0 Introduction

a. Motions take place in more than one dimension can be divided into separate motions in each dimension. This separation means that we can apply the laws that were developed for one dimension to many dimensions.

b. We will first look at two-dimensional motion – that is, motion confined to a flat surface.

c. Projection Motion: Projectile motion is a nonlinear motion that follows the curved path. The projectile curve is a combination of a constant-velocity horizontal motion and accelerated vertical motion.

d. The velocity of a projectile at any instant has two independent components of motion, and they don't affect each other. The only common thing links them together is the time.

1 Vector and Scalar Quantities

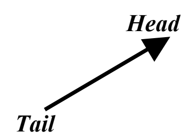
a. Physical quantities can be categorized as either scalar or vector quantities.

b. A ***scalar*** quantity has ***magnitude*** only, with no direction specified. Many quantities in physics, such as mass, volume, distance, speed and time, are scalar quantities.

c. A **vector** quantity has both ***magnitude*** and ***direction***. Many quantities in physics, such as force, velocity, displacement, and acceleration, are vector quantities.

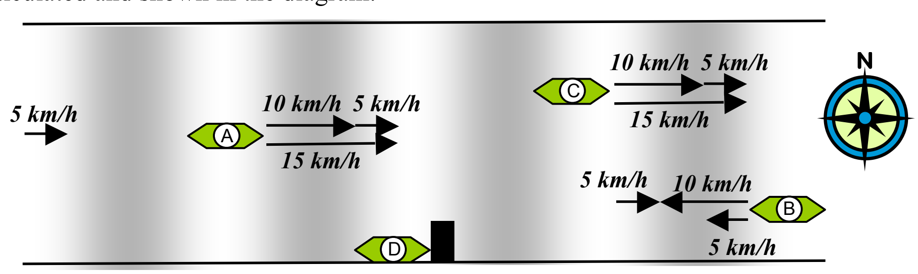
d. Scalars can be added, subtracted, multiplied, and divided like ordinary numbers.

2 Velocity Vectors

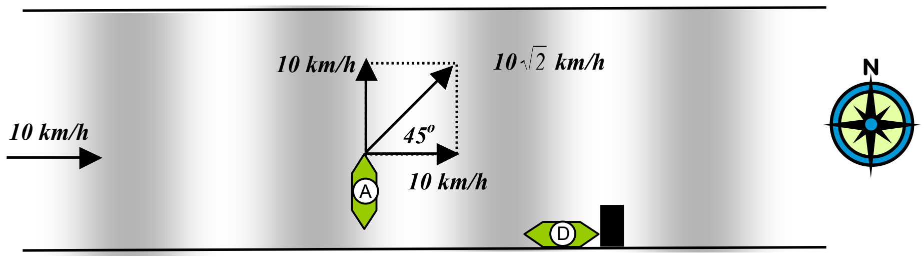


a. **Representation:** An arrow is used to represent the magnitude and direction of a vector quantity. The length of the arrow, drawn to scale, indicates the magnitude of the vector quantity. The direction of the arrow indicates the direction of the vector quantity.

b. **One-dimensional vector addition:** The result of adding two vectors is the sum or difference of the two speeds and the direction of the longer one. For example, in the following diagram, the velocity of the water current is 5 km/h east with respect to the observer D on the river bank. The velocities of boat A, B & C are 10 km/h east, 10 km/h west, and 10 km/h east respectively with respect to the water. The velocities of boat A, B & C with respect to the observer D are calculated and shown in the diagram.



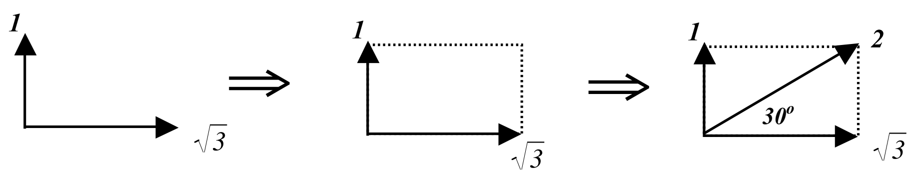
c. **Two-dimensional vector addition:** The result of adding two vectors, called the ***resultant***, is the diagonal of the parallelogram described by the two vectors. When the two vectors are perpendicular to each other, the resultant is the diagonal of the rectangle. For example, if the velocity of the water current is 10 km/h east with respect to the observer D and the velocity of the boat A is 10 km/h north with respect to the water. The velocity of the boat A with respect to the observer D is the resultant of the two vectors as shown in the diagram.



d. **Relative velocity**: ***Relative velocity*** is the vector difference between the velocities of two objects in the same coordinate system. For example, if the velocities of particles $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\boldsymbol{v_A}$ and $\boldsymbol{v_B}$ respectively in the same coordinate system, then the relative velocity of $\boldsymbol{A}$ with respect to $\boldsymbol{B}$ (also called the velocity of $\boldsymbol{A}$ relative to $\boldsymbol{B}$) is $\boldsymbol{v_A} - \boldsymbol{v_B}$. Conversely the velocity of $\boldsymbol{B}$ relative to $\boldsymbol{A}$ is $\boldsymbol{v_B} - \boldsymbol{v_A}$. The relative velocity vector calculation for both one- and two-dimensional motion are similar.

The velocity vector subtraction ($\boldsymbol{v_A} - \boldsymbol{v_B}$) can be viewed as vector addition ($\boldsymbol{v_A} + (-\boldsymbol{v_B})$).

e. **3-step (parallelogram) vector addition:**

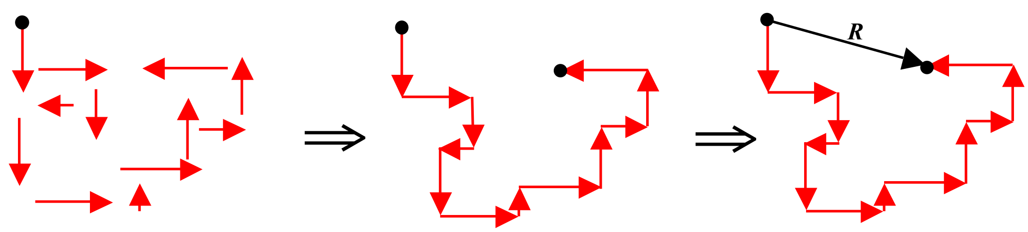


i) Draw two vectors with their tail touching.

ii) Draw a parallel projection of each vector with dashed lines to form a parallelogram.

iii) Draw the diagonal from the point where the two tails are touching.

f. **Head-tail method for vector addition:**



i) Choose a scale and indicate it on paper.

ii) Select a starting point and draw the first vector to scale in the indicated direction. Label the magnitude and direction of the scale on the diagram.

iii) Starting from the head of the first vector, draw the second vector to scale in the indicated direction. Label the magnitude and direction of the vector on the diagram.

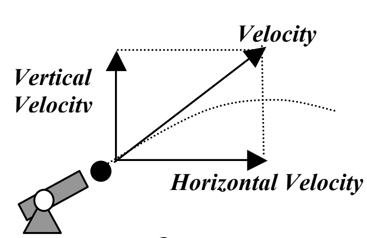
iv) Repeat steps (iii) for all vectors which are to be added. (The order is not important!)

v) Draw the resultant from the tail of the first vector to the head of the last vector.

vi) Using a ruler, measure the length of the resultant and determine its magnitude by converting to real units using the scale.

vii) Measure the direction of the resultant using the counterclockwise convention.

3 Components of Vectors



a. **Component:** Any vector can be resolved into two component vectors at right angle to each other. These two vectors are called the ***components*** of the vector. Any vector can be resolved into ***vertical*** and ***horizontal*** components.

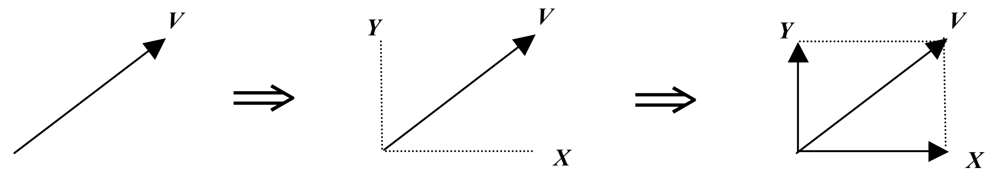
b. **Resolution:** The process of determining the components of a vector is called ***resolution***.

c. **Vector resolution:**

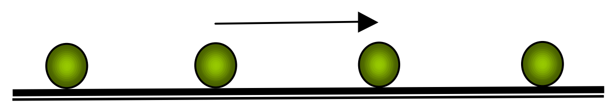
i) Vertical and horizontal lines are drawn from the tail of the vector.

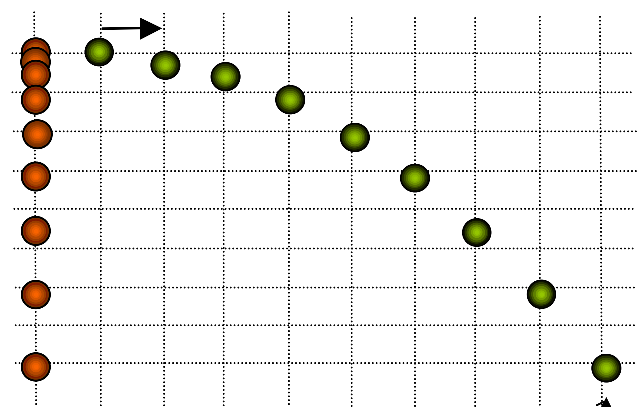
ii) A rectangle is drawn that encloses the vector as its diagonal.

iii) The sides of this rectangle are desired components, vector X and Y.



4 Projectile Motion

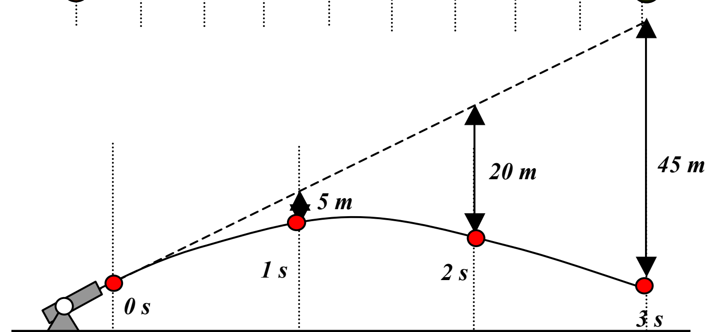




a. **Projectiles:** A cannonball shot from a cannon, a stone thrown into the air, a ball rolling of the edge of a table, a spacecraft circling Earth— all of these are examples *of* ***projectile.***

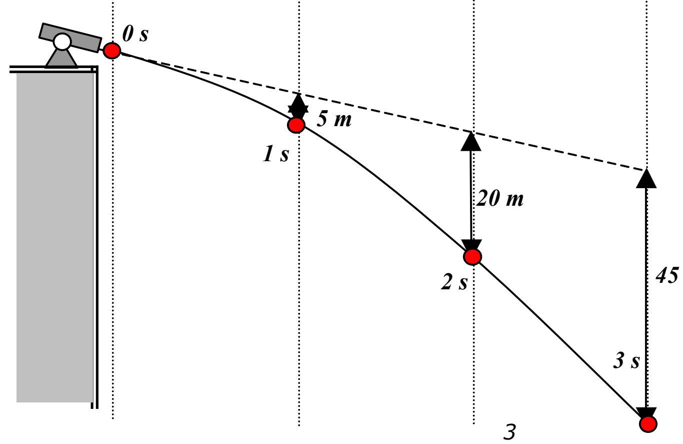
b. **Projectile motion:** When an object is thrown or lunched with initial velocity near Earth's surface, it experiences a constant vertical ***gravitational force*** and thereafter travels in a trajectory subject only to the force of gravity. Motion under these conditions is called ***projectile motion***.

c. The study of projectile motion is simplified because the motion can be treated as two mutually independent, perpendicular motions, one ***horizontal*** and the other ***vertical***.



d. The ***horizontal component*** of motion for a projectile is at ***constant speed***. It is like a rolling ball moving freely along a level surface. When friction is neglectable, the ball moves at constant velocity. The ball covers equal distances in equal intervals of time.

e. The ***vertical component*** of motion for a projectile is like a free falling object. There is a downward acceleration. The vertical component of velocity changes with time and causes a greater distance to be covered in each successive equal time interval.



f. For horizontally lunched projectiles, the time they take to reach the ground are only determined by the vertical motions, and nothing to do with the horizontal velocities. So, they will all take the same time to reach the ground.

g. The horizontal component of motion for a projectile is completely ***independent*** of the vertical component of motion. The only common factor links them together is the time.

h. The path traced by a projectile accelerating only in the vertical direction while moving at constant horizontal velocity is a parabola.

i. ***Reflection Question***: *Mathematically prove that the trajectory of a projectile is actually a parabola (i.e., a quadratic equation).*

5 Upwardly Launched Projectiles

a. **No gravity:** A cannonball shot from a cannon at an upward angle. If there is no gravity, the cannon ball will follow the ***straight-line path*** shown by the dashed line.

b. **With gravity:** The cannon ball will continually fall beneath this imaginary dashed line until it hits the ground. The ***vertical distance*** it falls beneath any point on the dashed line is the same ***vertical distance it would fall if it were dropped from rest*** and had been falling for the same amount of time. The falling distance can be calculated by the following formula (assume g = 10 m/s^2)

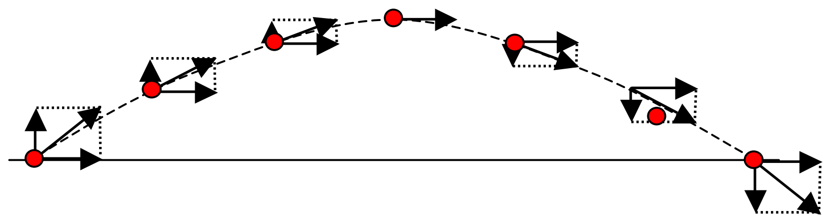
$$d = \frac{1}{2}gt^2 = 5t^2$$

c. **Horizontal velocity:** Since the cannon ball has ***no horizontal acceleration***, the horizontal component of velocity is always the same. The cannon ball moves equal distance in equal time interval. The only acceleration is the vertical acceleration due to gravity. The traveling distance can be calculated by the following formula

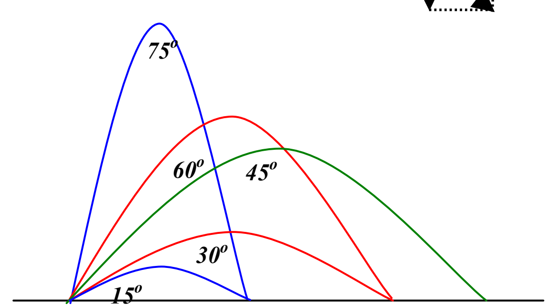
$$d = vt$$

d. **Downward launched projectile:** If the cannon were aimed downward instead of upward, the projectile ***displacements*** below the dashed line would be no difference.

e. **Vector representation:** The horizontal component and vertical components of velocity for a projectile on a parabolic path are shown in the graph. The horizontal component is always the same, and that only the vertical component changes. The resultant velocity vector is maximized at both ends and is a minimum at the top of the path. The velocity at the top of the path is equal to the horizontal component of velocity.



f. **Launching angles:** When a projectile is launching at ***same speed*** but at ***different angles***, the projectile will reach ***different vertical heights*** and travel ***different horizontal distances*** (***horizontal ranges***). When the launching angle is ***45°***, the projectile will reach its maximum horizontal range. The same range can be obtained by two different angles— $\boldsymbol{\theta}$ ***and*** $\boldsymbol{(90^o - \theta)}$. The projectile with smaller angle will remain in the air for a shorter time.



Proof: *Assume* $\boldsymbol{v}$ *is the initial speed of the projectile*

$\boldsymbol{\theta}$ *is the initial angle of the projectile*

$\boldsymbol{v_x}$ *is the horizontal component of* $\boldsymbol{v}$

$\boldsymbol{v_y}$ *is the vertical component of* $\boldsymbol{v}$

$\boldsymbol{d}$ *is the horizontal range of the projectile*

Then, use vector resolution and formulas of motion, we got:

$$\begin{cases} v_x = v\cos\theta \text{...........}(1) \\ v_y = v\sin\theta \text{...........}(2) \end{cases}$$

$$\begin{cases} d = v_x t \text{.................}(3) \\ 0 = v_y t - \frac{1}{2}gt^2 \text{........}(4) \end{cases}$$

Bases on (4) and solve for t, we got:

$$t(v_y - \tfrac{1}{2}gt) = 0$$

$$t = 0 \quad or \quad t = \frac{2v_y}{g}$$

So, $t = \frac{2v_y}{g}$ ....(5)

(5), (1) & (2) all plug into (3), and apply the double angle formula, then,

$$d = v_x(\frac{2v_y}{g}) = (v\cos\theta)(\frac{2v\sin\theta}{g}) = \frac{2v^2\sin\theta\cos\theta}{g} = \frac{v^2(2\sin\theta\cos\theta)}{g} = \frac{v^2}{g}\sin 2\theta$$

So, $d = \frac{v^2}{g}\sin 2\theta$ ....(6)

Since v and g are both constants, and the value of sine function is between [–1, 1], for d to have maximum value, we need to make sin2θ = 1, i.e., 2θ = 90° or θ = 45°.

So, when θ = 45°, the horizontal range is maximum and the value is $d = \frac{v^2}{g}$.

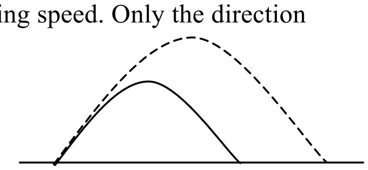
Since $d = \frac{v^2}{g}\sin 2\theta$, *if we replace θ by* $(\frac{\pi}{2} - \theta)$, *then*

$$d = \frac{v^2}{g}\sin 2(\frac{\pi}{2} - \theta) = \frac{v^2}{g}\sin(\pi - 2\theta) = \frac{v^2}{g}\sin 2\theta,$$

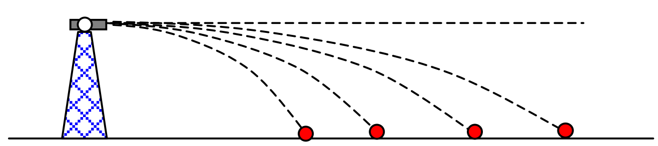
So, the same range can be obtained by two different angles— θ and (90° – θ).

g. **Falling speed**: For a projectile, the rising speed is the same as the falling speed. Only the direction is opposite.

h. **Air resistance**: When there is air resistance, the path of a projectile is no longer a parabola. For example, a batted baseball will travel only about 60% as far as in air as it would in a vacuum.



6 Fast-Moving Projectiles— Satellites



a. **Falling distance:** A cannonball shot horizontally from a cannon will follow the straight-line path if there is no gravity. But there is gravity and the ball falls below the straight-line path. The falling distances in certain time period are all the same for different horizontal speeds.

b. **Fast-moving projectiles:** When the cannonball is very fast, the curvature of Earth came into play. If it were shot fast enough so that its curved path matched the curve of Earth's surface, the ball would fall around Earth and become an Earth satellite (assume there is no air resistance).

c. **Satellite speed:** At 8 km/s, a projectile will travel fast enough to orbit Earth. Unfortunately, the air friction will burn any projectile at such speed. That's why satellites will be launched to altitude above 150 km to be free from the air resistance.

7 Two-Dimensional/Projectile Motion Example Problems

For all the following problems, assume the air resistance can be neglected. The acceleration due to gravity is $\boldsymbol{g}$.

a. A riverboat was to head straight north across a river with speed $\boldsymbol{v}$ while the river current's speed is $\boldsymbol{r}$ toward east. If the river's width is $\boldsymbol{d}$, (a) how long will it take for the boat to cross the river? (b) How far apart from the point straight across the river will the boat reach? (c) How far will the boat actually travel to cross the river? (d) If the riverboat were to go back to the original starting point with the same amount of time, what's the velocity is required in terms of speed $\boldsymbol{v}$ and angle $\boldsymbol{\theta}$?

b. A riverboat were to cross a river and reach the point straight north across the river in time $\boldsymbol{t}$. If the river current's speed is $\boldsymbol{r}$ toward the east and the width of the river is $\boldsymbol{d}$, what's the velocity of the riverboat is required in terms of speed $\boldsymbol{v}$ and angle $\boldsymbol{\theta}$?

c. A cannon ball was launched from a level ground with velocity $\boldsymbol{v}$ at angle $\boldsymbol{\theta}$ and fell back to the ground. Find (a) the maximum height $\boldsymbol{h}$ it will reach, (b) the flying time $\boldsymbol{t}$ of the cannon ball, (c) the horizontal distance $\boldsymbol{d}$ it will reach, and (d) the final velocity $\boldsymbol{v_f}$ right before hitting the ground.

d. A cannon ball was launched horizontally with velocity $\boldsymbol{v}$ from a cliff of height $\boldsymbol{h}$ and fell to the ground below the cliff. Find (a) the flying time $\boldsymbol{t}$ of the cannon ball, (b) the horizontal distance $\boldsymbol{d}$ it will reach, and (c) the final velocity $\boldsymbol{v_f}$ right before hitting the ground.

e. A cannon ball was launched upward from a cliff of height $\boldsymbol{h}$ with velocity $\boldsymbol{v}$ at angle $\boldsymbol{\theta}$ and fell to the ground below the cliff. Find (a) the maximum height $\boldsymbol{k}$ it will reach, (b) the flying time $\boldsymbol{t}$ of the cannon ball, (c) the horizontal distance $\boldsymbol{d}$ it will reach, and (d) the final velocity $\boldsymbol{v_f}$ right before hitting the ground.