

2.5 Inverse Matrices

Suppose A is a square matrix. We look for an “*inverse matrix*” A^{-1} of the same size, such that A^{-1} times A equals I . Whatever A does, A^{-1} undoes. Their product is the identity matrix—which does nothing to a vector, so $A^{-1}Ax = x$. But A^{-1} might not exist.

What a matrix mostly does is to multiply a vector x . Multiplying $Ax = b$ by A^{-1} gives $A^{-1}Ax = A^{-1}b$. **This is $x = A^{-1}b$.** The product $A^{-1}A$ is like multiplying by a number and then dividing by that number. A number has an inverse if it is not zero—matrices are more complicated and more interesting. The matrix A^{-1} is called “ A inverse.”

DEFINITION The matrix A is *invertible* if there exists a matrix A^{-1} such that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I. \quad (1)$$

Not all matrices have inverses. This is the first question we ask about a square matrix: Is A invertible? We don’t mean that we immediately calculate A^{-1} . In most problems we never compute it! Here are six “notes” about A^{-1} .

Note 1 *The inverse exists if and only if elimination produces n pivots* (row exchanges are allowed). Elimination solves $Ax = b$ without explicitly using the matrix A^{-1} .

Note 2 The matrix A cannot have two different inverses. Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this “proof by parentheses”:

$$B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{or} \quad B = C. \quad (2)$$

This shows that a *left-inverse* B (multiplying from the left) and a *right-inverse* C (multiplying A from the right to give $AC = I$) must be the *same matrix*.

Note 3 If A is invertible, the one and only solution to $Ax = b$ is $x = A^{-1}b$:

$$\text{Multiply } Ax = b \text{ by } A^{-1}. \text{ Then } x = A^{-1}Ax = A^{-1}b.$$

Note 4 (Important) *Suppose there is a nonzero vector x such that $Ax = 0$. Then A cannot have an inverse.* No matrix can bring 0 back to x .

If A is invertible, then $Ax = 0$ can only have the zero solution $x = A^{-1}0 = 0$.

Note 5 A 2 by 2 matrix is invertible if and only if $ad - bc$ is not zero:

$$\text{2 by 2 Inverse: } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \quad (3)$$

This number $ad - bc$ is the *determinant* of A . A matrix is invertible if its determinant is not zero (Chapter 5). The test for n pivots is usually decided before the determinant appears.

Note 6 A diagonal matrix has an inverse provided no diagonal entries are zero:

$$\text{If } A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \text{ then } A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}.$$

Example 1 The 2 by 2 matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ is not invertible. It fails the test in Note 5, because $ad - bc$ equals $2 - 2 = 0$. It fails the test in Note 3, because $A\mathbf{x} = \mathbf{0}$ when $\mathbf{x} = (2, -1)$. It fails to have two pivots as required by Note 1.

Elimination turns the second row of this matrix A into a zero row.

The Inverse of a Product AB

For two nonzero numbers a and b , the sum $a + b$ might or might not be invertible. The numbers $a = 3$ and $b = -3$ have inverses $\frac{1}{3}$ and $-\frac{1}{3}$. Their sum $a + b = 0$ has no inverse. But the product $ab = -9$ does have an inverse, which is $\frac{1}{3}$ times $-\frac{1}{3}$.

For two matrices A and B , the situation is similar. It is hard to say much about the invertibility of $A + B$. But the *product* AB has an inverse, if and only if the two factors A and B are separately invertible (and the same size). The important point is that A^{-1} and B^{-1} come in *reverse order*:

If A and B are invertible then so is AB . The inverse of a product AB is

$$(AB)^{-1} = B^{-1}A^{-1}. \quad (4)$$

To see why the order is reversed, multiply AB times $B^{-1}A^{-1}$. Inside that is $BB^{-1} = I$:

$$\text{Inverse of } AB \quad (AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I.$$

We moved parentheses to multiply BB^{-1} first. Similarly $B^{-1}A^{-1}$ times AB equals I . This illustrates a basic rule of mathematics: Inverses come in reverse order. It is also common sense: If you put on socks and then shoes, the first to be taken off are the _____. The same reverse order applies to three or more matrices:

$$\text{Reverse order} \quad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}. \quad (5)$$

Example 2 *Inverse of an elimination matrix.* If E subtracts 5 times row 1 from row 2, then E^{-1} adds 5 times row 1 to row 2:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiply EE^{-1} to get the identity matrix I . Also multiply $E^{-1}E$ to get I . We are adding and subtracting the same 5 times row 1. Whether we add and then subtract (this is EE^{-1}) or subtract and then add (this is $E^{-1}E$), we are back at the start.

For square matrices, an inverse on one side is automatically an inverse on the other side. If $AB = I$ then automatically $BA = I$. In that case B is A^{-1} . This is very useful to know but we are not ready to prove it.

Example 3 Suppose F subtracts 4 times row 2 from row 3, and F^{-1} adds it back:

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

Now multiply F by the matrix E in Example 2 to find FE . Also multiply E^{-1} times F^{-1} to find $(FE)^{-1}$. Notice the orders FE and $E^{-1}F^{-1}$!

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 20 & -4 & 1 \end{bmatrix} \quad \text{is inverted by} \quad E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}. \quad (6)$$

The result is beautiful and correct. The product FE contains “20” but its inverse doesn’t. E subtracts 5 times row 1 from row 2. Then F subtracts 4 times the *new* row 2 (changed by row 1) from row 3. **In this order FE , row 3 feels an effect from row 1.**

In the order $E^{-1}F^{-1}$, that effect does not happen. First F^{-1} adds 4 times row 2 to row 3. After that, E^{-1} adds 5 times row 1 to row 2. There is no 20, because row 3 doesn’t change again. **In this order $E^{-1}F^{-1}$, row 3 feels no effect from row 1.**

In elimination order F follows E . In reverse order E^{-1} follows F^{-1} .
 $E^{-1}F^{-1}$ is quick. The multipliers 5, 4 fall into place below the diagonal of 1’s.

This special multiplication $E^{-1}F^{-1}$ and $E^{-1}F^{-1}G^{-1}$ will be useful in the next section. We will explain it again, more completely. In this section our job is A^{-1} , and we expect some serious work to compute it. Here is a way to organize that computation.

Calculating A^{-1} by Gauss-Jordan Elimination

I hinted that A^{-1} might not be explicitly needed. The equation $Ax = b$ is solved by $x = A^{-1}b$. But it is not necessary or efficient to compute A^{-1} and multiply it times b . *Elimination goes directly to x .* Elimination is also the way to calculate A^{-1} , as we now show. The Gauss-Jordan idea is to solve $AA^{-1} = I$, *finding each column of A^{-1} .*

A multiplies the first column of A^{-1} (call that x_1) to give the first column of I (call that e_1). This is our equation $Ax_1 = e_1 = (1, 0, 0)$. There will be two more equations. Each of the columns x_1, x_2, x_3 of A^{-1} is multiplied by A to produce a column of I :

$$\mathbf{3 \text{ columns of } } A^{-1} \quad AA^{-1} = A[x_1 \ x_2 \ x_3] = [e_1 \ e_2 \ e_3] = I. \quad (7)$$

To invert a 3 by 3 matrix A , we have to solve three systems of equations: $Ax_1 = e_1$ and $Ax_2 = e_2 = (0, 1, 0)$ and $Ax_3 = e_3 = (0, 0, 1)$. Gauss-Jordan finds A^{-1} this way.

The *Gauss-Jordan method* computes A^{-1} by solving *all n equations together*. Usually the “augmented matrix” $[A \ b]$ has one extra column b . Now we have three right sides e_1, e_2, e_3 (when A is 3 by 3). They are the columns of I , so the augmented matrix is really the block matrix $[A \ I]$. I take this chance to invert my favorite matrix K , with 2's on the main diagonal and -1 's next to the 2's:

$$\begin{aligned}
 [K \ e_1 \ e_2 \ e_3] &= \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} && \text{Start Gauss-Jordan on } K \\
 &\rightarrow \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \frac{3}{2} & -1 & \frac{1}{2} & \mathbf{1} & \mathbf{0} \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} && (\frac{1}{2} \text{ row } 1 + \text{row } 2) \\
 &\rightarrow \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & \mathbf{1} \end{bmatrix} && (\frac{2}{3} \text{ row } 2 + \text{row } 3)
 \end{aligned}$$

We are halfway to K^{-1} . The matrix in the first three columns is U (upper triangular). The pivots $2, \frac{3}{2}, \frac{4}{3}$ are on its diagonal. Gauss would finish by back substitution. The contribution of Jordan is *to continue with elimination!* He goes all the way to the “*reduced echelon form*”. Rows are added to rows above them, to produce *zeros above the pivots*:

$$\begin{aligned}
 \left(\begin{array}{l} \text{Zero above} \\ \text{third pivot} \end{array} \right) &\rightarrow \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \frac{3}{2} & \mathbf{0} & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} && (\frac{3}{4} \text{ row } 3 + \text{row } 2) \\
 \left(\begin{array}{l} \text{Zero above} \\ \text{second pivot} \end{array} \right) &\rightarrow \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} & \frac{3}{2} & \mathbf{1} & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} && (\frac{2}{3} \text{ row } 2 + \text{row } 1)
 \end{aligned}$$

The last Gauss-Jordan step is to divide each row by its pivot. The new pivots are 1. We have reached I in the first half of the matrix, because K is invertible. *The three columns of K^{-1} are in the second half of $[I \ K^{-1}]$:*

$$\begin{aligned}
 & \left(\begin{array}{l} \text{divide by } 2 \\ \text{divide by } \frac{3}{2} \\ \text{divide by } \frac{4}{3} \end{array} \right) \begin{bmatrix} \mathbf{1} & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \mathbf{1} & 0 & \frac{1}{2} & \mathbf{1} & \frac{1}{2} \\ 0 & 0 & \mathbf{1} & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} = [I \ x_1 \ x_2 \ x_3] = [I \ K^{-1}].
 \end{aligned}$$

Starting from the 3 by 6 matrix $[K \ I]$, we ended with $[I \ K^{-1}]$. Here is the whole Gauss-Jordan process on one line for any invertible matrix A :

Gauss-Jordan *Multiply $[A \ I]$ by A^{-1} to get $[I \ A^{-1}]$.*

The elimination steps create the inverse matrix while changing A to I . For large matrices, we probably don't want A^{-1} at all. But for small matrices, it can be very worthwhile to know the inverse. We add three observations about this particular K^{-1} because it is an important example. We introduce the words *symmetric*, *tridiagonal*, and *determinant*:

1. K is *symmetric* across its main diagonal. So is K^{-1} .
2. K is *tridiagonal* (only three nonzero diagonals). But K^{-1} is a dense matrix with no zeros. That is another reason we don't often compute inverse matrices. The inverse of a band matrix is generally a dense matrix.
3. The *product of pivots* is $2(\frac{3}{2})(\frac{4}{3}) = 4$. This number 4 is the *determinant* of K .

$$K^{-1} \text{ involves division by the determinant} \quad K^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \quad (8)$$

This is why an invertible matrix cannot have a zero determinant.

Example 4 Find A^{-1} by Gauss-Jordan elimination starting from $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$. There are two row operations and then a division to put 1's in the pivots:

$$\begin{aligned} [A \ I] &= \begin{bmatrix} 2 & 3 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \quad (\text{this is } [U \ L^{-1}]) \\ &\rightarrow \begin{bmatrix} 2 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{7}{2} & -\frac{3}{2} \\ 0 & 1 & -2 & 1 \end{bmatrix} \quad (\text{this is } [I \ A^{-1}]). \end{aligned}$$

That A^{-1} involves division by the determinant $ad - bc = 2 \cdot 7 - 3 \cdot 4 = 2$. The code for $X = \text{inverse}(A)$ can use **rref**, the "row reduced echelon form" from Chapter 3:

```
I = eye (n);           % Define the n by n identity matrix
R = rref ([A I]);      % Eliminate on the augmented matrix [A I]
X = R(:, n + 1 : n + n) % Pick A^{-1} from the last n columns of R
```

A must be invertible, or elimination cannot reduce it to I (in the left half of R).

Gauss-Jordan shows why A^{-1} is expensive. We must solve n equations for its n columns.

To solve $Ax = b$ without A^{-1} , we deal with one column b to find one column x .

In defense of A^{-1} , we want to say that its cost is not n times the cost of one system $Ax = b$. Surprisingly, the cost for n columns is only multiplied by 3. This saving is because the n equations $Ax_i = e_i$ all involve the same matrix A . Working with the right sides is relatively cheap, because elimination only has to be done once on A .

The complete A^{-1} needs n^3 elimination steps, where a single x needs $n^3/3$. The next section calculates these costs.

Singular versus Invertible

We come back to the central question. Which matrices have inverses? The start of this section proposed the pivot test: A^{-1} *exists exactly when A has a full set of n pivots*. (Row exchanges are allowed.) Now we can prove that by Gauss-Jordan elimination:

1. With n pivots, elimination solves all the equations $Ax_i = e_i$. The columns x_i go into A^{-1} . Then $AA^{-1} = I$ and A^{-1} is at least a *right-inverse*.
2. Elimination is really a sequence of multiplications by E 's and P 's and D^{-1} :

$$\text{Left-inverse} \quad (D^{-1} \cdots E \cdots P \cdots E)A = I. \quad (9)$$

D^{-1} divides by the pivots. The matrices E produce zeros below and above the pivots. P will exchange rows if needed (see Section 2.7). The product matrix in equation (9) is evidently a *left-inverse*. With n pivots we have reached $A^{-1}A = I$.

The right-inverse equals the left-inverse. That was Note 2 at the start of in this section. So a square matrix with a full set of pivots will always have a two-sided inverse.

Reasoning in reverse will now show that A *must have n pivots if $AC = I$* . (Then we deduce that C is also a left-inverse and $CA = I$.) Here is one route to those conclusions:

1. If A doesn't have n pivots, elimination will lead to a *zero row*.
2. Those elimination steps are taken by an invertible M . *So a row of MA is zero.*
3. If $AC = I$ had been possible, then $MAC = M$. The zero row of MA , times C , gives a zero row of M itself.
4. An invertible matrix M can't have a zero row! A *must have n pivots if $AC = I$* .

That argument took four steps, but the outcome is short and important.

Elimination gives a complete test for invertibility of a square matrix. A^{-1} *exists (and Gauss-Jordan finds it) exactly when A has n pivots*. The argument above shows more:

$$\text{If } AC = I \text{ then } CA = I \text{ and } C = A^{-1}$$

Example 5 If L is lower triangular with 1's on the diagonal, so is L^{-1} .

A triangular matrix is invertible if and only if no diagonal entries are zero.

Here L has 1's so L^{-1} also has 1's. Use the Gauss-Jordan method to construct L^{-1} . Start by subtracting multiples of pivot rows from rows *below*. Normally this gets us halfway to the inverse, but for L it gets us all the way. L^{-1} appears on the right when I appears on the left. Notice how L^{-1} contains 11, from 3 times 5 minus 4.

$$\begin{array}{l}
 \text{Gauss-Jordan} \\
 \text{on triangular } L
 \end{array}
 \begin{array}{l}
 \left[\begin{array}{cccccc}
 1 & 0 & 0 & 1 & 0 & 0 \\
 3 & 1 & 0 & 0 & 1 & 0 \\
 4 & 5 & 1 & 0 & 0 & 1
 \end{array} \right] = [L \ I] \\
 \rightarrow \left[\begin{array}{cccccc}
 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & -3 & 1 & 0 \\
 0 & 5 & 1 & -4 & 0 & 1
 \end{array} \right] \begin{array}{l}
 \text{(3 times row 1 from row 2)} \\
 \text{(4 times row 1 from row 3)} \\
 \text{(then 5 times row 2 from row 3)}
 \end{array} \\
 \rightarrow \left[\begin{array}{cccccc}
 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & -3 & 1 & 0 \\
 0 & 0 & 1 & 11 & -5 & 1
 \end{array} \right] = [I \ L^{-1}].
 \end{array}$$

L goes to I by a product of elimination matrices $E_{32}E_{31}E_{21}$. So that product is L^{-1} . All pivots are 1's (a full set). L^{-1} is lower triangular, with the strange entry "11".

That 11 does not appear to spoil 3, 4, 5 in the good order $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = L$.

■ REVIEW OF THE KEY IDEAS ■

1. The inverse matrix gives $AA^{-1} = I$ and $A^{-1}A = I$.
2. A is invertible if and only if it has n pivots (row exchanges allowed).
3. If $A\mathbf{x} = \mathbf{0}$ for a nonzero vector \mathbf{x} , then A has no inverse.
4. The inverse of AB is the reverse product $B^{-1}A^{-1}$. And $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.
5. The Gauss-Jordan method solves $AA^{-1} = I$ to find the n columns of A^{-1} . The augmented matrix $[A \ I]$ is row-reduced to $[I \ A^{-1}]$.

■ WORKED EXAMPLES ■

2.5 A The inverse of a triangular **difference matrix** A is a triangular **sum matrix** S :

$$\begin{aligned}
 [A \ I] &= \left[\begin{array}{ccc|ccc}
 1 & 0 & 0 & 1 & 0 & 0 \\
 -1 & 1 & 0 & 0 & 1 & 0 \\
 0 & -1 & 1 & 0 & 0 & 1
 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc}
 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 1 & 1 & 0 \\
 0 & -1 & 1 & 0 & 0 & 1
 \end{array} \right] \\
 &\rightarrow \left[\begin{array}{ccc|ccc}
 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 1 & 1 & 0 \\
 0 & 0 & 1 & 1 & 1 & 1
 \end{array} \right] = [I \ A^{-1}] = [I \ \text{sum matrix}].
 \end{aligned}$$

If I change a_{13} to -1 , then all rows of A add to zero. The equation $A\mathbf{x} = \mathbf{0}$ will now have the nonzero solution $\mathbf{x} = (1, 1, 1)$. A clear signal: ***This new A can't be inverted.***

2.5 B Three of these matrices are invertible, and three are singular. Find the inverse when it exists. Give reasons for noninvertibility (zero determinant, too few pivots, nonzero solution to $A\mathbf{x} = \mathbf{0}$) for the other three. The matrices are in the order A, B, C, D, S, E :

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 \\ 8 & 7 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 0 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution

$$B^{-1} = \frac{1}{4} \begin{bmatrix} 7 & -3 \\ -8 & 4 \end{bmatrix} \quad C^{-1} = \frac{1}{36} \begin{bmatrix} 0 & 6 \\ 6 & -6 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

A is not invertible because its determinant is $4 \cdot 6 - 3 \cdot 8 = 24 - 24 = 0$. D is not invertible because there is only one pivot; the second row becomes zero when the first row is subtracted. E is not invertible because a combination of the columns (the second column minus the first column) is zero—in other words $E\mathbf{x} = \mathbf{0}$ has the solution $\mathbf{x} = (-1, 1, 0)$.

Of course all three reasons for noninvertibility would apply to each of A, D, E .

2.5 C Apply the Gauss-Jordan method to invert this triangular “Pascal matrix” L . You see **Pascal’s triangle**—adding each entry to the entry on its left gives the entry below. The entries of L are “binomial coefficients”. The next row would be 1, 4, 6, 4, 1.

$$\text{Triangular Pascal matrix } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \text{abs(pascal}(4,1))$$

Solution Gauss-Jordan starts with $[L \ I]$ and produces zeros by subtracting row 1:

$$[L \ I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & -1 & 0 & 0 & 1 \end{array} \right].$$

The next stage creates zeros below the second pivot, using multipliers 2 and 3. Then the last stage subtracts 3 times the new row 3 from the new row 4:

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & -3 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array} \right] = [I \ L^{-1}].$$

All the pivots were 1! So we didn’t need to divide rows by pivots to get I . The inverse matrix L^{-1} looks like L itself, except odd-numbered diagonals have minus signs.

The same pattern continues to n by n Pascal matrices, L^{-1} has “alternating diagonals”.

Problem Set 2.5

- 1 Find the inverses (directly or from the 2 by 2 formula) of A, B, C :

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

- 2 For these “permutation matrices” find P^{-1} by trial and error (with 1’s and 0’s):

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

- 3 Solve for the first column (x, y) and second column (t, z) of A^{-1} :

$$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- 4 Show that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is not invertible by trying to solve $AA^{-1} = I$ for column 1 of A^{-1} :

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \left(\text{For a different } A, \text{ could column 1 of } A^{-1} \right. \\ \left. \text{be possible to find but not column 2?} \right)$$

- 5 Find an upper triangular U (not diagonal) with $U^2 = I$ which gives $U = U^{-1}$.

- 6 (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.

(b) If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two different matrices such that $AB = AC$.

- 7 (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $A\mathbf{x} = (1, 0, 0)$ cannot have a solution.

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\mathbf{x} = \mathbf{b}$?

(c) What happens to row 3 in elimination?

- 8 If A has column 1 + column 2 = column 3, show that A is not invertible:

(a) Find a nonzero solution \mathbf{x} to $A\mathbf{x} = \mathbf{0}$. The matrix is 3 by 3.

(b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot.

- 9 Suppose A is invertible and you exchange its first two rows to reach B . Is the new matrix B invertible and how would you find B^{-1} from A^{-1} ?

- 10 Find the inverses (in any legal way) of

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.$$

- 11 (a) Find invertible matrices A and B such that $A + B$ is not invertible.
 (b) Find singular matrices A and B such that $A + B$ is invertible.
- 12 If the product $C = AB$ is invertible (A and B are square), then A itself is invertible. Find a formula for A^{-1} that involves C^{-1} and B .
- 13 If the product $M = ABC$ of three square matrices is invertible, then B is invertible. (So are A and C .) Find a formula for B^{-1} that involves M^{-1} and A and C .
- 14 If you add row 1 of A to row 2 to get B , how do you find B^{-1} from A^{-1} ?

Notice the order. The inverse of $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} A \end{bmatrix}$ is _____.

- 15 Prove that a matrix with a column of zeros cannot have an inverse.
- 16 Multiply $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ times $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. What is the inverse of each matrix if $ad \neq bc$?
- 17 (a) What matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.
 (b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.
- 18 If B is the inverse of A^2 , show that AB is the inverse of A .
- 19 Find the numbers a and b that give the inverse of $5 * \text{eye}(4) - \text{ones}(4,4)$:

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

What are a and b in the inverse of $6 * \text{eye}(5) - \text{ones}(5,5)$?

- 20 Show that $A = 4 * \text{eye}(4) - \text{ones}(4,4)$ is *not* invertible: Multiply $A * \text{ones}(4,1)$.
- 21 There are sixteen 2 by 2 matrices whose entries are 1's and 0's. How many of them are invertible?

Questions 22–28 are about the Gauss-Jordan method for calculating A^{-1} .

- 22 Change I into A^{-1} as you reduce A to I (by row operations):

$$[A \ I] = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [A \ I] = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}$$

- 23 Follow the 3 by 3 text example but with plus signs in A . Eliminate above and below the pivots to reduce $[A \ I]$ to $[I \ A^{-1}]$:

$$[A \ I] = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

- 24 Use Gauss-Jordan elimination on $[U \ I]$ to find the upper triangular U^{-1} :

$$UU^{-1} = I \quad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- 25 Find A^{-1} and B^{-1} (if they exist) by elimination on $[A \ I]$ and $[B \ I]$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

- 26 What three matrices E_{21} and E_{12} and D^{-1} reduce $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ to the identity matrix? Multiply $D^{-1}E_{12}E_{21}$ to find A^{-1} .

- 27 Invert these matrices A by the Gauss-Jordan method starting with $[A \ I]$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

- 28 Exchange rows and continue with Gauss-Jordan to find A^{-1} :

$$[A \ I] = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

- 29 True or false (with a counterexample if false and a reason if true):

- (a) A 4 by 4 matrix with a row of zeros is not invertible.
- (b) Every matrix with 1's down the main diagonal is invertible.
- (c) If A is invertible then A^{-1} and A^2 are invertible.

- 30 For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

- 31 Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or A^{-1}):

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}.$$

- 32 This matrix has a remarkable inverse. Find A^{-1} by elimination on $[A \ I]$. Extend to a 5 by 5 “alternating matrix” and guess its inverse; then multiply to confirm.

$$\text{Invert } A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and solve } A\mathbf{x} = (1, 1, 1, 1).$$

- 33 Suppose the matrices P and Q have the same rows as I but in any order. They are “permutation matrices”. Show that $P - Q$ is singular by solving $(P - Q)\mathbf{x} = \mathbf{0}$.
- 34 Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

- 35 Could a 4 by 4 matrix A be invertible if every row contains the numbers 0, 1, 2, 3 in some order? What if every row of B contains 0, 1, 2, -3 in some order?
- 36 In the Worked Example 2.5 C, the triangular Pascal matrix L has an inverse with “alternating diagonals”. Check that this L^{-1} is $DL D$, where the diagonal matrix D has alternating entries 1, -1 , 1, -1 . Then $LDL D = I$, so what is the inverse of $LD = \text{pascal}(4,1)$?
- 37 The Hilbert matrices have $H_{ij} = 1/(i + j - 1)$. Ask MATLAB for the exact 6 by 6 inverse `invhilb(6)`. Then ask it to compute `inv(hilb(6))`. How can these be different, when the computer never makes mistakes?
- 38 (a) Use `inv(P)` to invert MATLAB’s 4 by 4 symmetric matrix $P = \text{pascal}(4)$.
 (b) Create Pascal’s lower triangular $L = \text{abs}(\text{pascal}(4,1))$ and test $P = LL^T$.
- 39 If $A = \text{ones}(4)$ and $\mathbf{b} = \text{rand}(4,1)$, how does MATLAB tell you that $A\mathbf{x} = \mathbf{b}$ has no solution? For the special $\mathbf{b} = \text{ones}(4,1)$, which solution to $A\mathbf{x} = \mathbf{b}$ is found by $A \setminus \mathbf{b}$?

Challenge Problems

- 40 (Recommended) A is a 4 by 4 matrix with 1’s on the diagonal and $-a, -b, -c$ on the diagonal above. Find A^{-1} for this bidiagonal matrix.
- 41 Suppose E_1, E_2, E_3 are 4 by 4 identity matrices, except E_1 has a, b, c in column 1 and E_2 has d, e in column 2 and E_3 has f in column 3 (below the 1’s). Multiply $L = E_1 E_2 E_3$ to show that all these nonzeros are copied into L .
 $E_1 E_2 E_3$ is in the *opposite* order from elimination (because E_3 is acting first). But $E_1 E_2 E_3 = L$ is in the *correct* order to invert elimination and recover A .

- 42 Direct multiplications 1–4 give $MM^{-1} = I$, and I would recommend doing #3. M^{-1} shows the change in A^{-1} (useful to know) when a matrix is subtracted from A :

$$\begin{array}{ll}
 \mathbf{1} & M = I - \mathbf{uv} \quad \text{and} \quad M^{-1} = I + \mathbf{uv}/(1 - \mathbf{vu}) \quad (\text{rank 1 change in } I) \\
 \mathbf{2} & M = A - \mathbf{uv} \quad \text{and} \quad M^{-1} = A^{-1} + A^{-1}\mathbf{uv}A^{-1}/(1 - \mathbf{v}A^{-1}\mathbf{u}) \\
 \mathbf{3} & M = I - UV \quad \text{and} \quad M^{-1} = I_n + U(I_m - VU)^{-1}V \\
 \mathbf{4} & M = A - UW^{-1}V \quad \text{and} \quad M^{-1} = A^{-1} + A^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}
 \end{array}$$

The Woodbury-Morrison formula 4 is the “matrix inversion lemma” in engineering. The **Kalman filter** for solving block tridiagonal systems uses formula 4 at each step. The four matrices M^{-1} are in diagonal blocks when inverting these block matrices (\mathbf{v} is 1 by n , \mathbf{u} is n by 1, V is m by n , U is n by m).

$$\begin{bmatrix} I & \mathbf{u} \\ \mathbf{v} & 1 \end{bmatrix} \quad \begin{bmatrix} A & \mathbf{u} \\ \mathbf{v} & 1 \end{bmatrix} \quad \begin{bmatrix} I_n & U \\ V & I_m \end{bmatrix} \quad \begin{bmatrix} A & U \\ V & W \end{bmatrix}$$

- 43 Second difference matrices have beautiful inverses if they start with $T_{11} = 1$ (instead of $K_{11} = 2$). Here is the 3 by 3 tridiagonal matrix T and its inverse:

$$T_{11} = \mathbf{1} \quad T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

One approach is Gauss-Jordan elimination on $[T \ I]$. That seems too mechanical. I would rather write T as the product of first differences L times U . The inverses of L and U in Worked Example 2.5 A are **sum matrices**, so here are T and T^{-1} :

$$LU = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 0 & -1 & 1 & \\ \text{difference} & & & \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ & 1 & -1 \\ & & 1 \\ \text{difference} & & & \end{bmatrix} \quad U^{-1}L^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \\ \text{sum} & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ \text{sum} & & & \end{bmatrix}$$

Question. (4 by 4) What are the pivots of T ? What is its 4 by 4 inverse? The reverse order UL gives what matrix T^* ? What is the inverse of T^* ?

- 44 Here are two more difference matrices, both important. **But are they invertible?**

$$\text{Cyclic } C = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \text{Free ends } F = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

One test is elimination—the fourth pivot fails. Another test is the determinant, we don’t want that. The best way is much faster, and independent of matrix size:

Produce $\mathbf{x} \neq \mathbf{0}$ so that $C\mathbf{x} = \mathbf{0}$. Do the same for $F\mathbf{x} = \mathbf{0}$. Not invertible.

Show how both equations $C\mathbf{x} = \mathbf{b}$ and $F\mathbf{x} = \mathbf{b}$ lead to $0 = b_1 + b_2 + \dots + b_n$. There is no solution for other \mathbf{b} .

- 45 *Elimination for a 2 by 2 block matrix:* When you multiply the first block row by CA^{-1} and subtract from the second row, the “Schur complement” S appears:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix} \quad \begin{array}{l} A \text{ and } D \text{ are square} \\ S = D - CA^{-1}B. \end{array}$$

Multiply on the right to subtract $A^{-1}B$ times block column 1 from block column 2.

$$\begin{bmatrix} A & B \\ 0 & S \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = ? \quad \text{Find } S \text{ for } \begin{bmatrix} A & B \\ C & I \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}.$$

The block pivots are A and S . If they are invertible, so is $[A \ B; C \ D]$.

- 46 How does the identity $A(I + BA) = (I + AB)A$ connect the inverses of $I + BA$ and $I + AB$? Those are both invertible or both singular: not obvious.