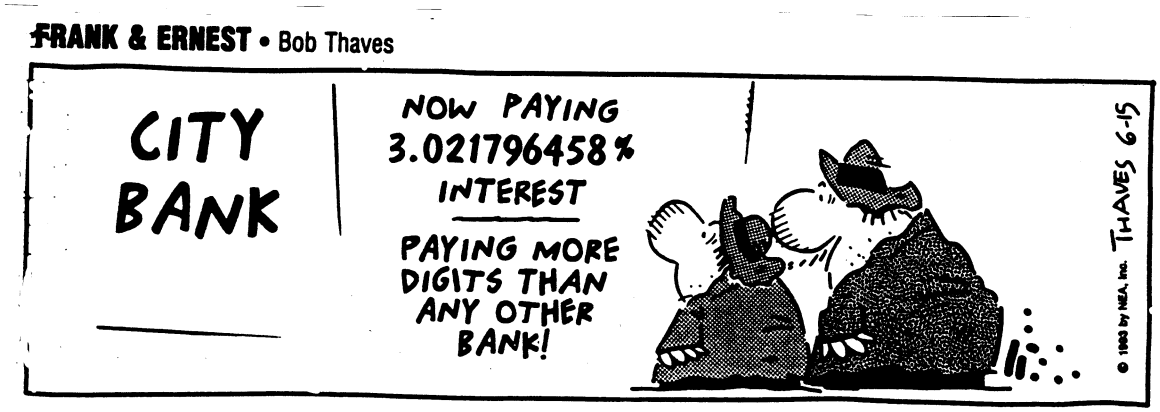
**Engineering 1N**
**THE NATURE OF ENGINEERING**

## Accuracy, Precision, Errors, and Significant Figures

*Errors like straws upon the surface flow;*
*He who would search for pearls must dive below.*

Dryden




## Errors

Measurements are *always* characterized by uncertainty. Whether because of the possibility of instrument drift, the need to interpolate visually an instrument scale, or the difficulty of defining exactly what we wish to measure, we are never certain that the value of what we have measured is the "true" value of what we intended to measure. Thus, we assume that all measurements include the possibility of "errors", and a measurement is not completely described without some indication of the nature of these errors (the uncertainty in the measurement). Measurement errors are unavoidable, so "error" in this context does not mean "mistake." We may have measured the true value, but we are never certain that we have done so.

Quite a bit of jargon has been developed to describe measurement errors. In previous classes you may have discussed the different meanings of the words “precision” and “accuracy.” “Precision” refers to the uncertainty in a measurement reading or observation. It is closely linked with the term “reproducibility.” A precise measurement is one which is characterized by high reproducibility. Repeated observation leads to nearly identical reported values. “Accuracy” is used to describe the closeness of an observation to the true value of the parameter being measured. It is independent of precision. Note that precision necessarily refers to the characteristics of a set of repeated observations, while accuracy can refer to a set of observations or to an individual observation. In other words, an observation from an imprecise instrument could very well be highly accurate, but a second observation has a high probability of being inaccurate since the instrument is imprecise. Whether from imprecision or inaccuracy, measurements are *always* characterized by errors, and the term “errors” is commonly used to describe both imprecision and inaccuracy collectively.

It is useful to think of measurement errors in two categories: *systematic* errors and *random* errors.

### Systematic errors

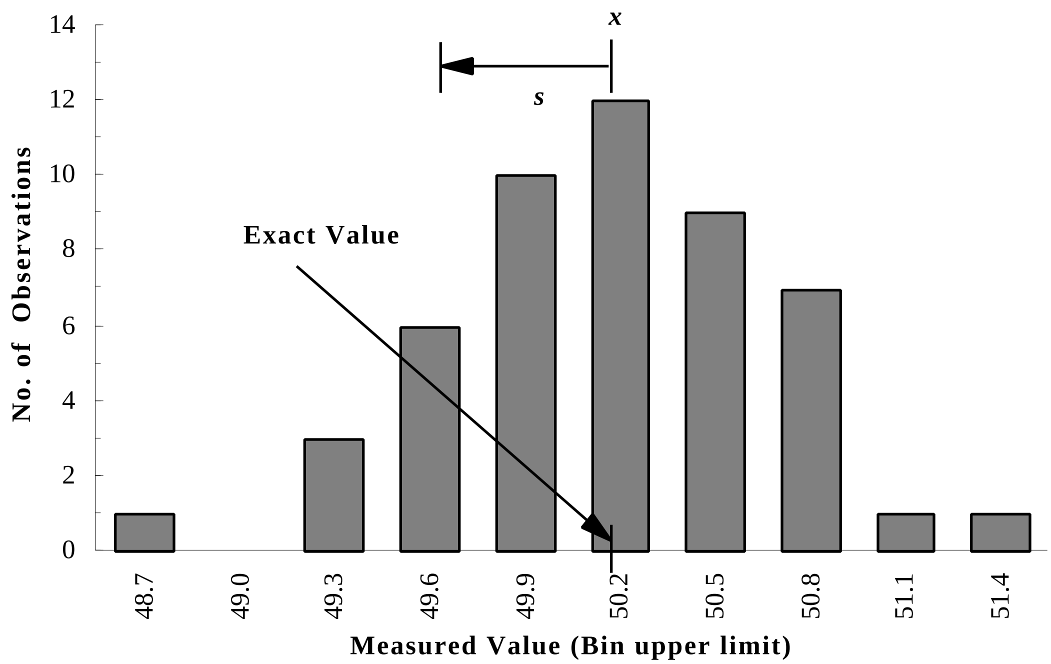
Systematic errors are those differences between an observation and the true value that are consistent from one observation to the next. For example, suppose the scale plate on a thermometer were shifted up or down. Then all of our observed temperatures would be off by the amount of the shift. Such *calibration errors* are the most common type of systematic error. Note that systematic errors, since they are consistent from one measurement to another, are most closely associated with inaccuracy. Also note that systematic errors are relatively easily managed, once they are detected. Detection, however, is nontrivial.

### Random errors

Random errors are more difficult to characterize and are usually more difficult to manage. By definition, they are *unpredictable* and change from one observation to another. Common sources of random errors include:

- different applications of the instrument and technique, for example, by different people during visual interpolation of instrument scales;
- inherent randomness in the instrumentation (usually electronic components);
- uncontrolled and unobserved external influences on the measurement. As an example of the latter, consider the effect of wind on a rain gage measurement. While wind essentially always reduces the measured amount of rain, the magnitude of that reduction depends on wind speed, direction, etc. These factors vary from event to event, day to day, leading to an unpredictable and varying error in the measurement.
- random differences in the quantity being measured, such as the differences between individual paper clips when measuring the number of bends required to break a paper clip.

Random errors manifest themselves as an *error distribution*, which is often represented graphically. Here is one example:




While such a distribution completely describes the nature of the errors, it is awkward to use and manipulate. Therefore, it is quite common to forego the complete information provided by the error distribution and instead to describe the errors by an *error* or *precision index*. We typically write:

$$x_{exact} = x_{observed} \pm \Delta x$$

where $\Delta x$ is the *precision index* or *error*. Note that the definition of $\Delta x$ can be ambiguous. It is a single number used to characterize the actual distribution of errors. Some choose to define $\Delta x$ in terms of the standard deviation of the distribution, $s$:

$$s = \frac{1}{n-1}\sum_{i=1}^{n}\left[x_i - \bar{x}\right]^2$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

As an example, the magnitudes of $\bar{x}$ and $s$ are indicated on the histogram shown above . The magnitude of $\Delta x$ can then be defined as some multiple of $s$. So a measurement might be reported as:

$$\bar{x} \pm 2s$$

The more conservative the observer, the greater the multiplier in front of $s$.

Others choose to define $\Delta x$ in terms of the maximum imaginable error. This is often the case when reading scales with tick marks, e.g., you are certain the observation is between one pair of tick marks and not another.

**Significant digits or figures**

In many cases engineers and scientists choose not to identify a precision index explicitly, but rather to use an implied precision via *significant digits*. For example, all of the following numbers have 3 significant digits:

3820
220.
6.47
0.190
0.00518

Significant digits carry with them an implied precision of ±1/2 unit in the rightmost significant digit, i.e.,

3280 ± 5
220. ± 0.5
6.47 ± 0.005
0.190 ± 0.0005
0.00518 ± 0.000005

This implied precision derives from the notion of the uncertainty in reading an instrument scale with tick marks corresponding to the rightmost significant digit. The implied precision then represents the half-way point between successive tick marks.

## Error Propagation

Our first problem was to define the errors associated with a measurement. Our next problem is to assess the impacts of these errors on derived variables. In other words, if we use a measurement to calculate some other variable, how does the error in the measurement propagate through the calculation? How big is the error in the derived variable? For example, suppose we measure the length and width of a rectangle. What will the error (uncertainty) be in the calculated area of the rectangle?

Suppose we have some derived variable $f$ which is a function of $n$ different measurements, $x_1$, $x_2$, $x_3$,…, $x_n$. So we can write:

$$f = f(x_1, x_2, x_3, \cdots, x_n)$$

The error in $f$, $\Delta f$, is bounded by (is no bigger than):

$$\Delta f \le \left|\frac{\partial f}{\partial x_1}\right|\Delta x_1 + \left|\frac{\partial f}{\partial x_2}\right|\Delta x_2 + \left|\frac{\partial f}{\partial x_3}\right|\Delta x_3 + \cdots + \left|\frac{\partial f}{\partial x_n}\right|\Delta x_n$$

The partial derivative of $f$ with respect to $x_1$

$$\frac{\partial f}{\partial x_1}$$

is the rate of change in the value of $f$ with respect to a change in $x_1$, with all other $x_i$ held fixed. It is found by taking the ordinary derivative of $f$ assuming that all $x_i$ other than $x_1$ are constants.

The bound given above is often quite large. If we wish a tighter error estimate, we need to be able to make an assumption. Let's assume that the errors in the measurements, $\Delta x_1$, $\Delta x_2$, $\Delta x_3$,…, $\Delta x_n$, are *independent*.. In other words, our uncertainty about $x_1$ isn't related to or influenced by our uncertainty about $x_2$, or any of the other measurements. This is not an outrageous assumption in many cases. For example, suppose you are going to estimate a velocity, $V$, by timing how long it takes, $t$, to move a given distance, $x$. The uncertainty, or imprecision, in your measurements of time and distance will be independent—you are using different measurement instruments.

If the errors are independent and random and are defined consistently as the same multiple of $s_i$, then:

$$\Delta f = \sqrt{\left[\frac{\partial f}{\partial x_1}\right]^2 \Delta x_1^2 + \left[\frac{\partial f}{\partial x_2}\right]^2 \Delta x_2^2 + \left[\frac{\partial f}{\partial x_3}\right]^2 \Delta x_3^2 + \cdots + \left[\frac{\partial f}{\partial x_n}\right]^2 \Delta x_n^2}$$

Using the velocity example and introducing numerical values:

$$f = V = \frac{x}{t}$$

with measured values: $x = 5.1 \pm 0.05$ $t = 2.3 \pm 0.08$

Finding the derivatives we need:

$$\frac{\partial f}{\partial x} = \frac{1}{t} \qquad \frac{\partial f}{\partial t} = -\frac{x}{t^2}$$

So, in general

$$\Delta f \le \left|\frac{\Delta x}{t}\right| + \left|-\frac{x\Delta t}{t^2}\right|$$

and, if the errors are random and independent:

$$\Delta f = \sqrt{\frac{\Delta x^2}{t^2} + \frac{x^2 \Delta t^2}{t^4}} = \frac{x}{t}\sqrt{\frac{\Delta x^2}{x^2} + \frac{\Delta t^2}{t^2}}$$

After substituting numerical values, we find:

$$f = \frac{x}{t} = 2.217 \qquad \Delta f = 0.080 \leq 0.099$$

Many people would choose to write the final result as:

$$f = 2.22 \pm 0.08$$

It is worth noting that if

$$\Delta t = 0.05$$

then

$$\Delta f = 0.053\,.$$

The error in the calculated velocity is clearly dominated by the error in measuring $t$.

**Error propagation for simple operations**

The general formulas given above for the upper bound on $\Delta f$ and for the case of independent errors reduce to very simple results in the case of the four basic arithmetic operations of addition, subtraction, multiplication, and division.

If $\quad f = x_1 \pm x_2$

then $\quad \Delta f \leq \Delta x_1 + \Delta x_2$

and for independent, random errors

$$\Delta f = \sqrt{\Delta x_1^2 + \Delta x_2^2}\,.$$

If $\quad f = x_1 * x_2$ or $x_1 / x_2$

then $\quad \dfrac{\Delta f}{|f|} \leq \dfrac{\Delta x_1}{|x_1|} + \dfrac{\Delta x_2}{|x_2|}$

and for independent, random errors

$$\frac{\Delta f}{|f|} = \sqrt{\left[\frac{\Delta x_1}{x_1}\right]^2 + \left[\frac{\Delta x_2}{x_2}\right]^2}$$

**Error propagation for significant figures**

When an error index is not explicitly provided and precision is therefore implied by significant digits, there are some useful rules of thumb that approximate the results of the strict application of error propagation theory. These rules of thumb are almost always adequate.

1) When adding or subtracting, the sum or difference is rounded to the last *decimal place* in the least precise number.

   Example:

   $$\begin{array}{r} 1.004 \\ 4.2 \\ 0.144 \\ \hline 5.348 \end{array} \qquad \underline{\underline{5.3}}$$

2) When multiplying or dividing, the product or quotient is rounded to the number of *significant digits* in the number with the least number of significant figures.

   Examples:

   4.9178 * 2.03 = 9.98313 $\underline{\underline{9.98}}$

   456.212 / 2.17 = 210.2359 $\underline{\underline{210.}}$

**Implied precision vs. explicit error propagation**

The use of implied precision to represent the effects of uncertainty is an *alternative* to using formal error propagation. In other words, the rules of error propagation (usually) lead to a statistically justified estimate of the uncertainty (precision) in a calculated variable that is a function of one or more other uncertain variables. The rules of thumb of implied precision (significant figures) are an approximation to the rules of error propagation. Let's consider an example.

$$V = \frac{4Q}{\pi D^2}$$

$V$ happens to be the velocity of water flowing in a pipe of diameter $D$ when the volume of water per unit time flowing through the pipe is $Q$. $V$ can be measured in ft / sec, $D$ in feet, and $Q$ in ft$^3$ / sec (cfs).

Suppose that: $Q = 0.22$ cfs, $\Delta Q = 0.005$ cfs
$D = 0.667$ ft, $\Delta D = 0.0005$ ft

In other words, let's begin by assuming the specified precision index is consistent with the implied precision in the data as given. Since for an arbitrary function, $f = f(x_1, x_2, x_3, \ldots, x_n)$, with independent errors:

$$\Delta f = \sqrt{\left[\frac{\partial f}{\partial x_1}\right]^2 \Delta x_1^2 + \left[\frac{\partial f}{\partial x_2}\right]^2 \Delta x_2^2 + \left[\frac{\partial f}{\partial x_3}\right]^2 \Delta x_3^2 + \cdots + \left[\frac{\partial f}{\partial x_n}\right]^2 \Delta x_n^2} \qquad (1)$$

then in this particular case:

$$V = 0.630 \text{ ft / sec}$$

$$\Delta V = \sqrt{\left[\frac{4}{\pi D^2}\right]^2 \Delta Q^2 + \left[\frac{-8Q}{\pi D^3}\right]^2 \Delta D^2} = 0.014 \text{ ft / sec}$$

The significant figures rule of thumb for this same calculation yields a two-significant-digit result, i.e., $V = 0.63$ ft/sec, with an implied precision of $\Delta V = 0.005$ ft/sec.

So, in this case, the rule of thumb yields an implied precision in the calculated velocity that is somewhat smaller than the correctly propagated error. The difference is small enough (a factor of 2) that we can judge the rule of thumb as adequate in this case. Another way of saying this is that the correctly propagated error results in more than 1, but less than 2 significant digits, while the rule of thumb yields 2 significant digits.

You might find it interesting to verify that the propagated error will be 0.005 ft/sec if $\Delta Q$ were 0.0017 cfs ($\Delta D$ as given above).

## Averaging

One of the reasons that we average replicate measurements is to increase precision. That is, averaging is one way to increase the number of significant digits. Let's see how this works.

Using our functional notation, we can write out the definition of the average:

$$f(x_1, x_2, \ldots, x_n) = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

so

$$\frac{\partial f}{\partial x_i} = \frac{1}{n}$$

Assuming that the precision index for all $x_i$ is the same, e.g., $\Delta x$, then substitution into Equation (1) above gives

$$\Delta f = \Delta\bar{x} = \frac{\Delta x}{\sqrt{n}}.$$

This means that 100 replicates gets us 1 additional significant digit and 10,000 replicates get us 2 additional significant digits.

Here is a numerical example. Suppose we take 4 replicate measurements and obtain values of

104, 106, 106, 105

The average is 105.25 (exactly). Using the implied precision for each of the measurements, 0.5, we find that the precision of the average is 0.5/2 = 0.25. We could either report that value as a precision index, i.e., 105.25 ± 0.25, or, if we choose to use implied precision and not show a precision index, then we are left with 105 as the most appropriate reported value. This is equivalent to what the rules of thumb for significant figures would yield.

**Units conversions**

Units conversions require some special care when considering error propagation or significant figures. In general, units conversions involve multiplication (or division) by a constant. In other words, they are a simple linear scaling. In many cases the scaling constant is not uncertain or is known to a large number of digits. Therefore, a units conversion is not the same as the multiplication of two uncertain numbers, and the rules of thumb of significant figures do not apply.

Note first that a simple scaling propagates through an error analysis as a simple scaling of the precision index:

$$f(x) = cx$$

$$\Delta f = \sqrt{c^2 \Delta x^2} = c\Delta x$$

Now, consider this example: we are provided with a measured length of 1.5 ft and we wish to convert to units of inches. The multiplicative conversion factor is 12 in/ft. So,

1.5 ± 0.05 ft = 12*1.5 ± 12*0.05 in = 18.0 ± 0.6 in

The converted length could be reported this way, or you would be justified in using implied precision and just giving the length as 18 inches (0.6 is pretty close to 0.5).

Let's compare this with the multiplication of two uncertain numbers:

1.5 ± 0.05 * 12 ± 0.5 = 18.0 ± 0.96

The precision index of the product was calculated using the error propagation equation, (1). So the rules of thumb do all right here. They indicate that the product could be given to 2 significant figures with an implied precision of 0.5. Although smaller than the more appropriate value of 0.96, 0.5 is the closest of the possible implied precisions (5, 0.5, 0.05), so giving the product as 18 would be satisfactory.

Now, let's return to a units conversion. This time consider a measured length of 9.5 ft converted to inches.

$$9.5 \pm 0.05 \text{ ft} = 12{*}9.5 \pm 12{*}0.05 \text{ in} = 114.0 \pm 0.6 \text{ in}$$

So the converted length should be reported as 114 if using implied precision. Therefore, we have added a significant figure, and *the rules of thumb do not work here*, since they would specify two significant figures.

It is worth noting that the rules of thumb must be applied equally carefully when dealing with other functions in addition to units conversions. For example, consider the sine function.

$$f(x) = \sin(x)$$
$$\Delta f = \cos(x)\Delta x$$

Take a look at the following table.

| $x$ (radians) | sin($x$) | cos($x$) | $\Delta$sin($x$) |
|---|---|---|---|
| 0.000 | 0.0000 | 1.0000 | 0.00050 |
| 0.785 | 0.7068 | 0.7074 | 0.00035 |
| 1.571 | 1.0000 | -0.0002 | 0.0000001 |
| 2.356 | 0.7072 | -0.7070 | 0.00035 |
| 3.142 | -0.0004 | -1.0000 | 0.00050 |

With $x$ specified in radians to ±0.0005, we find that the maximum (worst) precision index for sin($x$) is 0.0005. This means that 4 digits in $x$ yield 3 digits in sin($x$) *for some values of* $x$. It is important to note that in this case the precision index of sin($x$) is a function of $x$. For some other values of $x$ (1.571) the precision index is very small so that sin($x$) has many more than 7 significant digits!

## Interpreting reported data

In a perfect world, all data would be specified with their precision indices. In a somewhat less perfect world, all data would be specified to their appropriate significant figures. Unfortunately, in our world, we often rely on context and assumed knowledge of the data user to streamline the presentation of data. As a result, we often must make informed assumptions about the precision with which particular data are specified. Let me illustrate with a few examples.

Suppose you are provided with a set of measurements as follows: 15, 37, 91, 80, 56, …. It would be appropriate to assume that each of these numbers is specified to two significant digits, including the '80', in spite of the fact that there is no decimal point shown after the '0' in 80. While the decimal point would be formally required, the context indicates that the '0' is significant. A similar example would be reporting the dimensions of a rectangular area as 235' x 100'. The context indicates that both of the 0's in '100' are significant.

Another common situation is illustrated by the specification of pipe sizes, e.g., a "6-inch pipe." Taken literally, this specification indicates an uncertainty of ±0.5 in. for the pipe diameter. However, considering that pipe is machine-manufactured, it seems unlikely that the uncertainty would be this large. In fact, for standard steel screw pipe the internal diameter of 6-in pipe is specified as 6.065 in. So in reality, a reasonable precision to be applied to "6-in pipe" might be ± 0.05 or even ± 0.005 in. Of course, if we knew from the context that we were talking about standard steel screw pipe, then we would use the value 6.065 in. with its implied precision of 0.0005 in.

## Further Reading

Taylor, John R., *An Introduction to Error Analysis*, University Science Books, Sausalito, CA, 1982.