

## **A General Formula for the Discount for Lack of Marketability**

### **Abstract**

Stock transfer restrictions limit a share's marketability and reduce its value. The DLOM has been modeled as the value of an average-strike put option based on a lognormal approximation. The approximation's accuracy worsens as the restriction period lengthens. I generalize the average-strike put DLOM model first to restriction periods of any length  $L$  by modeling the  $L$ -year DLOM as the value of the one-year DLOM compounded over  $L$  years and then to restriction periods of uncertain length by assuming the restriction period is exponentially distributed. My model allows for a DLOM term premium, to reflect a risk averse investor's more prolonged exposure to the risk of an increasingly negatively skewed fat-tailed return distribution or greater exposure to investment-specific agency costs, or a term discount, to reflect value added due to special fund manager investment skill or strategic equity investment value.

## **A General Formula for the Discount for Lack of Marketability**

The growth of private equity (PE) investing has spawned secondary trading in unregistered common stocks. PE funds selling shares of portfolio companies, PE fund investors selling limited partnership units, and employees and other shareholders of privately owned firms selling shares to investors seeking such investments are just three examples.<sup>1</sup> Private stock transactions are infrequent, and the markets for these shares are consequently relatively small, due to the legal resale restrictions the Securities Act of 1933 (1933 Act) imposes on unregistered common stock.<sup>2</sup> The desire to transact in unregistered common stock coupled with the lack of regular market prices to facilitate price discovery or at least permit investors to gauge an appropriate discount for lack of marketability (DLOM) have generated interest in DLOM models that can be used reliably in pricing restricted shares.

Resale and transfer restrictions entail a loss of timing flexibility because the sale or transfer has to be postponed until the restrictions lapse. This loss of flexibility imposes a cost, which can be modeled as the value of a foregone put option that will expire when the marketability restriction lapses (Finnerty, 2012, 2013a; Longstaff, 1995, 2001; and Kahl, Liu, and Longstaff, 2003). Financial economists and business appraisers typically value restricted stock by calculating the freely traded value the stock would have if it traded in a liquid market free of restrictions and subtracting a DLOM to reflect the loss of timing flexibility due to any resale or transfer restrictions (Pratt and Grabowski, 2014).

The early DLOM papers reported the results of empirical studies of the discounts reflected in the prices of letter stock,<sup>3</sup> unregistered shares of common stock sold privately by public firms, which purchasers could later resell subject to the resale restrictions imposed by Rule 144 under the 1933 Act (SEC, 1971; Wruck, 1989; and Silber, 1991).<sup>4</sup> More recent research has focused on developing and empirically testing DLOM option models. At least three DLOM put option models have been proposed in the finance literature (Finnerty, 2013b). The seminal BSM put option pricing model (Black and Scholes, 1973; and Merton, 1973) was the basis for the earliest put option DLOM model proposed by Alli and Thompson (1991). Some appraisers still use it (Chaffee, 1993; and Abbott, 2009). It measures the DLOM with respect to the loss of the flexibility to sell shares at today's freely traded market price when the shareholder cannot sell the stock until the end of some restriction period. It cannot measure the DLOM accurately because its assumption of a fixed strike price for the duration of the restriction period

gives the restricted stockholder absolute downside protection (barring default by the option writer) whereas the potential sale prices are not actually fixed with restricted stock. Since the price of the otherwise identical unrestricted asset will change during the restriction period due to normal market forces, the asset holder's loss of timing flexibility should be measured relative to the opportunity she would have to sell at any of these market prices were there no transferability restrictions.

The lookback put and average-strike put option models avoid this shortcoming by adjusting the strike price to reflect the changing price of the unrestricted asset during the period of non-marketability. Longstaff (1995) obtains an upper bound on the DLOM by modeling the value of marketability as the price of a lookback put option, which assumes that investors have perfect market-timing ability. This assumption is generally consistent with empirical evidence that private information enables insiders to time the market and realize excess returns (Gompers and Lerner, 1998). However, it is inconsistent with evidence that outside investors, at least on average, do not have any special ability to outperform the market (Graham and Harvey, 1996; and Barber and Odean, 2000). Thus, while the lookback put option model may be appropriate in the presence of asymmetric information for equity placed with insiders who possess valuable private information, it will overstate the discount when investors do not have valuable private information about the stock (Finnerty, 2013a).

Finnerty (2012, 2013a) models the DLOM as the value of an average-strike put option. The investor is not assumed to have any special market-timing ability; instead, he assumes that the investor would, in the absence of any transfer restrictions, be equally likely to sell the shares anytime during the restriction period. To be useful, a DLOM model should produce discounts that are consistent with the DLOMs observed in the marketplace. Numerous studies have documented average discounts between 13.5 percent and 33.75 percent in private placements of letter stock, which is not freely transferable because of the Rule 144 resale restrictions (Hertzel and Smith, 1993; and Hertzel et al., 2002). Business appraisers typically apply DLOMs between 25 and 35 percent for a two-year restriction period and between 15 and 25 percent for a one-year restriction period (Finnerty, 2012, 2013a). Finnerty (2012) compares the model-predicted discounts to the discounts observed in a sample of 208 discounted common stock private placements and finds that the average-strike put option model discounts are generally consistent with the observed private placement discounts after adjusting for the information, ownership concentration, and overvaluation effects that accompany a stock private placement. However,

the model appears to be increasingly less accurate for longer restriction periods beyond two years, especially for high-volatility stocks.<sup>5</sup>

Stock transfer restrictions, such as those imposed by Rule 144, are one of the more often-cited factors responsible for a (restricted) share's relative lack of liquidity, in this case, due to its lack of marketability. Liquidity refers to an asset holder's flexibility to transfer asset ownership through a market transaction. It is the relative ease with which an asset holder can convert the asset into cash without losing some of the asset's intrinsic value. A liquid market gives an asset holder the flexibility to sell the asset at any time she chooses without sacrificing any intrinsic value. Lack of liquidity imposes a loss of timing flexibility because an asset holder cannot dispose of the asset quickly unless she is willing to accept a reduction in value. It takes more time to find a buyer in an illiquid market. The consequent loss of flexibility to sell an asset freely, or equivalently, the ability to sell it quickly only if there is some sacrifice of intrinsic value, can be modeled as the loss of value of a foregone put option.

An asset that lacks marketability also lacks liquidity because the marketability restrictions inhibit the holder from selling the asset quickly for its full intrinsic value. But lack of marketability can be distinguished from lack of liquidity. An asset's marketability refers to an asset holder's legal and contractual ability to sell or otherwise transfer ownership of the asset. Lack of marketability arises when legal or contractual restrictions on transfer prevent, or at least severely impair, an asset holder's ability to sell the asset or transfer it until the restriction period lapses (Finnerty, 2013b). The 1933 Act's restrictions on offering unregistered securities to investors who are not accredited (i.e., meet certain income and net worth tests) exemplify such legal restrictions, and the almost absolute prohibition on transferring employee stock options exemplifies such contractual restrictions on marketability.

It is important to appreciate that even when securities are unregistered, there may not be an absolute prohibition on transfer. For example, a security holder can still transfer unregistered shares of common stock by relying on one of the exemptions from registration under the 1933 Act. Also, specialized secondary markets have developed to enable holders of unregistered common stock to find potential buyers.<sup>6</sup> Nevertheless, a buyer of unregistered shares faces the same restrictions as the seller. Thus, it may still be reasonable to assume that the original holder cannot transfer the security for the duration of the initial restriction period even though that is not absolutely correct, unless a secondary market platform offers a viable means of selling the security. Secondary markets for restricted shares transform a security that lacks marketability

into one that is marketable but lacks liquidity. Lack of marketability and lack of liquidity thus entail similar losses of resale or transfer flexibility but with different root causes.

The DLOM plays an equivalent economic role to the one Amihud and Mendelson's (1986) liquidity risk premium plays in illiquid asset pricing. Investors require a liquidity risk premium in the required return, and by implication a discount in price, to purchase illiquid securities. Their work has stimulated a large and growing literature documenting the importance of liquidity in asset pricing (Chordia, Roll, and Subrahmanyam, 2000; Amihud, 2002; Pastor and Stambaugh, 2003; Acharya and Pedersen, 2005; Liu, 2006; Sadka, 2006; and Bekaert, Harvey, and Lundblad, 2007). It has also spawned research devoted to developing methodologies for measuring this premium and investigating the factors that affect its size and variability (Chalmers and Kadlec, 1998; Chen, Lesmond, and Wei, 2007; Koziol and Sauerbier, 2007; Goyenko, Subrahmanyam, and Ukhov, 2011; and Kempf, Korn, and Uhrig-Homburg, 2012). The model developed in this paper could also be used to estimate liquidity discounts that are due to factors other than marketability restrictions.

The DLOM option formulation is more challenging when there is no legal or contractual restriction on the holder's ability to sell or transfer the asset because the length of the restriction period is less clear.<sup>7</sup> For example, the market for an asset may be poorly developed, making it difficult, time-consuming, and therefore expensive to find a buyer for it, but the asset is still marketable. The restrictions are financial, rather than legal or contractual, and there is no expiration date. Applying a put-option-based DLOM model requires more judgment in that case, in particular, to estimate how long the pseudo-restriction period should be expected to last.

Likewise, the expected length of the restriction period when a privately held firm's unregistered common shares are being valued must be estimated because the lack of registration imposes a marketability restriction of uncertain length. If the private firm's business is sufficiently well developed, then it may be reasonable to estimate the amount of time that is likely to elapse before an initial public offering (IPO) or a change-of-control transaction might occur, or a probability distribution for the length of that period might be assumed. I generalize my DLOM model to restriction periods of uncertain length by assuming that the length of the restriction period is exponentially distributed.

The rest of the paper is organized as follows. Section I generalizes the average-strike put option DLOM model to restriction periods of any length  $L$  by modeling the  $L$ -year DLOM as the value of the one-year DLOM compounded over  $L$  years. Section II extends the multi-period

DLOM model to allow for a DLOM term premium or discount and to accommodate restriction periods of uncertain length. Section III concludes.

### **I. Generalized Average-Strike Put Option DLOM Model**

Finnerty (2012) derives the average-strike put option DLOM model by employing a lognormal approximation, which results in an effective cap on the volatility and an effective cap on the discount. Exhibit 1 in Finnerty (2012) reveals that the lognormal approximation can become problematic as the length of the restriction period increases beyond about four years especially for high-volatility stocks. For example, the discount decreases when the dividend yield is  $q = 0.02$  and the restriction period is between  $T = 4$  and  $T = 5$ . The model is best suited to calculating discounts for restriction periods up to two years in length, which corresponds to the longest Rule 144 restriction period in the sample of discounts Finnerty (2012) tested. Exhibit 6 in Finnerty (2012) indicates that the model is more accurate for 1-year than for 2-year restriction periods. I will build on this feature to extend the average-strike put option model to accommodate longer restriction periods. Such an extension is important because there are numerous forms of selling or liquidity restrictions that can prevent a shareholder from freely selling her shares for a multi-year period. Many are designed to resolve moral hazard and adverse selection problems (Kahl, Liu, and Longstaff, 2003).

Selling and liquidity restrictions can inhibit share sales by corporate officers, directors, entrepreneurs, private equity investors, venture capitalists, and certain other classes of shareholders for periods that sometimes extend over several years. Such resale restrictions have caused stockholders to suffer large losses during severe market downturns (Kahl, Liu, and Longstaff, 2003). Executive restricted stock plans typically prohibit share sales for between three- and five-year vesting periods (Simon, 2016). Rule 144 places resale restrictions on shareholders of public firms who hold unregistered shares and on affiliates who hold control shares (SEC, 2016). The SEC has progressively relaxed the restrictions and shortened the restriction periods on Rule 144 sales over the past 20 years making them less problematical but they are still economically significant because of the DLOM involved. Private equity investors and venture capitalists hold unregistered shares which they cannot sell until they either register the stock for an initial public offering or sell them to another private investor or in a change-of-control transaction. Private equity investors and venture capitalists typically hold an investment for between five and seven years, and investors in a private equity fund typically hold their fund investment for 10 years (Metrick and Yasuda, 2010). Entrepreneurs may hold their equity

investments for even longer periods.

Even some contractual restrictions that are usually designed for relatively short periods can in some cases restrict share sales for much longer periods. For example, initial public offering (IPO) lock-up periods typically extend for 180 days from the date of the IPO but the issuer and its underwriters sometimes agree to a much longer lock-up period (Kahl, Liu, and Longstaff, 2003). Merger agreements typically impose multi-year lock-up periods on the selling firm's corporate officers and key employees to discourage them from leaving the target firm and taking with them their valuable human capital, which may have been the main motivation behind the merger (Kahl, Liu, and Longstaff, 2003). Lastly, Rule 144 restrictions are generally much more onerous for 'affiliates,' who include corporate executives, directors, and 5% or greater shareholders who exercise some degree of control by having the power to direct the management and policies of the firm, than other shareholders (SEC, 2016). An affiliate cannot sell her shares for at least six months after acquiring her stock.<sup>8</sup> Thereafter, the number of shares she can sell during any three-month period is limited to one percent of the firm's outstanding common stock or, if the shares are listed on a stock exchange, the greater of one percent of the firm's outstanding common stock and the stock's reported average weekly trading volume during the four weeks preceding the sale. Consequently, for smaller firms with less actively traded shares, it may take several years before an affiliate can sell her entire shareholding. In general, for privately held firms, resale limitations restrict share sales until an IPO, a change-of-control transaction, or some other liquidity event occurs, and the timing of such an event, and even whether it will ever occur, are uncertain.

This section generalizes the average-strike put option DLOM model to restriction periods of any length  $L$  by modeling the  $L$ -year DLOM as the value of the one-year DLOM compounded over  $L$  years. I begin with a discussion of the economic rationale for using the value of an average-strike put option to measure the one-year DLOM.<sup>9</sup>

#### **A. Average-Strike Put Option DLOM Model for One-Year Restriction Period**

I assume the firm's unrestricted shares trade continuously in a frictionless market. The firm also has restricted shares outstanding, and all the firm's shares are identical except for the transfer restrictions. Transfer restrictions prevent the investor from selling the restricted shares for a period of length  $T$ . Any cash dividends are paid continuously during the time interval  $[0, T]$  at a continuously compounded rate  $q \geq 0$  that is proportional to  $V$ .  $V(t)$  is the value of a share of common stock without transfer restrictions. I assume  $V(t)$  follows a geometric diffusion process

of the form

$$dV = (\mu - q)Vdt + \sigma VdZ \quad (1)$$

where  $\mu$  and  $\sigma$  are constants that measure the continuously compounded mean return and volatility, respectively, and  $Z$  is a standard Wiener process. The continuously compounded riskless interest rate  $r$  is constant and the same for all maturities during  $[0, T]$ . No shareholder has any special market-timing ability. In the absence of resale restrictions, investors would be equally likely to sell the shares anytime during the restriction period. I assume the shareholder can sell the registered shares at  $t$ ,  $0 < t < T$ , and reinvest the proceeds in the riskless asset until  $T$  and that the investor would want to sell the unregistered shares prior to  $T$  were it not for the resale restrictions.

Suppose the shareholder can sell the registered shares at  $t$ ,  $0 < t < T$ , and reinvest the proceeds in the riskless asset until  $T$ . In a risk-neutral world, the investor would be indifferent between selling the share immediately for  $V(t)$  and selling it forward for delivery at  $T$  with forward price  $e^{(r-q)(T-t)}V(t)$ . Suppose further that the investor would want to sell the unregistered shares prior to  $T$  were it not for the resale restrictions. Since the investor lacks any special timing ability, assume that the investor would be equally likely to sell unrestricted shares at  $N + 1$  discrete points in time and that these points are equally spaced, so that the investor considers selling at  $t = 0, t = T/N, t = 2T/N, \dots, t = NT/N = T$ . In a risk-neutral world, such an investor would be indifferent between holding a registered share and holding an unregistered share plus a series of forward contracts all expiring at  $T$ . If the investor's transfer restriction risk exposure could be perfectly hedged, or if this risk is idiosyncratic with respect to the investor's securities portfolio, then the unregistered shares would be priced on a risk-neutral basis.

While price risk is hedgeable, liquidity risk is not. Because of the restrictions on hedging, the investor bears an opportunity cost due to the transfer restrictions if

$$\frac{I}{N+1} \sum_{j=0}^N [e^{(r-q)T(N-j)/N} V(jT/N)] > V(T) \quad (2)$$

but realizes an opportunity gain if the inequality is reversed. Inequality (2) suggests that the DLOM should be zero if the value of the investor's potential for economic gain exactly offsets the cost of the investor's potential for economic loss during the restriction period  $T$ . In particular, if (a) the stock's return distribution is symmetrical, (b) the investor is risk neutral or can costlessly fully hedge her risk exposure from holding the stock, and (c) she has adequate



liquidity from other sources such that the transfer restrictions do not cause her to miss any positive-NPV investment opportunities or to bear any other illiquidity costs, then the value of the potential upside and the cost of the potential downside offset, and the investor would not require a DLOM to purchase the restricted share. In effect, the restricted share would be equivalent to a long forward contract for the delivery at  $T$  of an otherwise identical unrestricted share when the transfer restrictions expire.

But actual common stock return distributions exhibit long-term negative skewness and fat tails, investors are generally risk averse, and liquidity risk cannot be hedged, and so inequality (2) should generally hold. Engle (2004) shows that long-term negative skewness is a consequence of asymmetric volatility; negative returns lead to higher volatility than comparable positive returns with the result that potential market declines are more extreme than possible market increases.<sup>10</sup> Skewness is negative for every horizon and increasingly negative for longer horizons. Harvey and Siddique (2000) furnish empirical evidence that U.S. equity returns for stocks in CRSP (NYSE, AMEX, and NASDAQ) are systematically negatively skewed, this skewness is economically important and commands an average risk premium of 3.60%, and the returns of the smallest stocks by market capitalization exhibit the most pronounced negative skewness based on stock return data for the period 1963 to 1993. Ze-To (2008) furnishes empirical evidence that the S&P 500 Index returns were negatively skewed and heavily tailed during the 1991 to 2005 period.<sup>11</sup> Bali (2007) and Bali and Theodossiou (2007) document the negatively skewed fat-tailed distributions of the returns on U.S. stocks as proxied by the Dow Jones Industrial Average for the period 1896-2000 and by the S&P 500 Index for the period 1950 to 2000, respectively. Cotter (2004) documents the negatively skewed fat-tailed distributions of European equity index returns. Given the economically significant negative skewness and fat tails that characterize common stock return distributions, it is reasonable to expect a risk averse investor to require a DLOM to compensate for the incremental cost of having to bear the excess of the cost of potential extreme downside loss over the value of potential extreme upside gain during the restriction period, except in the special case where he is committed to holding the stock for the entire restriction period and doing so does not impose any potential liquidity cost.

Finnerty (2012) models the discount for lack of marketability as the value of an average-strike put option for which the expected strike price is the arithmetic average of the risk-neutral forward prices. This average-strike put option can be characterized as the option to exchange a package of forward contracts on a share for the underlying share and evaluated as the value of an

option to exchange one asset for another. The option payoff function, equation (A5) in Appendix A, contains the sum of a set of correlated lognormal random variables. Its probability distribution can be approximated as the probability distribution for a lognormal random variable using Wilkinson's method. Finnerty (2012) derives the moment-generating function for the bivariate normal distribution for the average of the risk-neutral forward prices and the price of the underlying unrestricted share and then applies Hull's (2009) generalization of Margrabe's (1978) expression for the value of the option to exchange one asset for another when the stock is dividend-paying to obtain the following formula for the value of the marketability discount  $D(T)$ :

$$D(T) = V_0 e^{-qT} \left[ N\left(\frac{v\sqrt{T}}{2}\right) - N\left(-\frac{v\sqrt{T}}{2}\right) \right] \quad (3)$$

$$v\sqrt{T} = \left[ \sigma^2 T + \ln\left[2\left\{e^{\sigma^2 T} - \sigma^2 T - 1\right\}\right] - 2 \ln\left[e^{\sigma^2 T} - 1\right] \right]^{1/2} \quad (4)$$

where  $N(\cdot)$  is the cumulative standard normal distribution function.

$D(T)$  given by equations (3) - (4) is:

- Proportional to the current share price  $V_0$ .
- Directly related to the stock's price volatility  $\sigma$ .
- Directly related to the length of the transfer restriction period  $T$ .
- Inversely related to the stock's dividend yield  $q$ .
- Independent of the riskless interest rate  $r$ .

The volatility  $v$  in equation (4) depends directly on the volatility  $\sigma$  of the underlying unrestricted share of stock. Margrabe's (1978) expression for the value of the option to exchange one asset for another corresponds in the case of equations (3) and (4) to the exchange of a set of forward contracts to deliver a restricted share for the unrestricted share where the forward contracts have the same underlying unrestricted share. Accordingly, the volatility  $v$  in equation (4) is the volatility of the ratio of the average forward contract price to the price of the underlying unrestricted share. The volatility of this ratio is less than  $\sigma$  because the two components to the exchange have the same underlying share, and changes in the value of the unrestricted share being given up and changes in the value of restricted share being received in the assumed exchange pull the value of the ratio in opposite directions, making the value of the ratio less volatile than the price of the underlying share. Table 2 illustrates the relationship between  $v$  and  $\sigma$  for a range of assumed parameter values.

The logarithmic approximation utilized in deriving equations (3) - (4) results in an

important limitation on the model's usefulness in practice. As shown in Appendix B, the volatility parameter  $v^2$  behaves like  $\ln 2/T$  for large  $T$ , which approaches zero as  $T$  becomes very large. The average-strike put option DLOM model has a 32.28% effective upper bound on the percentage discount when  $T$  is large. The model may not generate reliable marketability discounts for the stock of a private firm when the liquidity event may not occur for several years or for valuing restricted shares when the specified restriction period extends for several years.

Set  $T = 1$  in equations (3) - (4) to obtain a one-period marketability discount formula:

$$D(1) = V_0 e^{-q} \left[ N\left(\frac{v}{2}\right) - N\left(-\frac{v}{2}\right) \right] \quad (5)$$

$$v = \left[ \sigma^2 + \ln \left[ 2 \left\{ e^{\sigma^2} - \sigma^2 - 1 \right\} \right] - 2 \ln \left[ e^{\sigma^2} - 1 \right] \right]^{1/2} \quad (6)$$

Next, I will use equations (5) - (6) to generalize the average-strike put option DLOM model to obtain a multi-period model that avoids the DLOM cap inherent in equations (3) - (4).

## B. Generalization to Longer Restriction Periods

Table 1 indicates that the marketability discounts calculated from the average-strike put option model are more consistent with empirical private placement discounts for 1-year than for 2-year (or presumably longer) restriction periods after adjusting the observed private placement discounts for the equity ownership concentration, information, and overvaluation effects that typically accompany common stock private placements. I build on this feature in this section to extend the average-strike put option model to accommodate longer restriction periods. I generalize the model by treating the  $L$ -restriction period marketability discount as the compounded value of  $L$  single-restriction-period marketability discounts for arbitrary  $L$ . I further extend the model to situations where  $L$  is uncertain and exponentially distributed in the next section.

I assume that the appropriate marketability discount for a restriction period of arbitrary length  $L$  can be expressed as the value of the one-period marketability discount  $\Delta$  compounded forward  $L$  periods when the market for restricted common stock is in equilibrium. Since the investor's ability to resell the stock is assumed to be restricted, investors are not able to arbitrage any mispricing of the discount by trading their restricted stock in the market.<sup>12</sup> Instead, this characterization of market equilibrium rests on the following assumptions. I assume that there is sufficient competition among prospective issuers and investors in the new issue market for restricted common stock to ensure that when this market is in equilibrium, the DLOMs on new

issues have the property that the DLOM for an  $L$ -year restriction period is equal to the cumulative compound DLOM from making a sequence of  $L$  one-year investments in the same restricted stock each at a one-year DLOM  $\Delta$ . I assume that there is no market segmentation or preferred habitat in the restricted stock new issue market; the market is complete in the sense that there are sufficient issuers and investors in the market for restricted stock for each restriction period of length  $L$  such that if this equilibrium condition is violated, there will always be sufficient issuers available to sell and sufficient investors available to purchase restricted stock with any length restriction period to restore market equilibrium.<sup>13</sup> I also assume that at the margin, issuers in the restricted stock market will not accept a term premium, and investors will not accept a term discount. Neither issuers nor investors face liquidity constraints that would make the marginal issuer or marginal investor amenable to a term premium or term discount, respectively. I relax this assumption in the next section to allow for a term premium or discount.

The percentage marketability discount is  $P = D(I)/V_0$  where  $D(I)$  is given by equation (5). The continuously compounded percentage marketability discount per year  $\Delta$  satisfies the equation

$$\Delta = -\ln[1 - P] = -\ln\left\{1 - e^{-q} \left[ N\left(\frac{v}{2}\right) - N\left(-\frac{v}{2}\right) \right]\right\} \quad (7)$$

One might think of the percentage discount per year for restriction periods of different lengths as forming a DLOM term structure. A DLOM term structure, or term structure of marketability restriction risk, is a natural application of Guidolin and Timmermann's (2006) term structure of risk. As Engle (2011) notes, risk measures are computed for horizons of varying length extending from one day to many years. Risk measures take into account that the losses that might result from extreme moves in the price of the underlying asset take time to unfold, which generally leads to long-term risks exceeding short-term risks. This phenomenon suggests that it is reasonable to expect liquidity risk to increase with the length of the marketability restriction period; by implication, the DLOM should also increase as the length of the restriction period increases.

As Engle (2011) also notes, the term structure of risk is analogous to the term structure of interest rates and the term structure of volatility (Cox, Ingersoll, and Ross, 1981; Fama, 1984, 1990; and Fabozzi, 2016). Given its prominence in finance, I will draw on the analogy to the term structure of interest rates in describing the DLOM term structure. If the percentage discount per year is the same for every length restriction period, then the DLOM term structure

implied by equation (7) is flat and equal to  $\Delta$ . Just as the interest rate term structure exhibits a variety of shapes (Fabozzi, 2016), the term structure of illiquidity premia can be upward-sloping, downward-sloping, or hump-shaped (Chen, Lesmond, and Wei, 2007). I generalize the DLOM model to allow for these shapes in the next section.

Compounding  $\Delta$  forward  $L$  periods leads to the following formula for the DLOM:

$$D^*(L) = 1 - e^{-\Delta L} = 1 - \left[1 - e^{-q} \left\{N\left(\frac{v}{2}\right) - N\left(-\frac{v}{2}\right)\right\}\right]^L \quad (8)$$

where  $v$  is given by equation (6). Note that  $D^*(0) = 0$  and  $\lim_{L \rightarrow \infty} D^*(L) = 1$ . As one would expect, the DLOM is zero when there are no marketability restrictions, and it approaches 100% in the limit as the length of the restriction period increases without bound. For the special case of a non-dividend-paying stock,  $q = 0$ , equation (8) simplifies to

$$D^*(L) = 1 - \left[2N\left(-\frac{v}{2}\right)\right]^L \quad (9)$$

Equation (8) provides the percentage discount, the loss of fair market value per dollar of freely traded asset value, attributable to the  $L$ -period transfer restrictions. It offers another way of characterizing the term structure of illiquidity premia by expressing these premia as percentage price discounts that vary as a function of  $L$ . Figure 1 illustrates the behavior of  $D^*(L)$  for different values of the stock price volatility  $\sigma$ .  $D^*(L)$  is a concave monotonically increasing function of  $L$ . For a restriction period of any particular length, the DLOM is larger for greater  $v$ , the volatility of the ratio of the stock price to the average futures price of the stock. One might interpret each  $D^*(L)$  curve in Figure 1 as the term structure of the DLOM for an issuer's restricted stock given the volatility  $\sigma$  of the issuer's unrestricted stock. When  $v = 0$ ,  $D^*(L) = 0$ ; the DLOM equals zero regardless of the length of the restriction period when the volatility of the price ratio is zero. As  $v$  increases, the DLOM increases and reaches the following maximum for given  $L$ , which is illustrated in Figure 1:

$$\lim_{v \rightarrow \infty} D^*(L) = 1 - [1 - e^{-q}]^L \quad (10)$$

$D^*(L)$  would approach, but not exceed, 100% in the limit as  $v$  becomes infinite because the dividend yield  $q$  is always a small nonnegative quantity.<sup>14</sup>

Figure 2 compares the DLOM obtained from the Finnerty [2012] average-strike put option model (3)-(4) with the DLOM obtained from the generalized model (8) with  $v$  given by equation (6). The two models give similar DLOM values when the restriction period is two years or less and the stock volatility is 30% or less. The two models agree on the DLOM at  $L = 1$  by design. For restriction periods less (greater) than one year, the Finnerty (2012) model DLOM

is greater (less) than the compound model DLOM. The Finnerty (2012) model exhibits greater concavity, which causes the DLOMs from the two models to diverge more widely for any given restriction period  $L$  as  $\sigma$  increases. The DLOMs implied by equation (8) exhibit a nearly linear relationship to  $L$ , which is a direct consequence of the flat term structure assumed for  $\Delta$ . It seems more consistent with how one would normally expect actual DLOMs to behave than the more extreme concavity of  $D(T)$ . The true behavior is, of course, an empirical question.

## II. Further Extensions of the DLOM Model

This section further extends the average-strike put option DLOM model first, to allow for a term premium to reflect a risk averse investor's more prolonged exposure to the risk of an increasingly negatively skewed fat-tailed return distribution or to idiosyncratic investment-specific agency costs, or term discount to reflect any idiosyncratic special issuer value added due to superior fund manager investment skill or strategic equity investment value, and second, to accommodate restriction periods of uncertain length under the assumption that the length of the restriction period is exponentially distributed.

### A. DLOM Term Premium

If the percentage discount  $\Delta$  is independent of  $L$ , as in equation (7), then the term structure of percentage marketability discounts is flat. Building on the analogy between the interest rate term structure and the DLOM term structure, various theories have been proposed to explain the various observed shapes of the interest rate term structure (Fabozzi, 2016). Under a pure expectations theory, equation (8) implies that market participants expect the one-year DLOM given by equation (7) to remain the same each year for  $L$  years. It is most consistent with either a broad formulation of the pure expectations theory (Lutz, 1940-1941) or a local expectations theory formulation of a pure expectations theory of the term structure (Cox, Ingersoll, and Ross, 1981, pp. 774-775).<sup>15</sup> However, the DLOM term structure might be more steeply upward-sloping if risk averse restricted stock investors require a liquidity term premium to compensate for the greater risk of an increasingly negatively skewed fat-tailed stock return distribution as the restriction period lengthens (Engle, 2009, 2011) and the greater uncertainty associated with being resale restricted for a longer period (Hicks, 1946). This term premium might not increase uniformly and might exhibit irregular shapes if restricted stock investors self-select into restriction-period preferred habitats (Modigliani and Sutch, 1966) or if legal investment restrictions or investment policy constraints effectively segment the market for

restricted stock into specific maturity sectors and prevent investors from moving from one sector to another to eliminate local supply-demand imbalances (Culbertson, 1957).

### 1. Term Structure of Illiquidity Yield Spreads

To allow for a more general DLOM term structure shape, I relax the restrictive assumption underlying equations (7) and (8) that the marginal issuers and investors in the new issue market for restricted stock do not require a term premium or discount and allow  $\Delta$  to increase (term premium) or decrease (term discount) as  $L$  increases. The finance literature describes the factors responsible for and quantifies illiquidity yield spreads, which can vary with the length of the restriction period (Amihud and Mendelson, 1986; Chalmers and Kadlec, 1998; Chen, Lesmond, and Wei, 2007; Koziol and Sauerbier, 2007; Goyenko, Subrahmanyam, and Ukhov, 2011; and Kempf, Korn, and Uhrig-Homburg, 2012). The illiquidity yield spread literature suggests that there is a term structure of illiquidity yield spreads, which can be upward-sloping, downward-sloping, or a hump-shaped function of maturity like the interest rate term structure (Chen, Lesmond, and Wei, 2007), that the illiquidity spread tends to increase with the volatility of security returns (Koziol and Sauerbier, 2007), and that the illiquidity spread, like the credit spread, is wider for lower-rated bonds (Chen, Lesmond, and Wei, 2007).

Koziol and Sauerbier (2007) apply Longstaff's (1995) lookback put option DLOM model and estimate a term structure for illiquidity spreads that is hump-shaped. Chen, Lesmond, and Wei (2007) empirically estimate illiquidity spreads for investment-grade bonds within three maturity ranges: between 8 and 35 basis points for short maturity (1-7 years), between 24 and 70 basis points for medium maturity (7-15 years), and between 59 and 67 basis points for long maturity (15-40 years) and for non-investment-grade bonds within the same maturity ranges: between 2.01% and 9.33% per year for short maturity (1-7 years), between 2.59% and 9.42% per year for medium maturity (7-15 years), and between 2.52% and 10.23% for long maturity (15-40 years). Non-investment-grade bond illiquidity yield spreads are more indicative of equity investment illiquidity term premia than investment-grade bond illiquidity yield spreads due to the greater overall riskiness of non-investment-grade bonds. Goyenko, Subrahmanyam, and Ukhov (2011), who focus on the Treasury bond market and the illiquidity premia reflected in the yields of the off-the-run issues, find that the term structure of illiquidity premia steepens during recessions and flattens during expansions. Kempf, Korn, and Uhrig-Homburg (2012) find substantial variation in the level and shape of the illiquidity premium term structure over the economic cycle.

The ranges of illiquidity term premia implied by the illiquidity yield spreads in Chen, Lesmond, and Wei (2007) are from -1.16% to +0.58% per year for the term premium between short maturity (1-7 years) and medium maturity (7-15 years) restriction periods and from -0.58% to +0.81% per year for the term premium between medium maturity and long maturity (15-40 years) restriction periods. The ranges of illiquidity term premia for equity securities are likely to be wider than these ranges because equity securities are riskier than bonds.

## 2. Economic Factors Mitigating the Term Premium

There are economic factors that can mitigate or even offset the illiquidity term premium. For example, certain restricted securities might be expected to provide superior risk-adjusted returns that cannot be duplicated in the public securities market or elsewhere in the private market. Investors should be willing to pay a premium price (accept a smaller liquidity spread) for such securities reflecting their scarcity value. There is empirical evidence that at least some illiquid investments, such as hedge fund or private equity (PE) fund ownership interests, can produce positive portfolio  $\alpha$  when fund managers are highly skilled at selecting and managing investments (Hasanhodzic and Lo, 2007; Kosowski, Naik, and Teo, 2007; Metrick and Yasuda, 2010; Fan, Fleming, and Warren, 2013; and Harris, Jenkinson, and Kaplan, 2014). Harris, Jenkinson, and Kaplan (2014) find that PE buyout funds consistently outperformed the public equity market on average by more than 3% annually net of fees and carried interest in the 1980s, 1990s, and 2000s.<sup>16</sup> They also find that PE venture capital (VC) funds outperformed the public equity market until the late 1990s; however, they have underperformed since by about 1% per year after fees and carried interest. Fan, Fleming, and Warren (2013) find that PE buyout funds generated positive  $\alpha$  between 5% and 6% per year net of fees and carried interest between 1993 and 2011, whereas PE VC funds underperformed the public equity market by 4% per year after fees on average between 1999 and 2011. Studies of hedge fund returns furnish similar empirical evidence of positive average  $\alpha$ . Hasanhodzic and Lo (2007) find 6.89% average hedge fund manager-specific  $\alpha$  net of fees and carried interest between 1986 and 2005 and average  $\alpha$  for certain investing styles as high as 15.97%. Kosowski, Naik, and Teo (2007) find that average hedge fund  $\alpha$  varied by decile from 2.40% to 8.21% net of fees and carried interest during the period 1990-2002.

Such superior investment skill could result in term discounts because investors should be willing to pay for supernormal investment rates of return, in effect, by being willing to pay (higher) prices that reflect smaller DLOMs. Manager-specific value added due to superior



investment skill (Metrick and Yasuda, 2010) or any special investment-specific value added due to strategic equity investment value (Allen and Phillips, 2000) could lead to a term discount. However, Phalippou and Gottschalg (2009) report that the PE fund fee structure typically absorbs about 6% per year of the value invested, which suggests that  $\alpha$  could be negative for some PE funds due to high fees as well as poor investment returns. This possibility would suggest that investment risk and the potential for negative returns net of fees could instead lead to a greater term premium instead of a smaller one after all.

In addition, restricted stock investments by their very nature expose equity investors to heightened agency risk because the controlling shareholders or fund managers could take actions that benefit themselves at the expense of the investors, whose ability to exit the investment may be severely limited by the security's resale restrictions. Investment-specific agency costs coupled with longer exposure to business and other risks due to investment lock-ups or illiquidity could lead to a term premium. In combination, these factors could cause actual marketability discounts to be greater than (term premium) or less than (term discount) the DLOM term structure implied by equations (7) and (8).

### 3. DLOM Model Term Premium or Discount

Denote a possible duration-specific illiquidity term premium (or discount, if negative)  $\tau$  and a possible investment-specific idiosyncratic additional term premium (or discount, if negative)  $\varepsilon$ . In keeping with the illiquidity yield spread literature, interpret  $\tau > 0$  as indicative of private equity investor aversion to the incremental riskiness of longer-duration restriction periods (including more prolonged exposure to the risk of an increasingly negatively skewed fat-tailed return distribution as  $L$  increases) and  $\tau < 0$  as indicative of any special benefits attributable to being able to access a particularly attractive class of longer-duration superior- $\alpha$  investments. The longer the restriction period, the longer the forced holding period, during which a greater number of value-decreasing events are possible;  $\tau > 0$  implies that investors are increasingly adverse to this negative event risk the longer the holding period, which increases the DLOM. A longer restriction period  $L$  for a stock investment means that it will have a longer duration, which increases the restricted stock's sensitivity to changes in stock prices generally, to the expected increase in downside investment risk (due to the longer restriction period's greater exposure to the risk of increasingly negative skewness), or to shifts in the equity risk premium or changes in the riskless rate (for example, as reflected in CAPM).

On the other hand, a willingness to accept a longer investment holding period might

create opportunities to invest in special situations that require longer holding periods but provide superior risk-adjusted returns (independent of idiosyncratic manager-specific skill or investment-specific agency risks, which are captured in  $\varepsilon$ ), which can decrease the DLOM. For example, investment opportunities that require time to accumulate assets, such as real estate, natural resource deposits, or agricultural land, and then additional time to manage the asset portfolio to optimize its value, could be associated with longer duration and superior returns that go hand in hand. The length of the restriction period  $L$  could interact with illiquidity risk to magnify the impact on the DLOM and increase the term premium (or discount)  $\tau$  as  $L$  increases.

In addition, investment managers and/or firm management can impart special idiosyncratic fund-manager-specific or special strategic benefits or expose investors to idiosyncratic investment-specific agency costs (independent of the benefits or costs attributable to investing in any particular class of longer-duration higher-yielding investments, which are captured in  $\tau$ ), which can affect the DLOM.  $\varepsilon$  increases due to increased exposure to any non-diversifiable idiosyncratic agency costs associated with the investment and decreases due to any idiosyncratic incremental value attributable to the special investment skills of the fund managers net of any economic rents they charge for their services or to any special strategic value associated with the investment. For example, if a firm has been the subject of some legal action, such as a lawsuit for securities fraud or antitrust violations, or has been disciplined for some regulatory infraction, such as a regulatory enforcement action for misstating its earnings, then the heightened agency risk will not only increase the firm's cost of capital (Hribar and Jenkins, 2004; Karpoff, Lee, and Martin, 2008; Murphy, Shrieves, and Tibbs, 2009; and Kravet and Shevlin, 2010) but will also likely increase the cost of any marketability restrictions, and  $\varepsilon$  would tend to be positive. The magnitude of  $\varepsilon$  would depend on the nature and scope of the infractions and could be even greater if some of the firm's key managers were found to be involved and were disciplined as a result (Karpoff, Lee, and Martin, 2008). The legal or regulatory difficulties suggest higher risk, and a longer restriction period would potentially expose investors to greater risk of repeat misbehavior.

On the other hand, suppose a private equity firm or a hedge fund is expected to generate a supernormal rate of return, based on its past investment performance, and that due to its unique proprietary investment strategy, this investment opportunity has no close substitutes. In that case,  $\varepsilon$  would tend to be negative because the private equity investment would provide special idiosyncratic private benefits that mitigate the cost of the marketability restriction. These special

benefits might include the opportunity to share in the fund manager's economic rent owing to her special investment skill that is expected to produce supernormal returns or to share in the firm's valuable option to participate in future joint business opportunities if it makes a strategic equity investment in another firm.

When a restricted equity investment exposes investors to heightened non-diversifiable idiosyncratic agency costs or investment-specific risks that outweigh the benefits expected due to superior investment skill or special strategic value,  $\varepsilon > 0$ , and the DLOM increases. When the idiosyncratic value added is expected to exceed the idiosyncratic agency costs,  $\varepsilon < 0$ , and the DLOM becomes smaller. The length of the restriction period  $L$  could interact with the investment-specific idiosyncratic risk to magnify the impact on the DLOM and increase the term premium (or discount)  $\varepsilon$  as  $L$  increases.

#### 4. Baseline Values for $\tau$ and $\varepsilon$

It is important to avoid double counting the valuation impact of those factors that directly determine the required rate of return but can also affect the DLOM. The general DLOM model developed in this paper allows an investor to take into account the possible interaction between an asset's lack of marketability on the one hand and the economic impact of restrictions specific to the asset class, idiosyncratic sources of value, or idiosyncratic risk factors on the other. For example, given a restriction period of any particular length, the DLOM is likely to be lower for an asset like a real estate investment that customarily requires such a long restriction period to optimize the value of the investment or for an investment managed by someone with superior skill who requires a long lock-up period as a condition for accepting the investment than would apply to an asset with close substitutes but without such restrictions. Likewise, the DLOM is likely to be greater when there are economically significant agency costs because the agency risk is potentially more costly to investors the longer the restriction period due to the more prolonged exposure to these costs. Values should be chosen for  $\tau$  and  $\varepsilon$  that capture the incremental effect of the interaction. The choice of values appropriate for any particular asset is an interesting research question, which will require empirical investigation. However, the empirical finance literature does suggest some general baseline ranges for  $\tau$  and  $\varepsilon$ .

The illiquidity yield spread literature provides a baseline for choosing a value for  $\tau$  when a DLOM term premium is appropriate. For example, using Chen, Lesmond, and Wei's (2007) estimated illiquidity premia for non-investment-grade bonds as a baseline, a reasonable range for  $\tau$  between short maturity (1-7 years) and medium maturity (7-15 years) restriction periods would

be between -1.16% and +0.58% per year, and a reasonable range for  $\tau$  between medium maturity and long maturity (15-40 years) restriction periods would be between -0.58% and +0.81% per year. However, the value of  $\tau$  could be lower (more negative), implying a smaller DLOM, if the investment gives investors access to a particular class of longer-duration restricted equity investments that generally provide positive  $\alpha$ . Koziol and Sauerbier's (2007) put-option-based model or Kempf, Korn, and Uhrig-Homburg's (2012) illiquidity term premium model could be applied to estimate a value for  $\tau$ .

The alternative investments literature suggests a way of estimating a reasonable value for  $\varepsilon$  to take into account any idiosyncratic superior manager-specific investment skill.<sup>17</sup> For example, the empirical evidence concerning the portfolio  $\alpha$  hedge fund and PE fund managers can earn suggests a baseline for choosing a value for  $\varepsilon$ . Extrapolating from Harris, Jenkinson, and Kaplan's (2014) and Fan, Fleming, and Warren's (2013) findings that average PE fund  $\alpha$  is between 3% and 6%, a reasonable value for  $\varepsilon$  for a PE fund investments would be the extent to which the specific investment's rate of return is expected to be above this range and is consequently unusually favorable (or in the opposite case, is expected to fall below the range and is therefore atypically unfavorable). Similarly, extrapolating from Kosowski, Naik, and Teo's (2007) finding that average hedge fund  $\alpha$  is between 2.40% and 8.21%, a reasonable value for  $\varepsilon$  for a hedge fund investment would again be the extent to which the rate of return is expected to be above this range (or in the opposite case, below it).

Any adjustment for investment-specific agency costs would reduce the magnitude of the term discount. Several papers have investigated the financial penalties on firms resulting from legal or regulatory infractions and documented that the cost to the firm is economically significant. For example, corporate fraud disclosure and the subsequent class action lawsuits and enforcement actions increase investors' perception of the defendant firms' systematic risk and their total risk and consequently raise the fraud firm's cost of capital (Hribar and Jenkins, 2004; Karpoff, Lee, and Martin, 2008; Murphy, Shrieves, and Tibbs, 2009; and Kravet and Shevlin, 2010). The increase in total risk for which investors require compensation would include any incremental liquidity risk following public disclosure of the adverse legal or regulatory action. If these idiosyncratic agency costs outweigh any idiosyncratic benefits attributable to the asset, then  $\varepsilon > 0$ , and the net impact would increase the DLOM.

##### 5. Term Premium-Adjusted DLOM Model

The marketability discount  $D^*(L)$  in equation (8) can be modified to incorporate a term

premium (or discount) by adjusting the continuously compounded percentage marketability discount per year  $\Delta$  by adding a duration-specific illiquidity term premium (or discount, if negative)  $\tau$  and an idiosyncratic investment-specific expected excess return premium (or discount, if negative)  $\varepsilon$ . The formula for the modified marketability discount,  $\widehat{D}(L)$ , is:

$$\widehat{D}(L) = 1 - e^{-(\Delta+\tau+\varepsilon)L} = 1 - e^{-(\tau+\varepsilon)L} \left[ 1 - e^{-q} \left\{ N\left(\frac{v}{2}\right) - N\left(-\frac{v}{2}\right) \right\} \right]^L \quad (11)$$

The expressions for  $D^*(L)$  for the special cases in equations (9) and (10) are similarly modified and the resulting formulas are interpreted similarly to equation (11).

Figure 3 illustrates how  $\widehat{D}$  varies as a function of  $\Delta$  and  $L$ .  $\widehat{D}$  is a concave increasing function of both parameters. Similarly,  $\widehat{D}$  is a concave increasing function of  $\tau$  and  $\varepsilon$ .

Equation (11) allows for the possibility that investor risk aversion could lead to greater illiquidity term premia for longer restriction periods ( $\tau > 0$ ) or investors' agency cost concerns could increase the marketability discount for a specific investment ( $\varepsilon > 0$ ), in which case  $\widehat{D}(L)$  exceeds  $D^*(L)$ . On the other hand, access to a particular class of investments offering superior investment returns but requiring a long holding period could reduce the DLOM that would normally be required given the length of the restriction period ( $\tau < 0$ ) or idiosyncratic superior manager-specific investment skill could partially offset the DLOM that would otherwise be required ( $\varepsilon < 0$ ). It is possible that superior investment management skill or any special strategic value of the equity investment could offset the effect of the investors' risk exposure to such an extent that it could not only reduce the marketability discount but possibly even more than offset the other components of the DLOM and lead to a premium in value and a term discount. This might happen if, for example, access to the restricted investment is limited and those who gain access are willing to pay the investment sponsor an economic rent to participate in the superior-return-lower-risk investment or if the private equity investment has great enough strategic corporate investment value (Allen and Phillips, 2000).

Some private equity placements take place at a premium to the stock's freely traded price. Allen and Phillips (2000) find that privately selling a block of stock to a nonfinancial firm leads to significant excess returns when the investment is coupled with a strategic product market relationship. They report that 59 percent of the strategic private placements take place at a premium stock price, strategic private buyers on average pay a 6 percent premium, and the private placement premium is comparable to the average premium paid in open market and other corporate equity block purchase transactions.

Due to the impossibility of arbitrage when trading in shares is restricted, there may be additional share-specific factors that are not captured in equation (11). Also, restricted stock investors may have preferred habitats due to their particular liquidity preferences or the liquidity characteristics of the securities in their investment portfolios, which could cause the DLOM term structure to deviate from the term structure implied by equation (11).

## **B. Restriction Period of Uncertain Length**

Equation (11) provides the DLOM when the length of the restriction period  $L$  is known, for example, due to particular contractual or legal restrictions on transfer of the security that will lapse on a date certain, or when  $L$  can be reasonably estimated and therefore can be assumed to take on a particular value  $L$  with certainty. The more challenging problem arises when  $L$  is uncertain. I further extend the average-strike put option DLOM model to handle such situations.

For example, suppose an investor wants to purchase the common stock of a privately held firm that intends to go public after the firm has developed its product line sufficiently that the IPO market would be receptive. The IPO date is uncertain. An investor might estimate an IPO date after researching similar IPOs in the past and assume that  $L$  corresponds to the date following the firm's IPO when the lockup on selling insider shares could be expected to expire.<sup>18</sup> Similarly, the venture capital backers of a young firm can plan for a liquidity event perhaps several years in the future, but the date is uncertain, and it will never occur if the firm fails before the liquidity event can take place. In either case, it might be reasonable to assume a particular probability distribution for  $L$  based on historical experience for similar firms under similar economic conditions. Nevertheless, the IPO market is notoriously fickle, and a firm's business fortunes could change for the worse before an IPO or a change-of-control transaction could take place. So even selecting a probability distribution for  $L$  might be difficult.

I assume that the date  $L$  the stock becomes unrestricted (exits the restricted period) follows an exponential distribution. Metrick (2007) models an analogous problem, the value of a venture capitalist's (VC) random-expiration call option. A VC holds a call option on the equity value of each VC portfolio company. The option expiration date is random because of the uncertainty concerning when an IPO or the sale of the portfolio company will force the expiration of the VC's call option (or whether the option will expire worthless before it ever goes into the money).

The instantaneous probability that the restriction period expires at  $L$  given that the stock is still restricted at  $L$  is  $\lambda$ , and the probability that the stock is still restricted at  $L$  is  $e^{-\lambda L}$ . The

probability the restriction period ends at  $L$  is the product of these two probabilities. The probability density function for  $L$  is  $f(L) = \lambda e^{-\lambda L}$  for  $L \geq 0$ , which is the exponential pdf. The exponential distribution covers the entire domain  $[0, \infty)$ .<sup>19</sup>

The parameter  $\lambda$  can also be interpreted as the rate at which the firm progresses toward achieving a liquidity event for its shares, such as an IPO or a change-of-control transaction. The mean restriction period is  $\bar{L} = 1/\lambda$ . Inverting this equation gives a method for estimating the parameter  $\lambda$ . If  $\bar{L}$  can be estimated from historical data for comparable restricted securities, then  $\lambda_{est} = 1/\bar{L}$ . The marketability discount  $\bar{D}$  is the exponentially weighted average of the marketability discounts  $\widehat{D}(L)$  for restriction periods of length  $L$ .

The restriction-period-weighted marketability discount is obtained by calculating the expected value of  $\widehat{D}(L)$  given by equation (11) when  $L$  is exponentially distributed with parameter  $\lambda$ :

$$\bar{D} = \int_0^{\infty} [1 - e^{-(\Delta + \tau + \varepsilon)L}] \lambda e^{-\lambda L} dL = 1 - \frac{\lambda}{\lambda + \Delta + \tau + \varepsilon} \quad (12)$$

where  $\Delta$  is given by equation (7).  $\bar{D}$  is an increasing function of  $\Delta$ ,  $\tau$ , and  $\varepsilon$  and a decreasing function of  $\lambda$ . In particular, increasing the rate  $\lambda$  at which the firm progresses toward achieving a liquidity event for its shares effectively shortens the expected length of the restriction period and reduces the DLOM.

Table 3 illustrates the range of values for the DLOM for a representative range of parameter values. Table 3 also shows the relationship between the volatility parameters  $\nu$  and  $\sigma$  according to equation (6).  $\bar{D}$  increases with the volatility of the security's returns  $\sigma$  and with the expected length of the restriction period  $\bar{L} = 1/\lambda$ , both of which are consistent with the option character of the DLOM. A 5% per year ( $\tau + \varepsilon = 0.05$ ) term premium increases the DLOM, for example, by about one-third from 19.25% to 25.29% for a two-year restriction period for a non-dividend-paying ( $q = 0$ ) stock with 50% volatility and by about one-sixth from 44.67% to 51.39% for a five-year restriction period for a stock yielding 3% ( $q = .03$ ) with 70% volatility.

Assume there is no term premium or discount and no idiosyncratic factors affecting the DLOM ( $\tau = \varepsilon = 0$ ). Figure 4 illustrates how the restriction-period-weighted marketability discount  $\bar{D}$  varies as a function of  $\lambda$  for different values of  $\sigma$ .  $\bar{D}$  is a convex function of  $\lambda$ . It is easily shown that the expression  $\ln\{ \}$  in equation (7) has a value in the range  $(-\infty, 0)$ . When  $\lambda = 0$ ,  $\bar{D} = 1$ . The restriction period is infinite and so the marketability discount is 100%, as we found previously.  $\bar{D}$  decreases as  $\lambda$  increases because the length of the probability-weighted

restriction period shortens. In the limit, the mean restriction period approaches zero and so does the marketability discount since  $\lim_{\lambda \rightarrow \infty} \bar{D} = 0$ , also as we found previously.

Figure 5 illustrates how  $\bar{D}$  varies as a function of  $\lambda$  and  $\sigma$ , and Figure 6 illustrates how  $\bar{D}$  varies as a function of  $\Delta$  and  $\bar{L}$ .  $\bar{D}$  is a convex decreasing function of  $\lambda$  and an increasing concave function of  $\Delta$ ,  $\tau$ ,  $\varepsilon$ , and  $\sigma$ . Doubling  $\bar{L}$  roughly doubles  $\bar{D}$  when  $\bar{L}$  is less than about 1.5 years and  $\sigma$  is less than about 40%. Beyond those ranges  $\bar{D}$  is a little less sensitive to increases in  $\bar{L}$  and  $\sigma$ .

### III. Conclusions

This paper generalized the average-strike put option DLOM model to restriction periods of any length  $L$ . It modeled the  $L$ -year DLOM as the value of the one-year average-strike put option DLOM compounded over  $L$  years. It extended the model to restriction periods of uncertain length by assuming the length of the restriction period is exponentially distributed. My model also allows for a DLOM term premium, to reflect a risk averse investor's more prolonged exposure to the risk of an increasingly negatively skewed fat-tailed return distribution or to idiosyncratic investment-specific agency costs, or a term discount, to reflect any special idiosyncratic issuer value added due to superior fund manager investment skill or strategic equity investment value.

Previous research has shown that the average-strike put option DLOM model calculates marketability discounts that are generally consistent with the discounts observed in letter stock private placements, which have relatively short restriction periods. Empirically testing the general DLOM model developed in this paper on a sample of private securities transactions with a wide range of restriction periods would be an area worthy of further investigation, which would likely improve our understanding of how liquidity factors affect asset pricing. A second potentially fruitful area for future research would be to embed the general DLOM model within a market equilibrium asset-pricing model. Such a model would likely lead to deeper insights regarding the nature of illiquidity premia, and more generally, a fuller understanding of the interplay of marketability restrictions (and more generally, liquidity) and the other factors relevant to asset pricing.

#### Appendix A. Derivation of the Average-Strike Put Option DLOM Model for One-Year Restriction Period

The derivation follows Finnerty (2012). The assumptions are stated in Section I. In



particular,  $V(t)$ , the value of a share of common stock without transfer restrictions follows the geometric diffusion process (1).

With any unhedged nonidiosyncratic risk exposure, a risk-averse investor would demand a risk premium, and the transferability discount is nonzero. Following the same reasoning as in Longstaff (1995), inequality (2) suggests that an upper bound on the investor's opportunity cost can be modeled as

$$\max\left\{0, \frac{1}{N+1} \sum_{j=0}^N \left[ e^{(r-q)T(N-j)/N} V(jT/N) \right] - V(T)\right\} \quad (\text{A1})$$

Expression (A1) is the payoff function for an average-strike put option in which the strike price

is the arithmetic average of the forward prices  $\frac{1}{N+1} \sum_{j=0}^N \left[ e^{(r-q)T(N-j)/N} V(jT/N) \right]$ .<sup>20</sup>

Finnerty (2012) obtains a formula for an upper bound on the value of the marketability discount by valuing the average-strike put option whose payoff is (A1). Assuming the stock price in the risk-neutral world can be described by the geometric diffusion process

$$dV = (r - q)Vdt + \sigma VdZ, \quad (\text{A2})$$

$\ln V(T)$  is normally distributed with mean  $\ln V_0 + (r - q - 1/2\sigma^2)T$  and standard deviation  $\sigma\sqrt{T}$ , where  $V_0$  is the stock price at  $t = 0$ . Similarly, the forward price  $F(t) = e^{(r-q)(T-t)}V(t)$  follows the martingale process

$$dF = \sigma FdZ \quad (\text{A3})$$

in a risk-neutral world, and  $\ln F(t)$  is normally distributed with mean  $\ln F_0 - 1/2\sigma^2 t$  and standard deviation  $\sigma\sqrt{t}$  where  $F_0 = F(0) = V_0 e^{(r-q)T}$ .

Finnerty (2012) models the discount for lack of marketability as the value of an average-strike put option for which the expected strike price is the arithmetic average of the risk-neutral forward prices:

$$\frac{1}{N+1} \sum_{j=0}^N F(t_j) \quad (\text{A4})$$

where  $t_j = jT/N$  and the forward prices follow the martingale process (A3). This average-strike put option can be characterized as the option to exchange a package of forward contracts on a share for the underlying share and evaluated as the value of an option to exchange one asset for another.

The option payoff function (A4) contains the sum of a set of correlated lognormal random variables. Its probability distribution can be approximated as the probability distribution for a lognormal random variable using Wilkinson's method, which matches the first and second moments (Finnerty, 2012).<sup>21</sup> Finnerty (2012) derives the moment-generating function for the bivariate normal distribution for the average of the risk-neutral forward prices and the price of the underlying unrestricted share and then applies Hull's (2009) generalization of Margrabe's (1978) expression for the value of the option to exchange one asset for another when the stock is dividend-paying to obtain equations (3) - (4) for the value of the marketability discount  $D(T)$ .

Set  $T = 1$  in equations (3) - (4) to obtain equations (5) - (6) for the one-period marketability discount formula.

### Appendix B. Upper Bound on the DLOM in the Average-Strike Put Option Model

Finnerty's (2012, 2013a) average-strike put option model has an effective upper bound on the percentage discount equal to 32.28% when  $T$  is large. Rewrite equation (4) as

$$\begin{aligned} v^2T &= \sigma^2T + \ln[2\{e^{\sigma^2T} - \sigma^2T - 1\}] - 2\ln[e^{\sigma^2T} - 1] \\ &= \sigma^2T + \ln 2 + \ln\{e^{\sigma^2T} - \sigma^2T - 1\} - 2\ln[e^{\sigma^2T} - 1] \\ &< \sigma^2T + \ln 2 + \ln\{e^{\sigma^2T} - 1\} - 2\ln[e^{\sigma^2T} - 1] \text{ for } T > 0 \\ &= \ln 2 + \sigma^2T - \ln[e^{\sigma^2T} - 1] \end{aligned}$$

Note that since  $\lim_{T \rightarrow \infty} \ln[e^{\sigma^2T} - 1] = \sigma^2T$  it follows that  $\lim_{T \rightarrow \infty} v^2T = \ln 2$ . Thus, for large  $T$ ,  $v^2$  behaves like  $\ln 2/T$ , which approaches zero as  $T$  becomes large.

Next rewrite equation (3) as

$$D(T) = V_0 e^{-qT} \left[ 2N\left(\frac{v\sqrt{T}}{2}\right) - 1 \right]. \quad (3)$$

For large  $T$ ,  $D(T)$  approaches  $V_0 e^{-qT} \left[ 2N\left(\frac{\sqrt{\ln 2}}{2}\right) - 1 \right] = 0.3228V_0 e^{-qT} < 0.3228V_0$ . The percentage discount cannot exceed 32.28% for large  $T$ , which is unrealistic.

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## Endnotes

<sup>1</sup> The growth of the PE industry has led to the development of secondary markets for PE fund investments and for PE fund LP units. PE fund managers have formed funds specifically to purchase the investments that other PE funds wish to cast off and the PE fund LP units that institutional investors wish to dispose of for liquidity or other reasons. BlackRock Private Opportunities Fund III, L.P. (2014) is one such example. Likewise, a secondary market has developed for shares of privately held firms that are expected to go public. Two secondary markets for such shares are Nasdaq Private Market and SharesPost.

<sup>2</sup> Nasdaq Private Market reported a secondary transaction volume of \$1.6 billion in 2015 whereas the NYSE reported annual trading volume of \$33,276 billion in 2015.

<sup>3</sup> Letter stock is not registered for resale under the Securities Act of 1933 and therefore cannot be freely traded in the public market. It must be placed privately with accredited (sophisticated) investors. Under Rule 144, the holder cannot sell the shares during a specified minimum holding period, which is measured from the issue date, except through another exempted transaction. Prior to February 1997, the minimum holding period for a stockholder who was not affiliated with the issuer was two years. The SEC shortened the Rule 144 minimum holding period in 1997 to one year (SEC, 1997) and again in December 2007 to six months (SEC, 2007). Longer restriction periods apply if the stockholder is an affiliate of the issuer or if the firm is not current with its SEC filings. After the minimum holding period, the shares can be sold in the public market without first registering them but only after the firm agrees to remove the restrictive stock legend on the share certificate, which usually requires an opinion from counsel that the sale complies with Rule 144.

<sup>4</sup> More recent noteworthy empirical studies include Hertz and Smith (1993) and Hertz et al. (2002). Collectively, these studies have documented significant discounts in private placements of unregistered common stock averaging between 13 percent and 34 percent. The more recent studies generally find smaller private placement discounts, which may reflect the fact that the early studies predated the development of organized stock option markets and the hedging opportunities they afford.

<sup>5</sup> There is also a tendency for the model to understate (overstate) the discount when the stock's volatility is less than 45 percent or greater than 75 percent (between 45 percent and 75 percent).

<sup>6</sup> There are at least two markets for the shares of private firms, both of which started in 2009. SecondMarket, based in New York and owned by NASDAQ Private Market since 2015, and SharesPost, based in Santa Monica, CA, both operate web-based auction markets for unregistered private firm shares. SecondMarket has completed more than \$2.5 billion of transactions since 2013, including \$1.6 billion in 2015. SharesPost has completed over \$1 billion of transactions since 2009. Sponsors of both markets state that they comply with U.S. securities laws, which provide exemptions from the registration requirements of the Securities Act of 1933, which limit potential purchasers to



institutional investors or accredited individual investors. Information concerning these markets is available at <https://www.secondmarket.com/> and <https://www.sharespost.com/>, respectively.

<sup>7</sup> Even with a marketability restriction, the length of the restriction period should be adjusted upward to reflect how long it is expected to take to sell all the shares after the resale restriction lapses. For example, the larger the block of shares, the longer it is likely to take to sell them, and the greater should be the length of the restriction period assumed in the DLOM calculation.

<sup>8</sup> The minimum holding period is six months so long as the firm is subject to the reporting requirements under the Securities Exchange Act of 1934 and is current with its SEC filings. If the firm is not subject to the reporting requirements under the 1934 Act or is not current with its filings, the minimum holding period is one year.

<sup>9</sup> I also briefly summarize the derivation of the average-strike put option DLOM model for the reader's convenience. The details are in Finnerty (2012).

<sup>10</sup> Guidolin and Timmermann (2006) define a term structure of risk. It has an important implication for stock return distributions: the risk of unexpected extreme stock price movements increases over longer time horizons and negative skewness consequently increases with longer time horizons (Engle, 2009, 2011).

<sup>11</sup> Ze-To (2008) finds that the returns on the Hang Seng Index and the Hang Seng China Enterprise Index were more heavily fat-tailed than the S&P 500 Index returns but that the two Hang Seng index return distributions exhibited positive skewness in contrast to the negative skewness of the S&P 500 return distribution.

<sup>12</sup> Investors can sell restricted shares subject to an exemption from registration under the Securities Act of 1933 (1933 Act) but such sales must be arranged privately, and the issuer will normally require the purchaser to sign an investment representation letter (Hicks, 1998). The issuer may also require an opinion of counsel that the sale is exempt from the 1933 Act's registration requirements. Both steps are time-consuming and involve legal fees and other transaction costs.

<sup>13</sup> Equivalently, I could assume that there are always sufficient numbers of issuers and investors in the restricted stock new issue market who are flexible enough to adjust their choice(s) of restriction period to restore market equilibrium.

<sup>14</sup> According to equation (10), when  $q = 0$ ,  $D^*(L) = 1$  in the limit for all values of  $L$  as  $v \rightarrow \infty$ .

<sup>15</sup> The local expectations formulation is the only one of the interpretations of the pure expectations theory that is consistent with a sustainable market equilibrium (Cox, Ingersoll, and Ross, 1981, pp. 774-775).

<sup>16</sup> Harris, Jenkinson, and Kaplan's (2014) results sharply contrast with Phalippou and Gottschalg's (2009) conclusion that PE fund average  $\alpha$  is -6% per year net of fees (implying  $\alpha = 0$  even before fees) but explain that the performance reported in the PE fund data base on which Phalippou and Gottschalg relied likely understates PE fund returns, especially for buyout funds, due to stale data.

<sup>17</sup> Recognize that choosing a value for  $\varepsilon$  requires careful judgment informed by truly comparable transactions and entails the risk that the parameter choice might be viewed as arbitrary. Lacking reliable transaction data or other credible empirical support, the value should be set to zero.

<sup>18</sup> A six-month post-IPO lockup period is typical.

<sup>19</sup> Since firms are not infinite-lived, a truncated exponential distribution could be assumed instead.

<sup>20</sup> Finnerty (2012) notes one other factor that needs to be considered. The transfer restrictions are costly only if the investor would sell the shares on or before  $T$  absent such restrictions. Suppose there is some likelihood  $p > 0$  that the investor would want to hold the stock past  $T$  even without resale restrictions. Again assuming that prior to  $T$ , any sale would be equally likely to occur anywhere in  $[0, T]$ , the payoff function becomes

$$\begin{aligned} & \max \left\{ 0, (1-p) \left( \frac{I}{N+1} \sum_{j=0}^N \left[ e^{(r-q)T(N-j)/N} V(jT/N) \right] - V(T) \right) \right\} \\ & = (1-p) \max \left\{ 0, \frac{I}{N+1} \sum_{j=0}^N \left[ e^{(r-q)T(N-j)/N} V(jT/N) \right] - V(T) \right\} \end{aligned}$$

In this case, the transferability discount equals  $1 - p$  times the discount calculated assuming the investor would otherwise always sell sometime prior to  $T$ .

<sup>21</sup> Ritchken, Sankarasubramanian, and Vijh (1993) show that the distribution of the average of a set of correlated lognormal stock prices or exchange rates can be approximated by a lognormal distribution with acceptable accuracy by applying Wilkinson's method. Beaulieu, Abu-Dayya, and McLane (1995) (BAM) describe four methods for analytically approximating the cumulative distribution function of a random variable that is the sum of  $n$  i.i.d. lognormal random variables and compare these approximations to a numerical simulation of the actual distribution. They do not find that any one approximation dominates the others.

**Table 1**  
**Market Test of the Accuracy of the Average-Strike Put Option Model**  
**of the Discount for Lack of Marketability**

**Panel A: Offerings Announced Prior to February 1997**  
**(Two-Year Minimum Restriction Period)**

$\sigma$ Range (%)	Number of Offerings	Mean Implied Marketability Discount (day prior)	Mean Model- Predicted Discount (T = 2)	Mean Model- Predicted Discount (T = 3)	Mean Model- Predicted Discount (T = 4)
0.0 - 29.9	4	19.47 %	7.84 %	9.54 %	10.95 %
30.0 - 44.9	0	-	-	-	-
45.0 - 59.9	4	10.51	16.97	20.15	22.55
60.0 - 74.9	9	13.82	19.18	22.03	23.96
75.0 - 89.9	10	24.15	23.16	26.52	28.64
90.0 - 104.9	2	34.97	26.10	29.07	30.64
105.0 - 120.0	1	61.51	28.40	30.73	31.70
> 120.0	6	44.10	30.88	31.95	32.20
<b>Average:</b>	<b>36</b>	<b>24.50 %</b>	<b>21.37 %</b>	<b>23.97 %</b>	<b>25.62 %</b>

**Panel B: Offerings Announced After February 1997**  
**(One-Year Minimum Restriction Period)**

$\sigma$ Range (%)	Number of Offerings	Mean Implied Marketability Discount (day prior)	Mean Model- Predicted Discount (T = 1)	Mean Model- Predicted Discount (T = 2)	Mean Model- Predicted Discount (T = 3)
0.0 - 29.9	7	12.66 %	5.22 %	6.99 %	8.12 %
30.0 - 44.9	19	18.56	7.95	10.75	12.64
45.0 - 59.9	17	15.92	11.74	16.15	19.22
60.0 - 74.9	28	19.21	14.83	19.84	23.00
75.0 - 89.9	25	21.37	17.82	23.50	26.83
90.0 - 104.9	16	21.61	20.28	26.07	29.05
105.0 - 120.0	6	24.89	22.74	28.29	30.65
> 120.0	28	29.71	27.72	31.20	32.02
<b>Average:</b>	<b>146</b>	<b>21.31 %</b>	<b>17.02 %</b>	<b>21.45 %</b>	<b>23.86 %</b>

Notes: The Model-Predicted Discount is calculated from the average-strike put option model (3) - (4), and the Implied Marketability Discount is from Exhibit 6 in Finnerty [2013]. Panel A applies to private placements of common stock that were announced prior to February 1997, and Panel B applies to private placements of common stock that were announced after February 1997. The sample includes 182 U.S. private placements between April 1, 1991 and March 8, 2007 that were priced at a discount to the price at which the unrestricted shares were trading in the market. T is the total effective restriction period after allowing for the additional time required to dispose of the shares when complying with Rule 144.

Source: Finnerty [2012], Exhibit 6.

**Table 2**  
**Relationship between the Volatility Parameters  $\nu$  and  $\sigma$  in**  
**the Average-Strike Put Option DLOM Model**

$\sigma$	$\nu$			
	$T = 0.5$	$T = 1.0$	$T = 2.0$	$T = 5.0$
<b>10%</b>	5.77%	5.77%	5.76%	5.75%
<b>20%</b>	11.53%	11.51%	11.47%	11.35%
<b>30%</b>	17.26%	17.19%	17.06%	16.67%
<b>40%</b>	22.94%	22.79%	22.47%	21.54%
<b>50%</b>	28.57%	28.26%	27.66%	25.83%
<b>60%</b>	34.12%	33.60%	32.54%	29.43%
<b>70%</b>	39.59%	38.75%	37.08%	32.26%
<b>80%</b>	44.95%	43.70%	41.22%	34.31%
<b>90%</b>	50.20%	48.42%	44.91%	35.67%
<b>100%</b>	55.31%	52.88%	48.12%	36.48%

Note: The volatility  $\nu$  is calculated from equation (6).

**Table 3**  
**Tabulated Values for  $\bar{D}$  Given by Equation (12)**

<b>Panel A: <math>\tau+\varepsilon=0</math></b>												
		<i>q = 0%</i>										
$\sigma$	$\nu$	$\lambda = 12.00$ <b>L = 0.083</b>	$\lambda = 6.00$ <b>L = 0.167</b>	$\lambda = 4.00$ <b>L = 0.250</b>	$\lambda = 2.00$ <b>L = 0.500</b>	$\lambda = 1.00$ <b>L = 1.000</b>	$\lambda = 0.67$ <b>L = 1.500</b>	$\lambda = 0.50$ <b>L = 2.000</b>	$\lambda = 0.33$ <b>L = 3.000</b>	$\lambda = 0.25$ <b>L = 4.000</b>	$\lambda = 0.20$ <b>L = 5.000</b>	$\lambda = 0.10$ <b>L = 10.000</b>
<b>10%</b>	<b>6%</b>	0.19%	0.39%	0.58%	1.15%	2.27%	3.37%	4.45%	6.53%	8.52%	10.43%	18.88%
<b>20%</b>	<b>12%</b>	0.39%	0.78%	1.16%	2.29%	4.49%	6.58%	8.59%	12.35%	15.82%	19.02%	31.96%
<b>30%</b>	<b>17%</b>	0.59%	1.17%	1.74%	3.43%	6.63%	9.62%	12.43%	17.55%	22.11%	26.19%	41.50%
<b>40%</b>	<b>23%</b>	0.79%	1.56%	2.32%	4.54%	8.68%	12.48%	15.98%	22.19%	27.55%	32.22%	48.74%
<b>50%</b>	<b>28%</b>	0.98%	1.95%	2.89%	5.63%	10.65%	15.17%	19.25%	26.34%	32.29%	37.35%	54.38%
<b>60%</b>	<b>34%</b>	1.18%	2.33%	3.46%	6.68%	12.52%	17.68%	22.26%	30.05%	36.42%	41.72%	58.88%
<b>70%</b>	<b>39%</b>	1.37%	2.70%	4.00%	7.70%	14.30%	20.01%	25.02%	33.35%	40.02%	45.48%	62.52%
<b>80%</b>	<b>44%</b>	1.56%	3.07%	4.53%	8.67%	15.96%	22.17%	27.53%	36.30%	43.17%	48.71%	65.51%
<b>90%</b>	<b>48%</b>	1.74%	3.42%	5.04%	9.60%	17.51%	24.16%	29.81%	38.91%	45.93%	51.50%	67.98%
<b>100%</b>	<b>53%</b>	1.91%	3.75%	5.52%	10.47%	18.95%	25.97%	31.87%	41.23%	48.33%	53.90%	70.05%
		<i>q = 3%</i>										
$\sigma$	$\nu$	$\lambda = 12.00$ <b>L = 0.083</b>	$\lambda = 6.00$ <b>L = 0.167</b>	$\lambda = 4.00$ <b>L = 0.250</b>	$\lambda = 2.00$ <b>L = 0.500</b>	$\lambda = 1.00$ <b>L = 1.000</b>	$\lambda = 0.67$ <b>L = 1.500</b>	$\lambda = 0.50$ <b>L = 2.000</b>	$\lambda = 0.33$ <b>L = 3.000</b>	$\lambda = 0.25$ <b>L = 4.000</b>	$\lambda = 0.20$ <b>L = 5.000</b>	$\lambda = 0.10$ <b>L = 10.000</b>
<b>10%</b>	<b>6%</b>	0.19%	0.37%	0.56%	1.12%	2.21%	3.28%	4.32%	6.35%	8.29%	10.15%	18.42%
<b>20%</b>	<b>12%</b>	0.38%	0.75%	1.13%	2.23%	4.36%	6.40%	8.35%	12.02%	15.41%	18.55%	31.30%
<b>30%</b>	<b>17%</b>	0.57%	1.13%	1.69%	3.32%	6.44%	9.35%	12.09%	17.11%	21.58%	25.59%	40.75%
<b>40%</b>	<b>23%</b>	0.76%	1.51%	2.25%	4.40%	8.44%	12.14%	15.56%	21.66%	26.93%	31.54%	47.95%
<b>50%</b>	<b>28%</b>	0.95%	1.89%	2.81%	5.46%	10.35%	14.76%	18.76%	25.73%	31.60%	36.60%	53.59%
<b>60%</b>	<b>34%</b>	1.14%	2.26%	3.35%	6.48%	12.18%	17.22%	21.71%	29.37%	35.67%	40.94%	58.10%
<b>70%</b>	<b>39%</b>	1.33%	2.62%	3.88%	7.47%	13.90%	19.50%	24.41%	32.63%	39.24%	44.67%	61.75%
<b>80%</b>	<b>44%</b>	1.51%	2.97%	4.39%	8.41%	15.52%	21.61%	26.87%	35.54%	42.36%	47.88%	64.76%
<b>90%</b>	<b>48%</b>	1.68%	3.31%	4.88%	9.31%	17.04%	23.55%	29.11%	38.12%	45.10%	50.66%	67.25%
<b>100%</b>	<b>53%</b>	1.85%	3.63%	5.35%	10.16%	18.44%	25.33%	31.14%	40.42%	47.49%	53.06%	69.33%

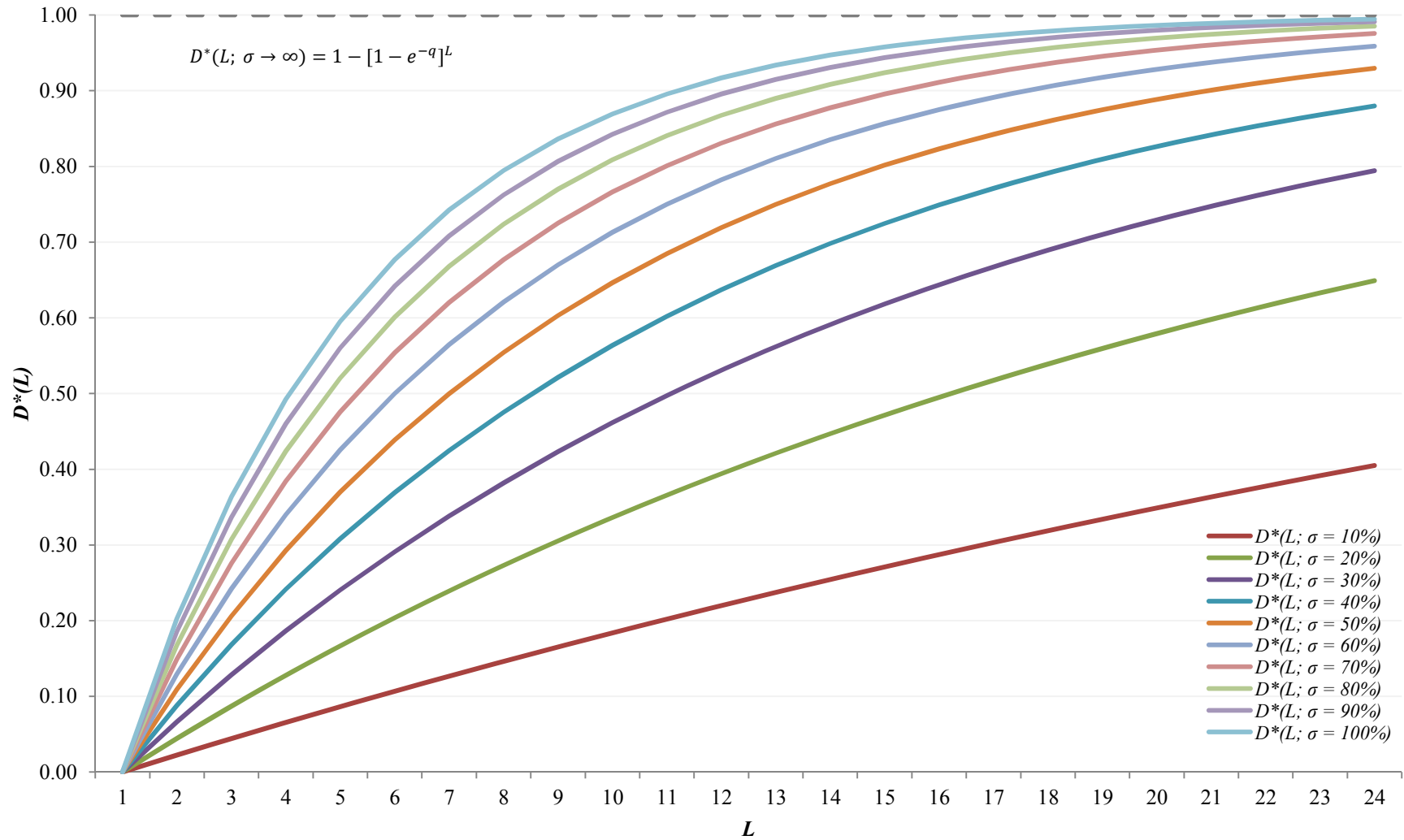
Note: The volatility  $\nu$  is calculated from equation (6).

**Table 3 - Continued**  
**Tabulated Values for  $\bar{D}$  Given by Equation (12)**

<b>Panel B: <math>\tau+\varepsilon=0.05</math></b>												
<i>q = 0%</i>												
$\sigma$	$\nu$	$\lambda = 12.00$ <b>L = 0.083</b>	$\lambda = 6.00$ <b>L = 0.167</b>	$\lambda = 4.00$ <b>L = 0.250</b>	$\lambda = 2.00$ <b>L = 0.500</b>	$\lambda = 1.00$ <b>L = 1.000</b>	$\lambda = 0.67$ <b>L = 1.500</b>	$\lambda = 0.50$ <b>L = 2.000</b>	$\lambda = 0.33$ <b>L = 3.000</b>	$\lambda = 0.25$ <b>L = 4.000</b>	$\lambda = 0.20$ <b>L = 5.000</b>	$\lambda = 0.10$ <b>L = 10.000</b>
<b>10%</b>	<b>6%</b>	0.61%	1.21%	1.80%	3.53%	6.83%	9.90%	12.78%	18.02%	22.67%	26.81%	42.29%
<b>20%</b>	<b>12%</b>	0.80%	1.59%	2.37%	4.62%	8.84%	12.70%	16.24%	22.54%	27.95%	32.65%	49.23%
<b>30%</b>	<b>17%</b>	1.00%	1.98%	2.94%	5.70%	10.79%	15.36%	19.48%	26.63%	32.61%	37.69%	54.74%
<b>40%</b>	<b>23%</b>	1.19%	2.36%	3.50%	6.76%	12.67%	17.87%	22.49%	30.33%	36.72%	42.04%	59.20%
<b>50%</b>	<b>28%</b>	1.39%	2.74%	4.06%	7.80%	14.47%	20.24%	25.29%	33.67%	40.36%	45.83%	62.85%
<b>60%</b>	<b>34%</b>	1.58%	3.12%	4.61%	8.81%	16.19%	22.47%	27.87%	36.69%	43.59%	49.13%	65.89%
<b>70%</b>	<b>39%</b>	1.77%	3.49%	5.14%	9.78%	17.82%	24.54%	30.25%	39.41%	46.44%	52.02%	68.44%
<b>80%</b>	<b>44%</b>	1.96%	3.84%	5.66%	10.71%	19.35%	26.46%	32.43%	41.85%	48.97%	54.54%	70.58%
<b>90%</b>	<b>48%</b>	2.14%	4.19%	6.15%	11.60%	20.78%	28.24%	34.41%	44.04%	51.20%	56.74%	72.40%
<b>100%</b>	<b>53%</b>	2.31%	4.52%	6.63%	12.43%	22.11%	29.86%	36.21%	45.99%	53.17%	58.66%	73.95%
<i>q = 3%</i>												
$\sigma$	$\nu$	$\lambda = 12.00$ <b>L = 0.083</b>	$\lambda = 6.00$ <b>L = 0.167</b>	$\lambda = 4.00$ <b>L = 0.250</b>	$\lambda = 2.00$ <b>L = 0.500</b>	$\lambda = 1.00$ <b>L = 1.000</b>	$\lambda = 0.67$ <b>L = 1.500</b>	$\lambda = 0.50$ <b>L = 2.000</b>	$\lambda = 0.33$ <b>L = 3.000</b>	$\lambda = 0.25$ <b>L = 4.000</b>	$\lambda = 0.20$ <b>L = 5.000</b>	$\lambda = 0.10$ <b>L = 10.000</b>
<b>10%</b>	<b>6%</b>	0.60%	1.20%	1.78%	3.50%	6.77%	9.82%	12.68%	17.88%	22.50%	26.63%	42.06%
<b>20%</b>	<b>12%</b>	0.79%	1.57%	2.33%	4.56%	8.72%	12.54%	16.04%	22.28%	27.65%	32.33%	48.86%
<b>30%</b>	<b>17%</b>	0.98%	1.94%	2.88%	5.61%	10.62%	15.12%	19.20%	26.27%	32.21%	37.26%	54.29%
<b>40%</b>	<b>23%</b>	1.17%	2.31%	3.43%	6.64%	12.45%	17.57%	22.14%	29.89%	36.25%	41.54%	58.70%
<b>50%</b>	<b>28%</b>	1.36%	2.68%	3.97%	7.64%	14.20%	19.89%	24.87%	33.17%	39.83%	45.28%	62.33%
<b>60%</b>	<b>34%</b>	1.55%	3.05%	4.50%	8.62%	15.87%	22.06%	27.39%	36.14%	43.01%	48.54%	65.35%
<b>70%</b>	<b>39%</b>	1.73%	3.40%	5.02%	9.56%	17.45%	24.08%	29.72%	38.81%	45.82%	51.39%	67.89%
<b>80%</b>	<b>44%</b>	1.91%	3.75%	5.52%	10.46%	18.95%	25.96%	31.86%	41.22%	48.32%	53.89%	70.04%
<b>90%</b>	<b>48%</b>	2.08%	4.08%	6.00%	11.32%	20.34%	27.70%	33.81%	43.38%	50.53%	56.08%	71.86%
<b>100%</b>	<b>53%</b>	2.25%	4.40%	6.46%	12.13%	21.64%	29.29%	35.57%	45.30%	52.48%	57.99%	73.41%

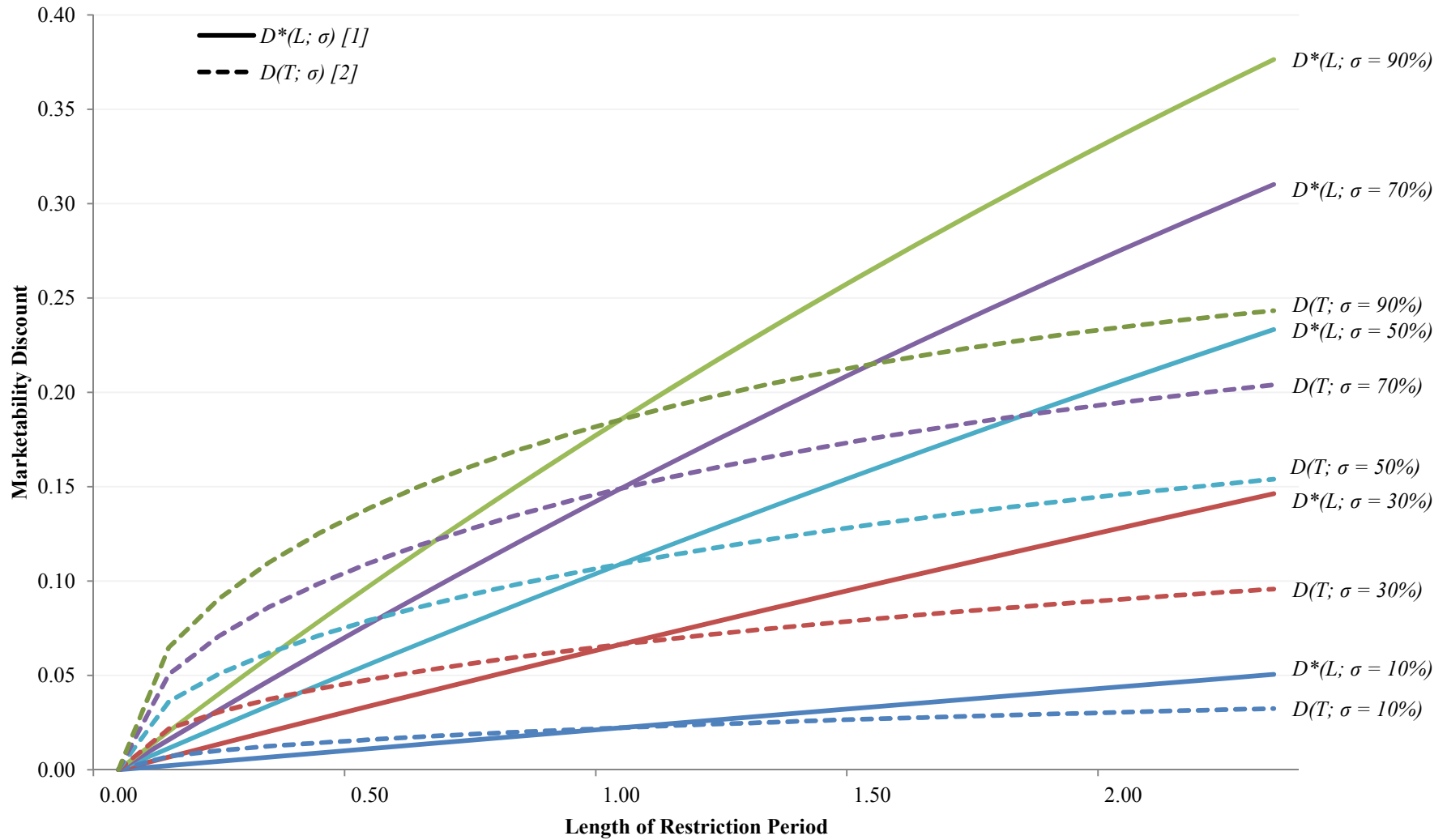
Note: The volatility  $\nu$  is calculated from equation (6).

**Figure 1**  
 **$D^*(L)$  Given by Equation (8)**



Note: Assumes  $q=0.03$  for illustrative purposes.

**Figure 2**  
**Comparison of the Two Average-Strike Put Option DLOM Models**



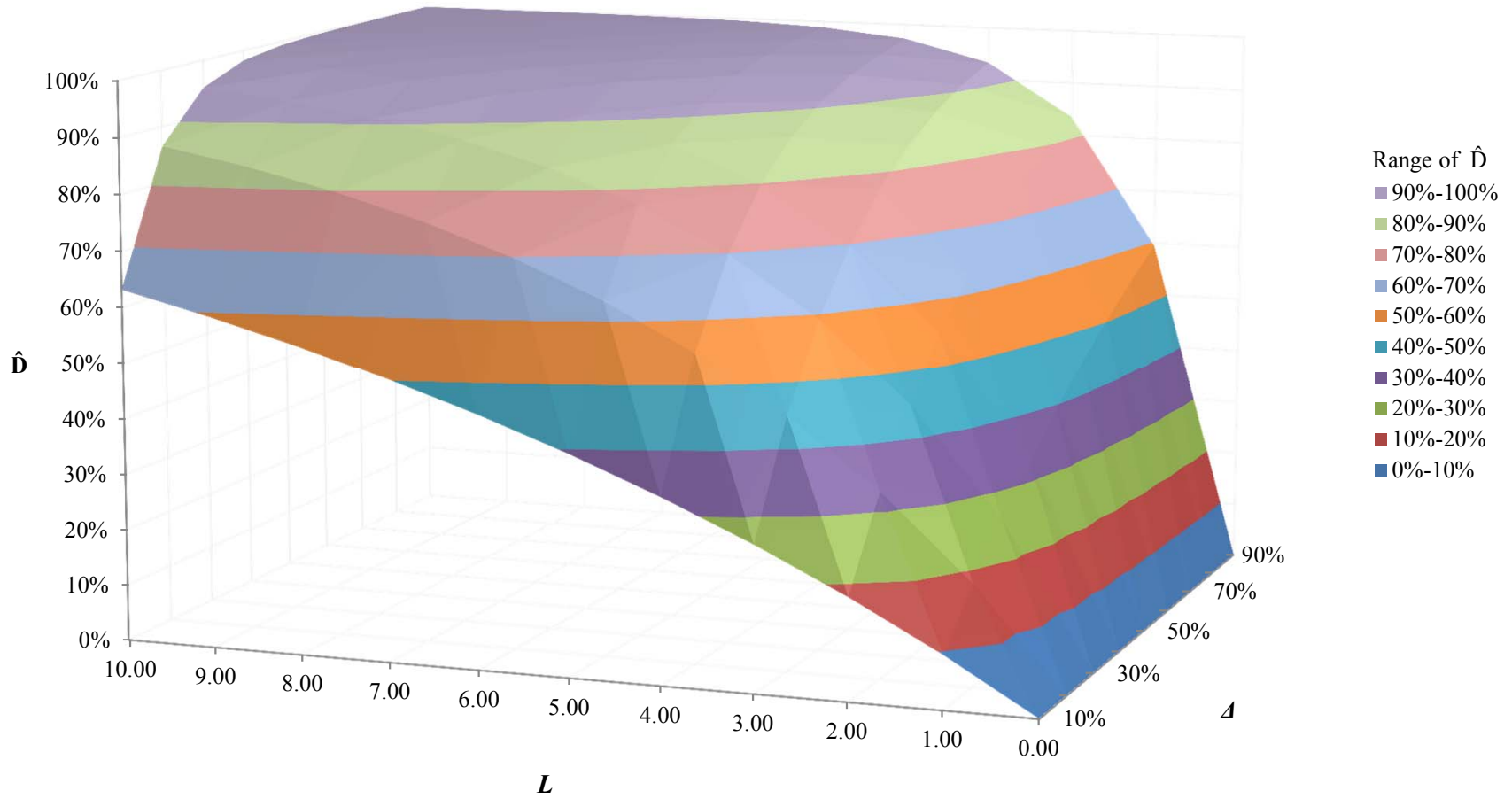
Note: Assumes  $q=0.03$  for illustrative purposes.

[1]  $D^*(L)$  is given by equation (8) with  $v$  given by equation (6), and  $\sigma$  is the stock price volatility.

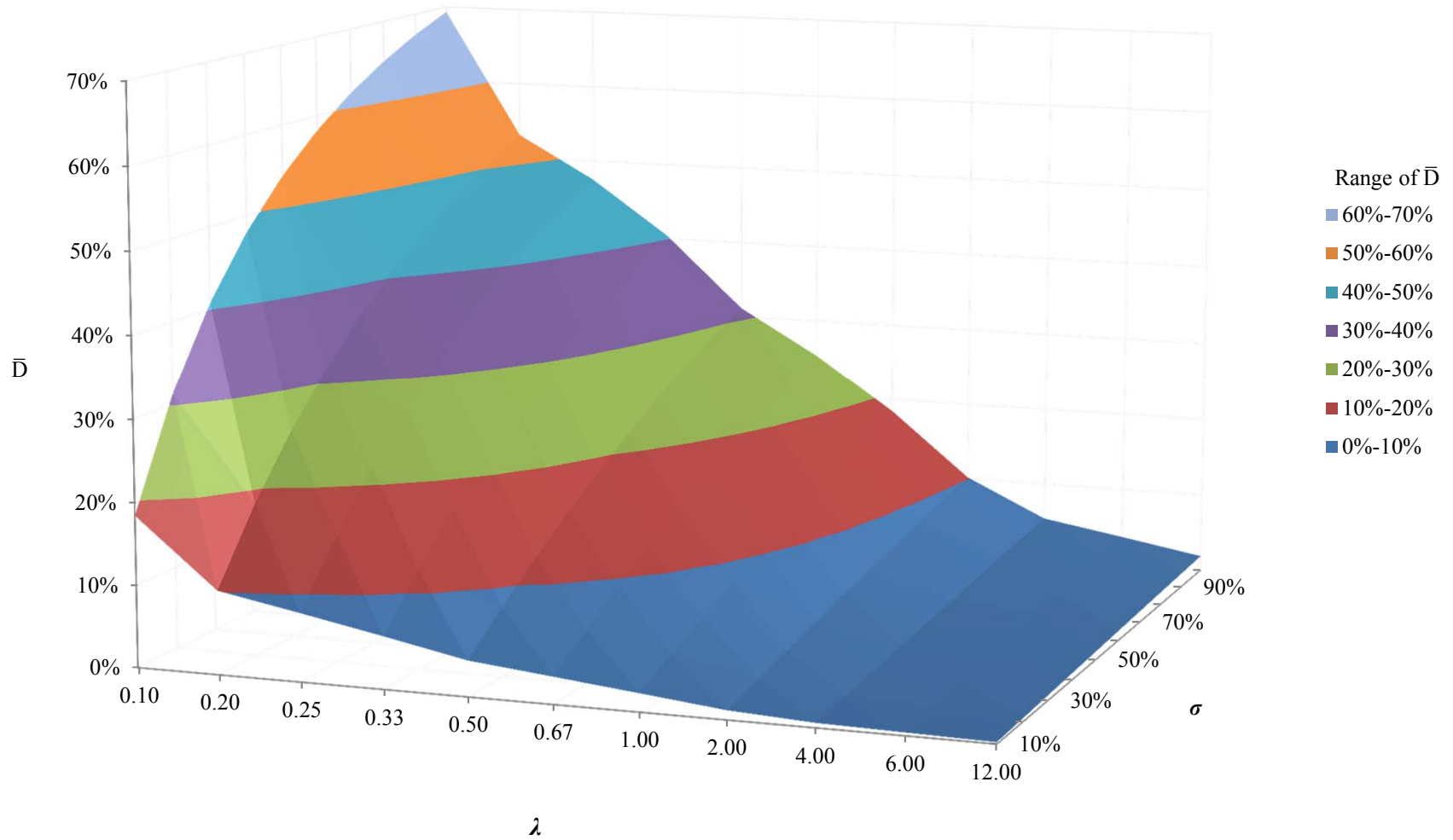
[2]  $D(T)$  is given by equation (3) with  $v\sqrt{T}$  given by equation (4), and  $\sigma$  is the stock price volatility.



**Figure 3**  
Sensitivity of  $\hat{D}$  Given by Equation (11) to  $L$  and  $\Delta$  When  $\tau = \varepsilon = 0$

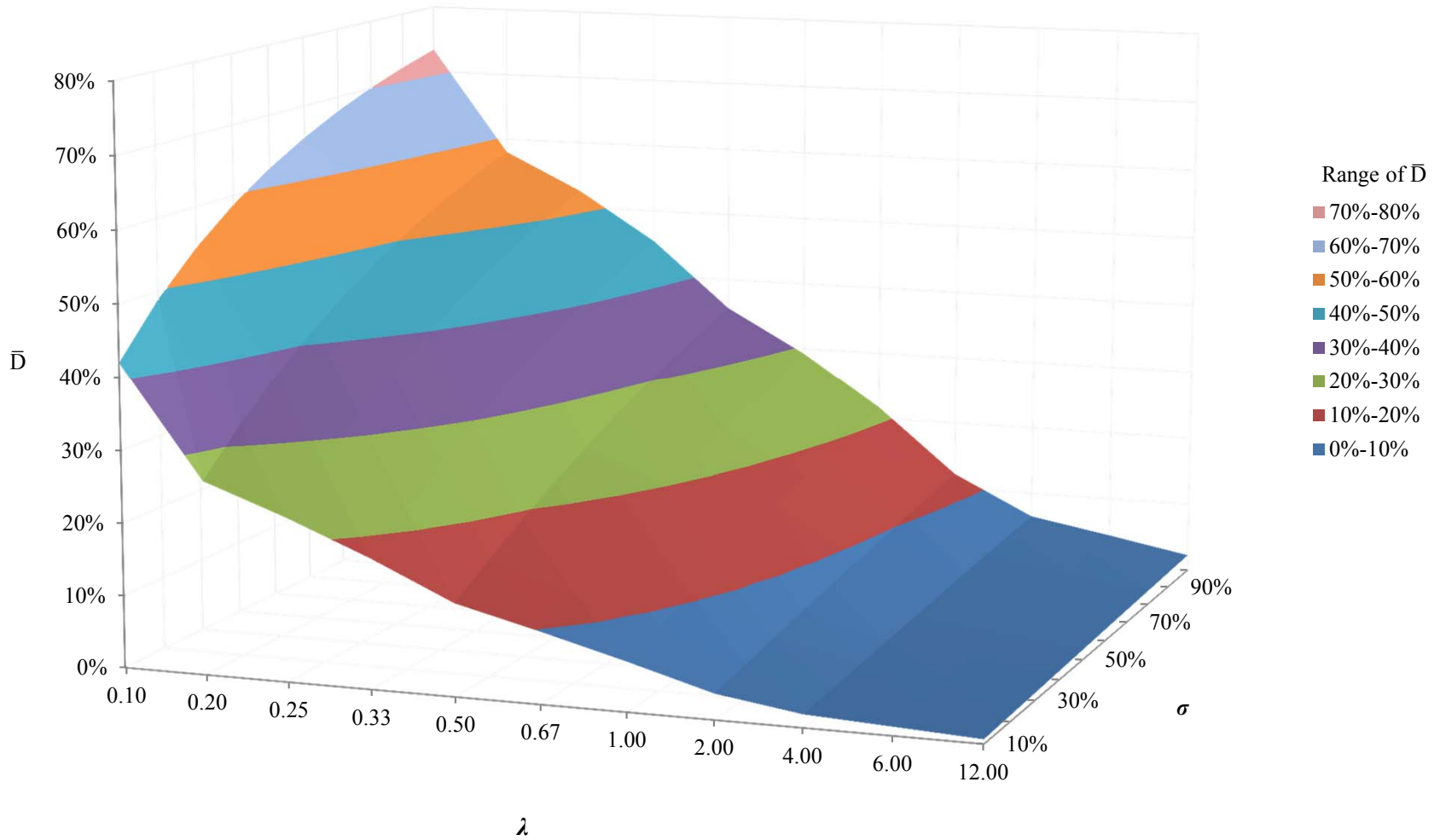


**Figure 4**  
Sensitivity of  $\bar{D}$  Given by Equation (12) to  $\lambda$  and  $\sigma$  When  $\tau = \varepsilon = 0$



Note: Assumes  $q = 0.03$  for illustrative purposes.

**Figure 5**  
**Sensitivity of  $\bar{D}$  Given by Equation (12) to  $\lambda$  and  $\sigma$  When  $\tau + \varepsilon = 0.05$**



Note: Assumes  $q = 0.03$  for illustrative purposes.

**Figure 6**

**Sensitivity of  $\bar{D}$  Given by Equation (12) to  $\bar{L}$  and  $\Delta$  When  $\tau = \varepsilon = 0$**

