There are then 5 intervals that we are concerned with $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, $(2, 3)$, and $(3, \infty)$.

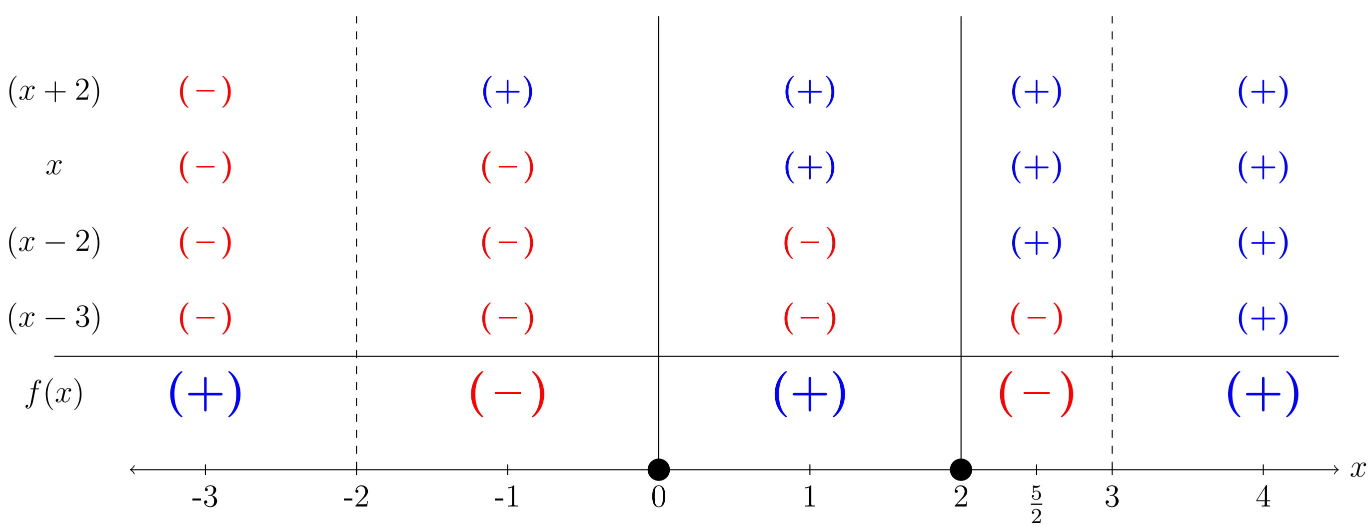
Test Point Method

We will need to test a point in each of the 5 intervals above. Below solid lines represent x-intercepts and dotted lines represent vertical asymptotes. We will choose test points $-3, -1, 1, \frac{5}{2}, 4$ (note that sometimes we cannot choose integer values).



Stacked Factors Method

The Stacked Factors method is the same as with polynomial inequalities with the addition of including the factors associated with vertical asymptotes (those in the denominator). Just as before linear factors are negative before the associated x-intercepts/vertical asymptotes and positive afterwards.



The question asks for when the function is greater than zero, so we want to identify the positive intervals which are $(-\infty, -2)$, $(0, 2)$, and $(3, \infty)$, which gives an answer:

ANSWER: $\boxed{(\text{-}\infty, -2) \cup (0, 2) \cup (3, \infty)}$

.

Example. Solve $\dfrac{6-x}{(x+2)(x-3)} \leq -1$.

Note that this is the same problem as above with $\leq$ instead of $>$. Thus we are looking at the same sign chart but identifying the negative intervals AND the x-intercepts $x = 0$ and $x = 2$. (Remember we are not including the vertical asymptotes because they are not included in the domain).

ANSWER: $\boxed{(\text{-}2,0\,] \cup [\,2, 3)}$

.

Things to Remember

- We use the same techniques as when solving polynomial inequalities but we need to add the factors associated with vertical asymptotes as well.
- We must have one side of the inequality be 0. If this is not the case, some algebra (often involving common denominators) to obatin this.
- If we have $\leq$ or $\geq$ we must include x-intercepts but NOT vertical asymptotes. This is why it is important to differentiate between them in the sign charts).
- Just like with polynomial inequalities, in order to determine the x-intercepts (and vertical asymptotes) we need to factor both the top and the bottom.