Integration that leads to logarithm functions

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The derivative of $\ln x$ is $\frac{1}{x}$. As a consequence, if we reverse the process, the integral of $\frac{1}{x}$ is $\ln x + c$. In this unit we generalise this result and see how a wide variety of integrals result in logarithm functions.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- recognise integrals in which the numerator is the derivative of the denominator.
- rewrite integrals in alternative forms so that the numerator becomes the derivative of the denominator.
- recognise integrals which can lead to logarithm functions.

Contents

1. Introduction

We already know that when we differentiate $y = \ln x$ we find $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{x}$. We also know that if we have $y = \ln f(x)$ and we differentiate it we find $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{f'(x)}{f(x)}$.

The point is that if we recognise that the function we are trying to integrate is the derivative of another function, we can simply reverse the process. So if the function we are trying to integrate is a quotient, and if the numerator is the derivative of the denominator, then the integral will involve a logarithm:

$$\text{if} \quad y = \ln f(x) \qquad \text{so that} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f'(x)}{f(x)}$$

and, reversing the process,

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln(f(x)) + c.$$

This procedure works if the function $f(x)$ is positive, because then we can take its logarithm. What happens if the function is negative? In that case, $-f(x)$ is positive, so that we can take the logarithm of $-f(x)$. Then:

$$\text{if} \quad y = \ln(-f(x)) \qquad \text{so that} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-f'(x)}{-f(x)} = \frac{f'(x)}{f(x)}$$

and, reversing the process,

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln(-f(x)) + c$$

when the function is negative.

We can combine both these results by using the modulus function. Then we can use the formula in both cases, or when the function takes both positive and negative values (or when we don't know).

Key Point

To integrate a quotient when the numerator is the derivative of the denominator, we use

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln|f(x)| + c.$$

2. Some examples

Example

Find $\int \tan x \, \mathrm{d}x$.

Recall that we can rewrite $\tan x$ as $\dfrac{\sin x}{\cos x}$. Observe that the derivative of $\cos x$ is $-\sin x$, so that the numerator is very nearly the derivative of the denominator. We make it so by rewriting $\dfrac{\sin x}{\cos x}$ as $-\dfrac{-\sin x}{\cos x}$ and the integral becomes

$$\begin{aligned} \int \tan x \, \mathrm{d}x &= \int \frac{\sin x}{\cos x} \mathrm{d}x \\ &= -\int \frac{-\sin x}{\cos x} \mathrm{d}x \\ &= -\ln|\cos x| + c. \end{aligned}$$

This result can be written in the alternative form $\ln|\sec x| + c$.

Example

Find $\int \dfrac{x}{1+x^2} \mathrm{d}x$.

The derivative of the denominator is $2x$. Note that the numerator is not quite the derivative of the denominator, but we can make it so by rewriting $\dfrac{x}{1+x^2}$ as $\dfrac{1}{2} \cdot \dfrac{2x}{1+x^2}$. Then

$$\begin{aligned} \int \frac{x}{1+x^2} \mathrm{d}x &= \frac{1}{2} \int \frac{2x}{1+x^2} \mathrm{d}x \\ &= \frac{1}{2} \ln|1+x^2| + c. \end{aligned}$$

Example

Find $\int \dfrac{1}{x \ln|x|} \mathrm{d}x$.

Remember that the derivative of $\ln|x|$ is $\dfrac{1}{x}$. So we rewrite the integrand slightly differently: $\dfrac{1}{x \ln|x|} = \dfrac{1/x}{\ln|x|}$. Now the numerator is the derivative of the denominator. So

$$\begin{aligned} \int \frac{1}{x \ln|x|} \mathrm{d}x &= \int \frac{1/x}{\ln|x|} \mathrm{d}x \\ &= \ln|\ln|x|| + c. \end{aligned}$$

Example

Find $\displaystyle\int \frac{x\cos x + \sin x}{x\sin x}\mathrm{d}x$.

First of all think about what we would obtain if we differentiated the denominator: let's do this first. If $y = x\sin x$, then using the product rule of differentiation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\cos x + \sin x.$$

So we see that in the integral we are trying to find, the numerator is the derivative of the denominator.

So

$$\int \frac{x\cos x + \sin x}{x\sin x}\mathrm{d}x = \ln|x\sin x| + c.$$

Exercises

1. Determine each of the following integrals

   a) $\displaystyle\int \frac{3}{2+3x}\,\mathrm{d}x$ b) $\displaystyle\int \frac{x}{1+2x^2}\,\mathrm{d}x$ c) $\displaystyle\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}\,\mathrm{d}x$

   d) $\displaystyle\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}-1}\,\mathrm{d}x$ e) $\displaystyle\int \cot x\,\mathrm{d}x$ f) $\displaystyle\int \frac{x^{-3}}{x^{-2}+4}\,\mathrm{d}x$

2. For each of the following integrals, use the result $\displaystyle\int \frac{f'}{f} = \ln|f| + c$ to determine the integral. Then repeat the integral, using algebra to simplify the integrand before integration. Check that the two answers obtained are the same.

   a) $\displaystyle\int \frac{3x^2}{x^3}\,\mathrm{d}x$ b) $\displaystyle\int \frac{\cos x\,\mathrm{e}^{sinx}}{\mathrm{e}^{\sin x}}\,\mathrm{d}x$ c) $\displaystyle\int \frac{4x^{-5}}{x^{-4}}\,\mathrm{d}x$

   d) $\displaystyle\int \frac{x^{-1/2}}{2x^{1/2}}\,\mathrm{d}x$ e) $\displaystyle\int \frac{5\mathrm{e}^{5x}}{e^{5x}}\,\mathrm{d}x$ f) $\displaystyle\int \frac{x^{-5/2}}{x^{-3/2}}\,\mathrm{d}x$

Answers

1. a) $\ln|2+3x| + c$ b) $\frac{1}{4}\ln\left|1+2x^2\right| + c$ c) $\frac{1}{2}\ln\left|\mathrm{e}^{2x}+1\right| + c$

   d) $\frac{1}{2}\ln\left|\mathrm{e}^{2x}-1\right| + c$ e) $\ln|\sin x| + c$ f) $-\frac{1}{2}\ln\left|x^{-2}+4\right| + c$

2. a) $\ln\left|x^3\right| + c = 3\ln|x| + c$ b) $\ln\left|\mathrm{e}^{\sin x}\right| + c = \sin x + c$

   c) $-\ln\left|x^{-4}\right| + c = 4\ln|x| + c$ d) $\ln\left|x^{1/2}\right| + c = \frac{1}{2}\ln|x| + c$

   e) $\ln\left|\mathrm{e}^{5x}\right| + c = 5x + c$ f) $-\frac{2}{3}\ln\left|x^{-3/2}\right| + c = \ln|x| + c$

