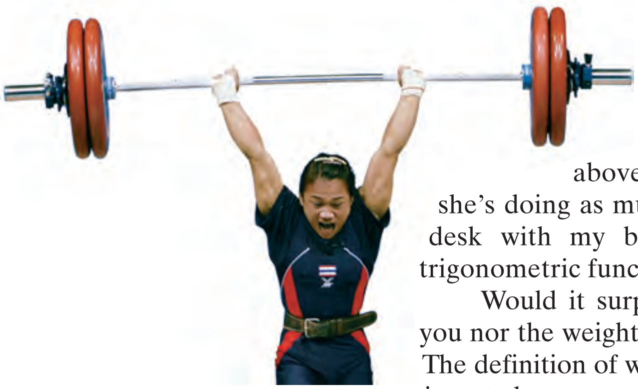
Section 7.7 The Dot Product

Objectives

1. Find the dot product of two vectors.
2. Find the angle between two vectors.
3. Use the dot product to determine if two vectors are orthogonal.
4. Find the projection of a vector onto another vector.
5. Express a vector as the sum of two orthogonal vectors.
6. Compute work.



Talk about hard work! I can see the weightlifter's muscles quivering from the exertion of holding the barbell in a stationary position above her head. Still, I'm not sure if she's doing as much work as I am, sitting at my desk with my brain quivering from studying trigonometric functions and their applications.

Would it surprise you to know that neither you nor the weightlifter are doing any work at all? The definition of work in physics and mathematics is not the same as what we mean by "work" in everyday use. To understand what is involved in real work, we turn to a new vector operation called the dot product.

The Dot Product of Two Vectors

1 Find the dot product of two vectors.

The operations of vector addition and scalar multiplication result in vectors. By contrast, the *dot product* of two vectors results in a scalar (a real number), rather than a vector.

Definition of the Dot Product

If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j}$ are vectors, the **dot product** $\mathbf{v} \cdot \mathbf{w}$ is defined as follows:

$$\mathbf{v} \cdot \mathbf{w} = a_1a_2 + b_1b_2.$$

The dot product of two vectors is the sum of the products of their horizontal components and their vertical components.

EXAMPLE 1 Finding Dot Products

If $\mathbf{v} = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{w} = -3\mathbf{i} + 4\mathbf{j}$, find each of the following dot products:

a. $\mathbf{v} \cdot \mathbf{w}$ **b.** $\mathbf{w} \cdot \mathbf{v}$ **c.** $\mathbf{v} \cdot \mathbf{v}$.

Solution To find each dot product, multiply the two horizontal components, and then multiply the two vertical components. Finally, add the two products.

a. $\mathbf{v} \cdot \mathbf{w} = 5(-3) + (-2)(4) = -15 - 8 = -23$

Multiply the horizontal components and multiply the vertical components of $\mathbf{v} = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{w} = -3\mathbf{i} + 4\mathbf{j}$.

b. $\mathbf{w} \cdot \mathbf{v} = -3(5) + 4(-2) = -15 - 8 = -23$

Multiply the horizontal components and multiply the vertical components of $\mathbf{w} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 2\mathbf{j}$.

c. $\mathbf{v} \cdot \mathbf{v} = 5(5) + (-2)(-2) = 25 + 4 = 29$

Multiply the horizontal components and multiply the vertical components of $\mathbf{v} = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 2\mathbf{j}$.

Check Point 1 If $\mathbf{v} = 7\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} - \mathbf{j}$, find each of the following dot products:

a. $\mathbf{v} \cdot \mathbf{w}$ **b.** $\mathbf{w} \cdot \mathbf{v}$ **c.** $\mathbf{w} \cdot \mathbf{w}$.

In Example 1 and Check Point 1, did you notice that $\mathbf{v} \cdot \mathbf{w}$ and $\mathbf{w} \cdot \mathbf{v}$ produced the same scalar? The fact that $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ follows from the definition of the dot product. Properties of the dot product are given in the following box. Proofs for some of these properties are given in the appendix.

Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3. $\mathbf{0} \cdot \mathbf{v} = 0$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

The Angle between Two Vectors

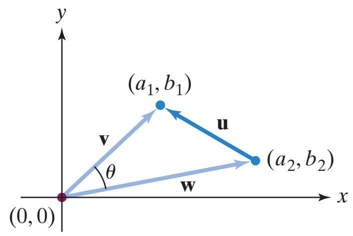


Figure 7.64

The Law of Cosines can be used to derive another formula for the dot product. This formula will give us a way to find the angle between two vectors.

Figure 7.64 shows vectors $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j}$. By the definition of the dot product, we know that $\mathbf{v} \cdot \mathbf{w} = a_1a_2 + b_1b_2$. Our new formula for the dot product involves the angle between the vectors, shown as θ in the figure. Apply the Law of Cosines to the triangle shown in the figure.

$$\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\,\|\mathbf{w}\| \cos\theta$$

Use the Law of Cosines.

$\mathbf{u} = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j}$
$\|\mathbf{u}\| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$

$\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$
$\|\mathbf{v}\| = \sqrt{a_1^2 + b_1^2}$

$\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j}$
$\|\mathbf{w}\| = \sqrt{a_2^2 + b_2^2}$

$$(a_1 - a_2)^2 + (b_1 - b_2)^2 = (a_1^2 + b_1^2) + (a_2^2 + b_2^2) - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$$

Substitute the squares of the magnitudes of vectors u, v, and w into the Law of Cosines.

$$a_1^2 - 2a_1a_2 + a_2^2 + b_1^2 - 2b_1b_2 + b_2^2 = a_1^2 + b_1^2 + a_2^2 + b_2^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$$

Square the binomials using $(A - B)^2 = A^2 - 2AB + B^2$.

$$-2a_1a_2 - 2b_1b_2 = -2\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$$

Subtract a_1^2, a_2^2, b_1^2, and b_2^2 from both sides of the equation.

$$a_1a_2 + b_1b_2 = \|\mathbf{v}\|\,\|\mathbf{w}\| \cos\theta$$

Divide both sides by −2.

By definition, $\mathbf{v} \cdot \mathbf{w} = a_1a_2 + b_1b_2$.

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$$

Substitute $\mathbf{v} \cdot \mathbf{w}$ for the expression on the left side of the equation.

Alternative Formula for the Dot Product

If $\mathbf{v}$ and $\mathbf{w}$ are two nonzero vectors and θ is the smallest nonnegative angle between them, then

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta.$$

2 Find the angle between two vectors.

Solving the formula in the box for $\cos\theta$ gives us a formula for finding the angle between two vectors:

Formula for the Angle between Two Vectors

If $\mathbf{v}$ and $\mathbf{w}$ are two nonzero vectors and θ is the smallest nonnegative angle between $\mathbf{v}$ and $\mathbf{w}$, then

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \quad \text{and} \quad \theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right).$$

EXAMPLE 2 Finding the Angle between Two Vectors

Find the angle θ between the vectors $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{w} = -\mathbf{i} + 4\mathbf{j}$, shown in **Figure 7.65**. Round to the nearest tenth of a degree.

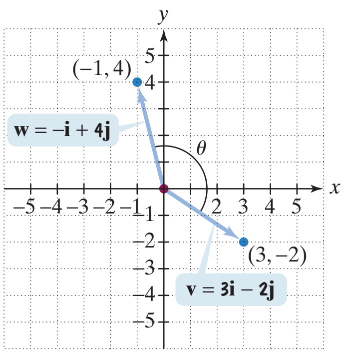


Figure 7.65 Finding the angle between two vectors

Solution Use the formula for the angle between two vectors.

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

Begin with the formula for the cosine of the angle between two vectors.

$$= \frac{(3\mathbf{i} - 2\mathbf{j})\cdot(-\mathbf{i} + 4\mathbf{j})}{\sqrt{3^2 + (-2)^2}\sqrt{(-1)^2 + 4^2}}$$

Substitute the given vectors in the numerator. Find the magnitude of each vector in the denominator.

$$= \frac{3(-1) + (-2)(4)}{\sqrt{13}\sqrt{17}}$$

Find the dot product in the numerator. Simplify in the denominator.

$$= -\frac{11}{\sqrt{221}}$$

Perform the indicated operations.

The angle θ between the vectors is

$$\theta = \cos^{-1}\left(-\frac{11}{\sqrt{221}}\right) \approx 137.7^\circ.$$

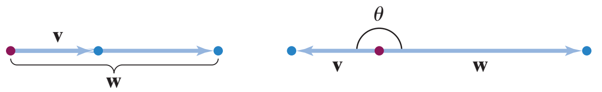
Use a calculator.

Check Point 2 Find the angle between the vectors $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j}$. Round to the nearest tenth of a degree.

3 Use the dot product to determine if two vectors are orthogonal.

Parallel and Orthogonal Vectors

Two vectors are **parallel** when the angle θ between the vectors is 0° or 180°. If $\theta = 0^\circ$, the vectors point in the same direction. If $\theta = 180^\circ$, the vectors point in opposite directions. **Figure 7.66** shows parallel vectors.



$\theta = 0^\circ$ and $\cos\theta = 1$. Vectors point in the same direction.

$\theta = 180^\circ$ and $\cos\theta = -1$. Vectors point in opposite directions.

Figure 7.66 Parallel vectors

Two vectors are **orthogonal** when the angle between the vectors is 90°, shown in **Figure 7.67**. (The word *orthogonal*, rather than *perpendicular*, is used to describe vectors that meet at right angles.) We know that $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$. If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, then

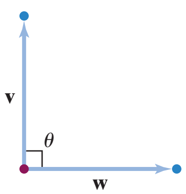


Figure 7.67 Orthogonal vectors: $\theta = 90^\circ$ and $\cos\theta = 0$

$$\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos 90^\circ = \|\mathbf{v}\|\|\mathbf{w}\|(0) = 0.$$

Conversely, if $\mathbf{v}$ and $\mathbf{w}$ are vectors such that $\mathbf{v}\cdot\mathbf{w} = 0$, then $\|\mathbf{v}\| = 0$ or $\|\mathbf{w}\| = 0$ or $\cos\theta = 0$. If $\cos\theta = 0$, then $\theta = 90^\circ$, so $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

The discussion at the bottom of the previous page is summarized as follows:

The Dot Product and Orthogonal Vectors

Two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v} \cdot \mathbf{w} = 0$. Because $\mathbf{0} \cdot \mathbf{v} = 0$, the zero vector is orthogonal to every vector $\mathbf{v}$.

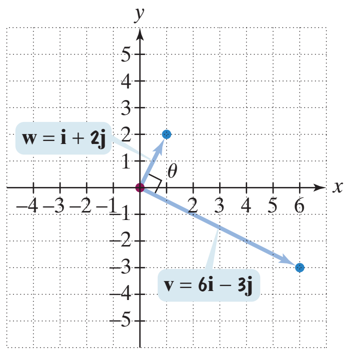


Figure 7.68 Orthogonal vectors

EXAMPLE 3 Determining Whether Vectors Are Orthogonal

Are the vectors $\mathbf{v} = 6\mathbf{i} - 3\mathbf{j}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j}$ orthogonal?

Solution The vectors are orthogonal if their dot product is 0. Begin by finding $\mathbf{v} \cdot \mathbf{w}$.

$$\mathbf{v} \cdot \mathbf{w} = (6\mathbf{i} - 3\mathbf{j}) \cdot (\mathbf{i} + 2\mathbf{j}) = 6(1) + (-3)(2) = 6 - 6 = 0$$

The dot product is 0. Thus, the given vectors are orthogonal. They are shown in **Figure 7.68**.

Check Point 3 Are the vectors $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 6\mathbf{i} - 4\mathbf{j}$ orthogonal?

4 Find the projection of a vector onto another vector.

Projection of a Vector Onto Another Vector

You know how to add two vectors to obtain a resultant vector. We now reverse this process by expressing a vector as the sum of two orthogonal vectors. By doing this, you can determine how much force is applied in a particular direction. For example, **Figure 7.69** shows a boat on a tilted ramp. The force due to gravity, $\mathbf{F}$, is pulling straight down on the boat. Part of this force, $\mathbf{F}_1$, is pushing the boat down the ramp. Another part of this force, $\mathbf{F}_2$, is pressing the boat against the ramp, at a right angle to the incline. These two orthogonal vectors, $\mathbf{F}_1$ and $\mathbf{F}_2$, are called the **vector components** of $\mathbf{F}$. Notice that

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2.$$

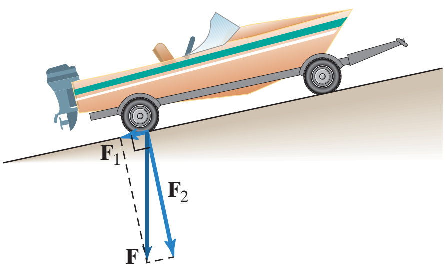


Figure 7.69

A method for finding $\mathbf{F}_1$ and $\mathbf{F}_2$ involves projecting a vector onto another vector.

Figure 7.70 shows two nonzero vectors, $\mathbf{v}$ and $\mathbf{w}$, with the same initial point. The angle between the vectors, θ, is acute in **Figure 7.70(a)** and obtuse in **Figure 7.70(b)**. A third vector, called the **vector projection of v onto w**, is also shown in each figure, denoted by $\text{proj}_{\mathbf{w}}\mathbf{v}$.

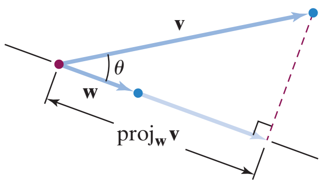


Figure 7.70(a)

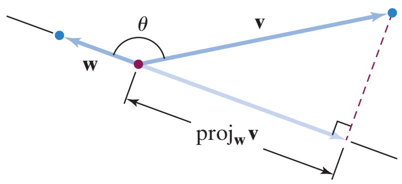


Figure 7.70(b)

How is the vector projection of $\mathbf{v}$ onto $\mathbf{w}$ formed? Draw the line segment from the terminal point of $\mathbf{v}$ that forms a right angle with a line through $\mathbf{w}$, shown in red. The projection of $\mathbf{v}$ onto $\mathbf{w}$ lies on a line through $\mathbf{w}$, and is parallel to vector $\mathbf{w}$. This vector begins at the common initial point of $\mathbf{v}$ and $\mathbf{w}$. It ends at the point where the dashed red line segment intersects the line through $\mathbf{w}$.

Our goal is to determine an expression for $\text{proj}_{\mathbf{w}}\mathbf{v}$. We begin with its magnitude. By the definition of the cosine function,

$$\cos\theta = \frac{\|\text{proj}_{\mathbf{w}}\mathbf{v}\|}{\|\mathbf{v}\|}$$ This is the magnitude of the vector projection of v onto w.

$$\|\mathbf{v}\|\cos\theta = \|\text{proj}_{\mathbf{w}}\mathbf{v}\|$$ Multiply both sides by $\|\mathbf{v}\|$.

$$\|\text{proj}_{\mathbf{w}}\mathbf{v}\| = \|\mathbf{v}\|\cos\theta.$$ Reverse the two sides.

We can rewrite the right side of this equation and obtain another expression for the magnitude of the vector projection of $\mathbf{v}$ onto $\mathbf{w}$. To do so, use the alternate formula for the dot product, $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$.

Divide both sides of $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$ by $\|\mathbf{w}\|$:

$$\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|} = \|\mathbf{v}\|\cos\theta.$$

The expression on the right side of this equation, $\|\mathbf{v}\|\cos\theta$, is the same expression that appears in the formula for $\|\text{proj}_{\mathbf{w}}\mathbf{v}\|$. Thus,

$$\|\text{proj}_{\mathbf{w}}\mathbf{v}\| = \|\mathbf{v}\|\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}.$$

We use the formula for the magnitude of $\text{proj}_{\mathbf{w}}\mathbf{v}$ to find the vector itself. This is done by finding the scalar product of the magnitude and the unit vector in the direction of **w**.

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}\right)\left(\frac{\mathbf{w}}{\|\mathbf{w}\|}\right) = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}$$

This is the magnitude of the vector projection of **v** onto **w**.

This is the unit vector in the direction of **w**.

The Vector Projection of v Onto w

If **v** and **w** are two nonzero vectors, the vector projection of **v** onto **w** is

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}.$$

EXAMPLE 4 Finding the Vector Projection of One Vector Onto Another

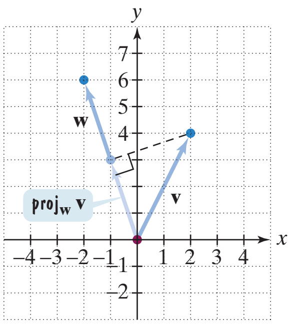


Figure 7.71 The vector projection of **v** onto **w**

If $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} + 6\mathbf{j}$, find the vector projection of **v** onto **w**.

Solution The vector projection of **v** onto **w** is found using the formula for $\text{proj}_{\mathbf{w}}\mathbf{v}$.

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{(2\mathbf{i} + 4\mathbf{j})\cdot(-2\mathbf{i} + 6\mathbf{j})}{\left(\sqrt{(-2)^2 + 6^2}\right)^2}\mathbf{w}$$

$$= \frac{2(-2) + 4(6)}{\left(\sqrt{40}\right)^2}\mathbf{w} = \frac{20}{40}\mathbf{w} = \frac{1}{2}(-2\mathbf{i} + 6\mathbf{j}) = -\mathbf{i} + 3\mathbf{j}$$

The three vectors, **v**, **w**, and $\text{proj}_{\mathbf{w}}\mathbf{v}$, are shown in **Figure 7.71**.

Check Point 4 If $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j}$ and $\mathbf{w} = \mathbf{i} - \mathbf{j}$, find the vector projection of **v** onto **w**.

5 Express a vector as the sum of two orthogonal vectors.

We use the vector projection of **v** onto **w**, $\text{proj}_{\mathbf{w}}\mathbf{v}$, to express **v** as the sum of two orthogonal vectors.

The Vector Components of v

Let **v** and **w** be two nonzero vectors. Vector **v** can be expressed as the sum of two orthogonal vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to **w** and $\mathbf{v}_2$ is orthogonal to **w**.

$$\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}, \quad \mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$$

Thus, $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are called the **vector components** of **v**. The process of expressing **v** as $\mathbf{v}_1 + \mathbf{v}_2$ is called the **decomposition** of **v** into $\mathbf{v}_1$ and $\mathbf{v}_2$.

EXAMPLE 5 Decomposing a Vector into Two Orthogonal Vectors

Let $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} + 6\mathbf{j}$. Decompose $\mathbf{v}$ into two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to $\mathbf{w}$ and $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$.

Solution These are the vectors we worked with in Example 4. We use the formulas in the box on the previous page.

$$\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = -\mathbf{i} + 3\mathbf{j}$$ *We obtained this vector in Example 4.*

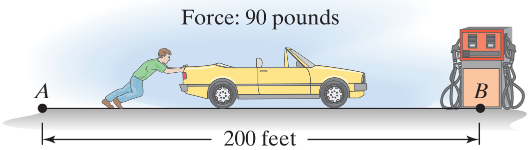
$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = (2\mathbf{i} + 4\mathbf{j}) - (-\mathbf{i} + 3\mathbf{j}) = 3\mathbf{i} + \mathbf{j}$$

Check Point 5 Let $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j}$ and $\mathbf{w} = \mathbf{i} - \mathbf{j}$. (These are the vectors from Check Point 4.) Decompose $\mathbf{v}$ into two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to $\mathbf{w}$ and $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$.

6 Compute work.

Work: An Application of the Dot Product

The bad news: Your car just died. The good news: It died on a level road just 200 feet from a gas station. Exerting a constant force of 90 pounds, and not necessarily whistling as you work, you manage to push the car to the gas station.



Although you did not whistle, you certainly did work pushing the car 200 feet from point A to point B. How much work did you do? If a constant force $\mathbf{F}$ is applied to an object, moving it from point A to point B in the direction of the force, the work, W, done is

$$W = (\text{magnitude of force})(\text{distance from } A \text{ to } B).$$

You pushed with a force of 90 pounds for a distance of 200 feet. The work done by your force is

$$W = (90 \text{ pounds})(200 \text{ feet})$$

or 18,000 foot-pounds. Work is often measured in foot-pounds or in newton-meters.

The photo on the left shows an adult pulling a small child in a wagon. Work is being done. However, the situation is not quite the same as pushing your car. Pushing the car, the force you applied was along the line of motion. By contrast, the force of the adult pulling the wagon is not applied along the line of the wagon's motion. In this case, the dot product is used to determine the work done by the force.

Definition of Work

The work, W, done by a force $\mathbf{F}$ moving an object from A to B is

$$W = \mathbf{F} \cdot \overrightarrow{AB}.$$

When computing work, it is often easier to use the alternative formula for the dot product. Thus,

$$W = \mathbf{F} \cdot \overrightarrow{AB} = \|\mathbf{F}\| \|\overrightarrow{AB}\| \cos\theta.$$

$\|\mathbf{F}\|$ is the magnitude of the force.

$\|\overrightarrow{AB}\|$ is the distance over which the constant force is applied.

θ is the angle between the force and the direction of motion.

It is correct to refer to W as either the work done or the work done by the force.

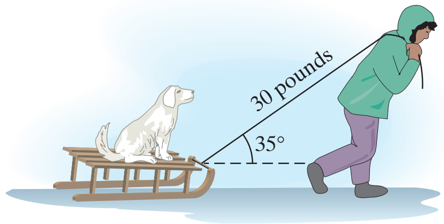


Figure 7.72 Computing work done pulling the sled 200 feet

EXAMPLE 6 Computing Work

A child pulls a sled along level ground by exerting a force of 30 pounds on a rope that makes an angle of 35° with the horizontal. How much work is done pulling the sled 200 feet?

Solution The situation is illustrated in **Figure 7.72**. The work done is

$$W = \|\mathbf{F}\| \|\overrightarrow{AB}\| \cos\theta = (30)(200)\cos 35° \approx 4915.$$

Magnitude of the force is 30 pounds.

Distance is 200 feet.

The angle between the force and the sled's motion is 35°.

Thus, the work done is approximately 4915 foot-pounds.

Check Point 6 A child pulls a wagon along level ground by exerting a force of 20 pounds on a handle that makes an angle of 30° with the horizontal. How much work is done pulling the wagon 150 feet?

Exercise Set 7.7

Practice Exercises

In Exercises 1–8, use the given vectors to find $\mathbf{v}\cdot\mathbf{w}$ *and* $\mathbf{v}\cdot\mathbf{v}$.

1. $\mathbf{v} = 3\mathbf{i} + \mathbf{j}, \quad \mathbf{w} = \mathbf{i} + 3\mathbf{j}$

2. $\mathbf{v} = 3\mathbf{i} + 3\mathbf{j}, \quad \mathbf{w} = \mathbf{i} + 4\mathbf{j}$

3. $\mathbf{v} = 5\mathbf{i} - 4\mathbf{j}, \quad \mathbf{w} = -2\mathbf{i} - \mathbf{j}$

4. $\mathbf{v} = 7\mathbf{i} - 2\mathbf{j}, \quad \mathbf{w} = -3\mathbf{i} - \mathbf{j}$

5. $\mathbf{v} = -6\mathbf{i} - 5\mathbf{j}, \quad \mathbf{w} = -10\mathbf{i} - 8\mathbf{j}$

6. $\mathbf{v} = -8\mathbf{i} - 3\mathbf{j}, \quad \mathbf{w} = -10\mathbf{i} - 5\mathbf{j}$

7. $\mathbf{v} = 5\mathbf{i}, \quad \mathbf{w} = \mathbf{j}$

8. $\mathbf{v} = \mathbf{i}, \quad \mathbf{w} = -5\mathbf{j}$

In Exercises 9–16, let

$$\mathbf{u} = 2\mathbf{i} - \mathbf{j}, \quad \mathbf{v} = 3\mathbf{i} + \mathbf{j}, \quad \text{and} \quad \mathbf{w} = \mathbf{i} + 4\mathbf{j}.$$

Find each specified scalar.

9. $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w})$

10. $\mathbf{v}\cdot(\mathbf{u} + \mathbf{w})$

11. $\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

12. $\mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{w}$

13. $(4\mathbf{u})\cdot\mathbf{v}$

14. $(5\mathbf{v})\cdot\mathbf{w}$

15. $4(\mathbf{u}\cdot\mathbf{v})$

16. $5(\mathbf{v}\cdot\mathbf{w})$

In Exercises 17–22, find the angle between **v** *and* **w**. *Round to the nearest tenth of a degree.*

17. $\mathbf{v} = 2\mathbf{i} - \mathbf{j}, \quad \mathbf{w} = 3\mathbf{i} + 4\mathbf{j}$

18. $\mathbf{v} = -2\mathbf{i} + 5\mathbf{j}, \quad \mathbf{w} = 3\mathbf{i} + 6\mathbf{j}$

19. $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j}, \quad \mathbf{w} = 4\mathbf{i} - \mathbf{j}$

20. $\mathbf{v} = \mathbf{i} + 2\mathbf{j}, \quad \mathbf{w} = 4\mathbf{i} - 3\mathbf{j}$

21. $\mathbf{v} = 6\mathbf{i}, \quad \mathbf{w} = 5\mathbf{i} + 4\mathbf{j}$

22. $\mathbf{v} = 3\mathbf{j}, \quad \mathbf{w} = 4\mathbf{i} + 5\mathbf{j}$

In Exercises 23–32, use the dot product to determine whether **v** *and* **w** *are orthogonal.*

23. $\mathbf{v} = \mathbf{i} + \mathbf{j}, \quad \mathbf{w} = \mathbf{i} - \mathbf{j}$

24. $\mathbf{v} = \mathbf{i} + \mathbf{j}, \quad \mathbf{w} = -\mathbf{i} + \mathbf{j}$

25. $\mathbf{v} = 2\mathbf{i} + 8\mathbf{j}, \quad \mathbf{w} = 4\mathbf{i} - \mathbf{j}$

26. $\mathbf{v} = 8\mathbf{i} - 4\mathbf{j}, \quad \mathbf{w} = -6\mathbf{i} - 12\mathbf{j}$

27. $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j}, \quad \mathbf{w} = -\mathbf{i} + \mathbf{j}$

28. $\mathbf{v} = 5\mathbf{i} - 5\mathbf{j}, \quad \mathbf{w} = \mathbf{i} - \mathbf{j}$

29. $\mathbf{v} = 3\mathbf{i}, \quad \mathbf{w} = -4\mathbf{i}$

30. $\mathbf{v} = 5\mathbf{i}, \quad \mathbf{w} = -6\mathbf{i}$

31. $\mathbf{v} = 3\mathbf{i}, \quad \mathbf{w} = -4\mathbf{j}$

32. $\mathbf{v} = 5\mathbf{i}, \quad \mathbf{w} = -6\mathbf{j}$

In Exercises 33–38, find $\text{proj}_{\mathbf{w}}\mathbf{v}$. *Then decompose* **v** *into two vectors,* $\mathbf{v}_1$ *and* $\mathbf{v}_2$, *where* $\mathbf{v}_1$ *is parallel to* **w** *and* $\mathbf{v}_2$ *is orthogonal to* **w**.

33. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}, \quad \mathbf{w} = \mathbf{i} - \mathbf{j}$

34. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}, \quad \mathbf{w} = 2\mathbf{i} + \mathbf{j}$

35. $\mathbf{v} = \mathbf{i} + 3\mathbf{j}, \quad \mathbf{w} = -2\mathbf{i} + 5\mathbf{j}$

36. $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j}, \quad \mathbf{w} = -3\mathbf{i} + 6\mathbf{j}$

37. $\mathbf{v} = \mathbf{i} + 2\mathbf{j}, \quad \mathbf{w} = 3\mathbf{i} + 6\mathbf{j}$

38. $\mathbf{v} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{w} = 6\mathbf{i} + 3\mathbf{j}$

Practice Plus

In Exercises 39–42, let

$$\mathbf{u} = -\mathbf{i} + \mathbf{j}, \quad \mathbf{v} = 3\mathbf{i} - 2\mathbf{j}, \quad \text{and} \quad \mathbf{w} = -5\mathbf{j}.$$

Find each specified scalar or vector.

39. $5\mathbf{u}\cdot(3\mathbf{v} - 4\mathbf{w})$

40. $4\mathbf{u}\cdot(5\mathbf{v} - 3\mathbf{w})$

41. $\text{proj}_{\mathbf{u}}(\mathbf{v} + \mathbf{w})$

42. $\text{proj}_{\mathbf{u}}(\mathbf{v} - \mathbf{w})$

In Exercises 43–44, find the angle, in degrees, between **v** *and* **w**.

43. $\mathbf{v} = 2\cos\frac{4\pi}{3}\mathbf{i} + 2\sin\frac{4\pi}{3}\mathbf{j}, \quad \mathbf{w} = 3\cos\frac{3\pi}{2}\mathbf{i} + 3\sin\frac{3\pi}{2}\mathbf{j}$

44. $\mathbf{v} = 3\cos\frac{5\pi}{3}\mathbf{i} + 3\sin\frac{5\pi}{3}\mathbf{j}, \quad \mathbf{w} = 2\cos\pi\mathbf{i} + 2\sin\pi\mathbf{j}$

In Exercises 45–50, determine whether **v** *and* **w** *are parallel, orthogonal, or neither.*

45. $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j}, \quad \mathbf{w} = 6\mathbf{i} - 10\mathbf{j}$

46. $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}, \quad \mathbf{w} = -6\mathbf{i} + 9\mathbf{j}$

47. $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j}, \quad \mathbf{w} = 6\mathbf{i} + 10\mathbf{j}$

48. $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}, \quad \mathbf{w} = -6\mathbf{i} - 9\mathbf{j}$

49. $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j}, \quad \mathbf{w} = 6\mathbf{i} + \frac{18}{5}\mathbf{j}$

50. $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}, \quad \mathbf{w} = -6\mathbf{i} - 4\mathbf{j}$

Application Exercises

51. The components of $\mathbf{v} = 240\mathbf{i} + 300\mathbf{j}$ represent the respective number of gallons of regular and premium gas sold at a station. The components of $\mathbf{w} = 2.90\mathbf{i} + 3.07\mathbf{j}$ represent the respective prices per gallon for each kind of gas. Find $\mathbf{v}\cdot\mathbf{w}$ and describe what the answer means in practical terms.

52. The components of $\mathbf{v} = 180\mathbf{i} + 450\mathbf{j}$ represent the respective number of one-day and three-day videos rented from a video store. The components of $\mathbf{w} = 3\mathbf{i} + 2\mathbf{j}$ represent the prices to rent the one-day and three-day videos, respectively. Find $\mathbf{v}\cdot\mathbf{w}$ and describe what the answer means in practical terms.

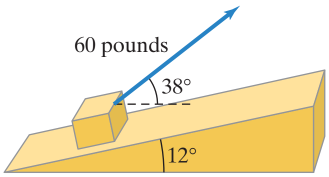
53. Find the work done in pushing a car along a level road from point A to point B, 80 feet from A, while exerting a constant force of 95 pounds. Round to the nearest foot-pound.

54. Find the work done when a crane lifts a 6000-pound boulder through a vertical distance of 12 feet. Round to the nearest foot-pound.

55. A wagon is pulled along level ground by exerting a force of 40 pounds on a handle that makes an angle of 32° with the horizontal. How much work is done pulling the wagon 100 feet? Round to the nearest foot-pound.

56. A wagon is pulled along level ground by exerting a force of 25 pounds on a handle that makes an angle of 38° with the horizontal. How much work is done pulling the wagon 100 feet? Round to the nearest foot-pound.

57. A force of 60 pounds on a rope is used to pull a box up a ramp inclined at 12° from the horizontal. The figure shows that the rope forms an angle of 38° with the horizontal. How much work is done pulling the box 20 feet along the ramp?



58. A force of 80 pounds on a rope is used to pull a box up a ramp inclined at 10° from the horizontal. The rope forms an angle of 33° with the horizontal. How much work is done pulling the box 25 feet along the ramp?

59. A force is given by the vector $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j}$. The force moves an object along a straight line from the point (4, 9) to the point (10, 20). Find the work done if the distance is measured in feet and the force is measured in pounds.

60. A force is given by the vector $\mathbf{F} = 5\mathbf{i} + 7\mathbf{j}$. The force moves an object along a straight line from the point (8, 11) to the point (18, 20). Find the work done if the distance is measured in meters and the force is measured in newtons.

61. A force of 4 pounds acts in the direction of 50° to the horizontal. The force moves an object along a straight line from the point (3, 7) to the point (8, 10), with distance measured in feet. Find the work done by the force.

62. A force of 6 pounds acts in the direction of 40° to the horizontal. The force moves an object along a straight line from the point (5, 9) to the point (8, 20), with the distance measured in feet. Find the work done by the force.

63. Refer to **Figure 7.69** on page 736. Suppose that the boat weighs 700 pounds and is on a ramp inclined at 30°. Represent the force due to gravity, $\mathbf{F}$, using

$$\mathbf{F} = -700\mathbf{j}.$$

a. Write a unit vector along the ramp in the upward direction.
b. Find the vector projection of $\mathbf{F}$ onto the unit vector from part (a).
c. What is the magnitude of the vector projection in part (b)? What does this represent?

64. Refer to **Figure 7.69** on page 736. Suppose that the boat weighs 650 pounds and is on a ramp inclined at 30°. Represent the force due to gravity, $\mathbf{F}$, using

$$\mathbf{F} = -650\mathbf{j}.$$

a. Write a unit vector along the ramp in the upward direction.
b. Find the vector projection of $\mathbf{F}$ onto the unit vector from part (a).
c. What is the magnitude of the vector projection in part (b)? What does this represent?

Writing in Mathematics

65. Explain how to find the dot product of two vectors.

66. Using words and no symbols, describe how to find the dot product of two vectors with the alternative formula

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos \theta.$$

67. Describe how to find the angle between two vectors.

68. What are parallel vectors?

69. What are orthogonal vectors?

70. How do you determine if two vectors are orthogonal?

71. Draw two vectors, $\mathbf{v}$ and $\mathbf{w}$, with the same initial point. Show the vector projection of $\mathbf{v}$ onto $\mathbf{w}$ in your diagram. Then describe how you identified this vector.

72. How do you determine the work done by a force $\mathbf{F}$ in moving an object from A to B when the direction of the force is not along the line of motion?

73. A weightlifter is holding a barbell perfectly still above his head, his body shaking from the effort. How much work is the weightlifter doing? Explain your answer.

74. Describe one way in which the everyday use of the word *work* is different from the definition of work given in this section.

Critical Thinking Exercises

Make Sense? *In Exercises 75–78, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

75. Although I expected vector operations to produce another vector, the dot product of two vectors is not a vector, but a real number.

76. I've noticed that whenever the dot product is negative, the angle between the two vectors is obtuse.

77. I'm working with a unit vector, so its dot product with itself must be 1.

78. The weightlifter does more work in raising 300 kilograms above her head than Atlas, who is supporting the entire world.

In Exercises 79–81, use the vectors

$$\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}, \quad \mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}, \quad \text{and} \quad \mathbf{w} = a_3\mathbf{i} + b_3\mathbf{j},$$

to prove the given property.

79. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

80. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$

81. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

82. If $\mathbf{v} = -2\mathbf{i} + 5\mathbf{j}$, find a vector orthogonal to $\mathbf{v}$.

83. Find a value of b so that $15\mathbf{i} - 3\mathbf{j}$ and $-4\mathbf{i} + b\mathbf{j}$ are orthogonal.

84. Prove that the projection of $\mathbf{v}$ onto $\mathbf{i}$ is $(\mathbf{v} \cdot \mathbf{i})\mathbf{i}$.

85. Find two vectors $\mathbf{v}$ and $\mathbf{w}$ such that the projection of $\mathbf{v}$ onto $\mathbf{w}$ is $\mathbf{v}$.

Group Exercise

86. Group members should research and present a report on unusual and interesting applications of vectors.

Preview Exercises

Exercises 87–89 will help you prepare for the material covered in the first section of the next chapter.

87. **a.** Does $(4, -1)$ satisfy $x + 2y = 2$?

b. Does $(4, -1)$ satisfy $x - 2y = 6$?

88. Graph $x + 2y = 2$ and $x - 2y = 6$ in the same rectangular coordinate system. At what point do the graphs intersect?

89. Solve: $5(2x - 3) - 4x = 9$.

Chapter 7 Summary, Review, and Test

Summary

DEFINITIONS AND CONCEPTS	EXAMPLES
7.1 and 7.2 The Law of Sines; The Law of Cosines	
a. The Law of Sines $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$	Ex. 1, p. 665; Ex. 2, p. 666; Ex. 3, p. 667; Ex. 4, p. 668;
b. The Law of Sines is used to solve SAA, ASA, and SSA (the ambiguous case) triangles. The ambiguous case may result in no triangle, one triangle, or two triangles; see the box on page 667.	Ex. 5, p. 669
c. The area of a triangle equals one-half the product of the lengths of two sides times the sine of their included angle.	Ex. 6, p. 670
d. The Law of Cosines $$a^2 = b^2 + c^2 - 2bc \cos A$$ $$b^2 = a^2 + c^2 - 2ac \cos B$$ $$c^2 = a^2 + b^2 - 2ab \cos C$$	
e. The Law of Cosines is used to find the side opposite the given angle in an SAS triangle; see the box on page 677. The Law of Cosines is also used to find the angle opposite the longest side in an SSS triangle; see the box on page 678.	Ex. 1, p. 678; Ex. 2, p. 679
f. Heron's Formula for the Area of a Triangle The area of a triangle with sides a, b, and c is $\sqrt{s(s - a)(s - b)(s - c)}$, where s is one-half its perimeter: $s = \frac{1}{2}(a + b + c)$.	Ex. 4, p. 680
7.3 and 7.4 Polar Coordinates; Graphs of Polar Equations	
a. A point P in the polar coordinate system is represented by (r, θ), where r is the directed distance from the pole to the point and θ is the angle from the polar axis to line segment OP. The elements of the ordered pair (r, θ) are called the polar coordinates of P. See Figure 7.20 on page 684. When r in (r, θ) is negative, a point is located $\lvert r \rvert$ units along the ray opposite the terminal side of θ. Important information about the sign of r and the location of the point (r, θ) is found in the box on page 685.	Ex. 1, p. 685
b. Multiple Representations of Points If n is any integer, $(r, \theta) = (r, \theta + 2n\pi)$ or $(r, \theta) = (-r, \theta + \pi + 2n\pi)$.	Ex. 2, p. 686
c. Relations between Polar and Rectangular Coordinates $$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = r^2, \quad \tan\theta = \frac{y}{x}$$	
d. To convert a point from polar coordinates (r, θ) to rectangular coordinates (x, y), use $x = r\cos\theta$ and $y = r\sin\theta$.	Ex. 3, p. 687
e. To convert a point from rectangular coordinates (x, y) to polar coordinates (r, θ), use the procedure in the box on page 688.	Ex. 4, p. 688; Ex. 5, p. 689
f. To convert a rectangular equation to a polar equation, replace x with $r\cos\theta$ and y with $r\sin\theta$.	Ex. 6, p. 690