Taxicab Geometry

UCI Math Circle
October 24, 2016

In this Math Circle, we will ask a simple question:
What is the *distance* a taxicab driver travels between two points within a city?

Between any given two points, there are usually many paths that connect them. We think of the *distance* between two points as the length of the shortest path connecting them. On a plane, the shortest path connecting two points is given by a line segment. The length of this line segment gives what is called the *Euclidean* distance. The usual way that we think about points, lines and angles on a plane is known as *Euclidean geometry.*

How about the distance as seen by a taxicab driver? Suppose we have a city where the streets are laid out on a square grid. The shortest paths between two points are no longer straight lines. This is because a straight line might go through buildings but a taxicab certainly can not do that! So how does a taxi driver measure the distance travelled between two points?

Adapted from Math Circle worksheet by Olga Radko.

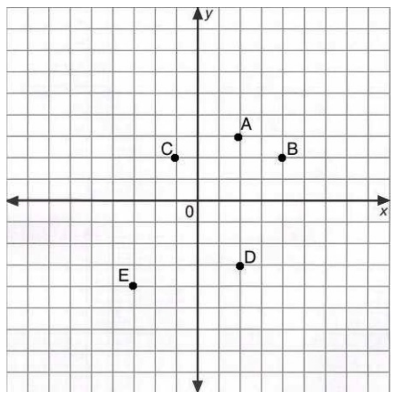
Manhattan Midtown:

Plotting Points

Let our city be a plane that extends infinitely in all directions where

- the center of the city is marked by point O;
- the horizontal (West-East) line going through O is called the x-axis;
- the vertical (South-North) line going through O is called the y-axis;

Example. Point A shown below has coordinates $(2, 3)$.



In the example above, $A = (2, 3)$.

- The first number tells you the distance to the y-axis. The distance is *positive* if you are on the right of the y-axis. The distance is *negative* if you are on the left side of the y-axis.

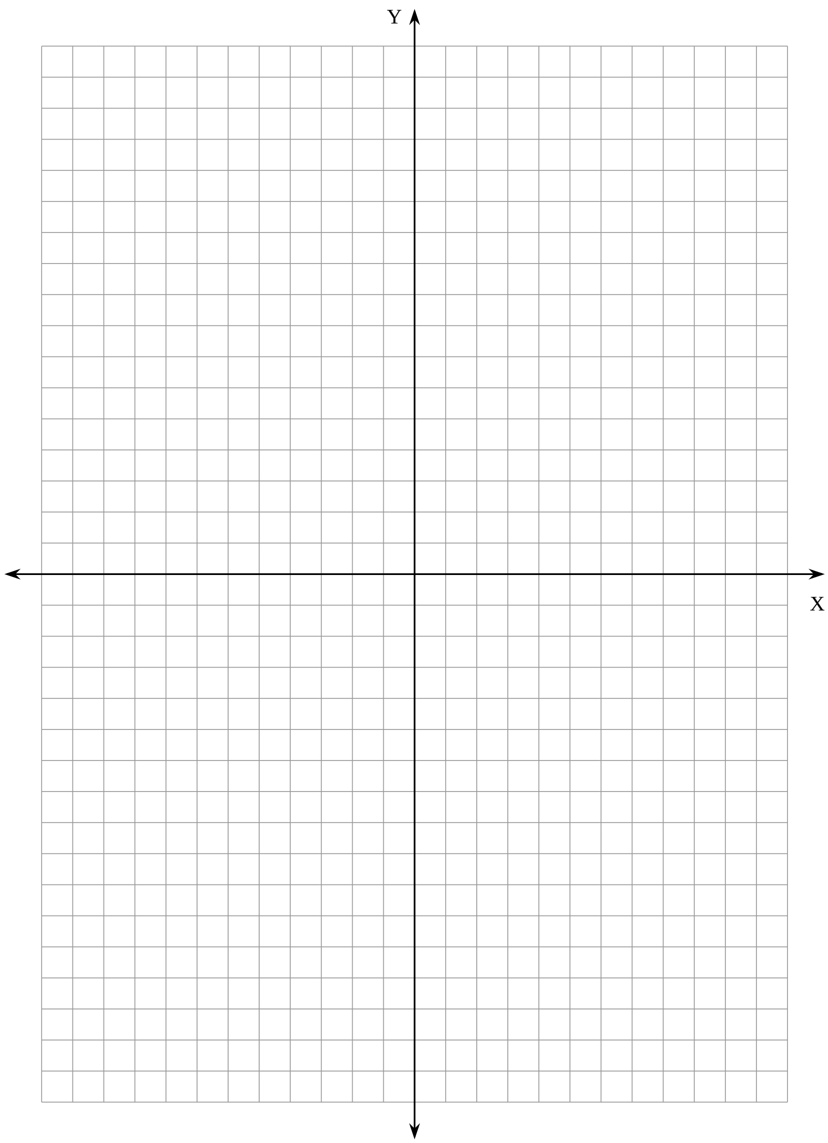
- The second number tells you the distance to the x-axis. The distance is *positive* if you are above the x-axis. The distance is *negative* if you are below the y-axis.

Concept Check.
Problem 1. Find the coordinates of several points in the city above:

1). Point B has address (,); Point C has address (,);

2). Point D has address (,); Point E has address (,).

Problem 2. On the graph above, plot the points with coordinates $(2,-6),(-1,3)$ and $(\frac{1}{2},\frac{3}{2})$.



Problem 3. Refer to the map of Manhattan Midtown, if Time Square is the origin, find the coordinates of the following places

1. Empire State Building: (,);
2. Carnegie Hall: (,);
3. Museum of Modern art: (,).

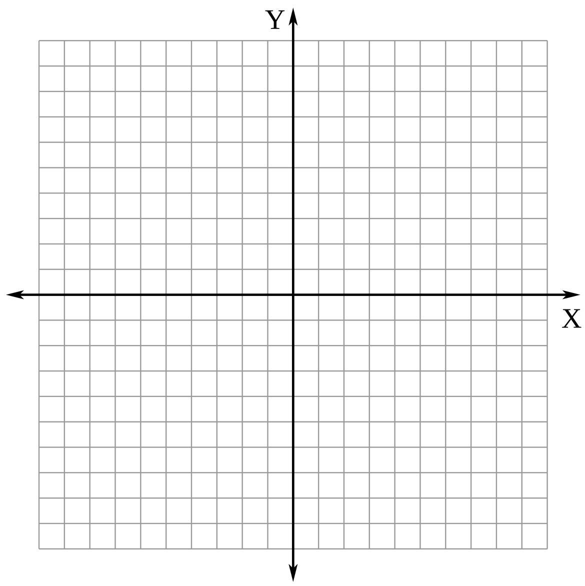
Dispatch the Firetruck

Problem 3. A fire starts at some intersection in the city. There are two firetrucks nearby. Decide which firetruck should be sent to the fire site based on the coordinates of the fire and the current positions of the firetrucks. Remember, the firetrucks can only travel along the vertical or horizontal streets in the city. We also assume that both firetrucks travel with the same speed.

Start by drawing the routes the routes the firetrucks will be taking on the coordinate planes on the next page.

1. Fire site: (0, 0);
   First firetruck: (5,0);
   Second firetruck: (-4, 0).
2. Fire site: (0, 0);
   First firetruck: (5,0);
   Second firetruck: (0, 9).
3. Fire site: (0, 0);
   First firetruck: (4, 3);
   Second firetruck: (2, 6).

Can you give instructions to the fire department dispatcher on how to decide which of the firetrucks should be sent to the fire in general?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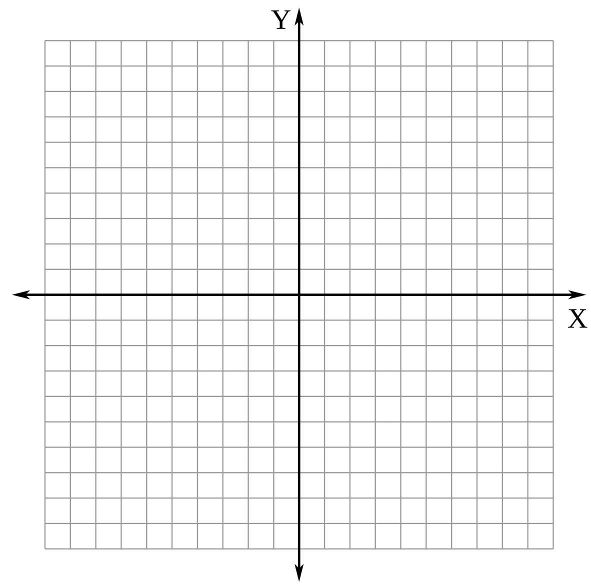
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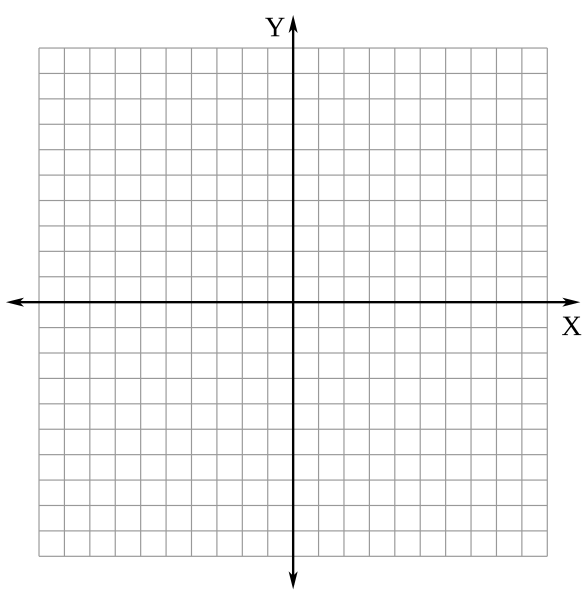
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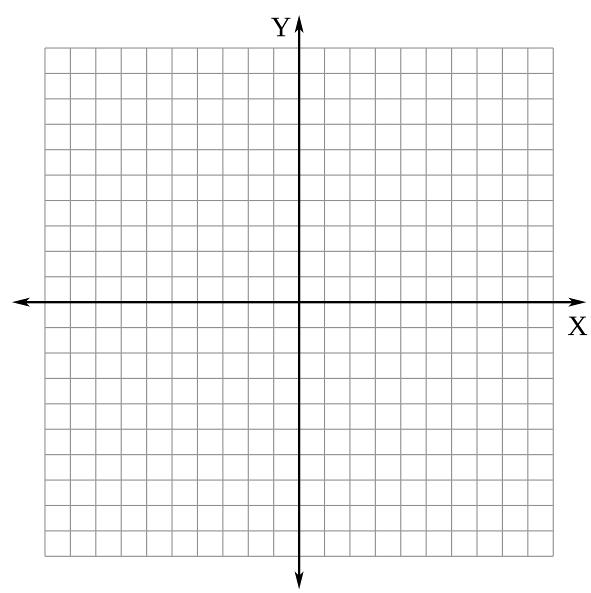
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Taxicab Distance

Imagine that you are only allowed to move along vertical lines and along horizontal lines, such as a city which only has streets running in the north-south and in the east-west direction. Let us call such a route a *taxi route*.

Problem 4.

1. Draw the shortest possible *taxi route* from point $A = (0, 3)$ to point $B = (4, 0)$.

2. Find the length of this route:

$$d_{taxi}(A, B) =$$

3. Is there another taxi route that also gives you the shortest possible distance between the two points?

We shall call the distance between points A and B obtained by going along one of the shortest taxi routes the *taxicab distance*.

Problem 5. Find the taxicab distances between the following points:

1. $(1, 0)$ and $(1, 7)$.

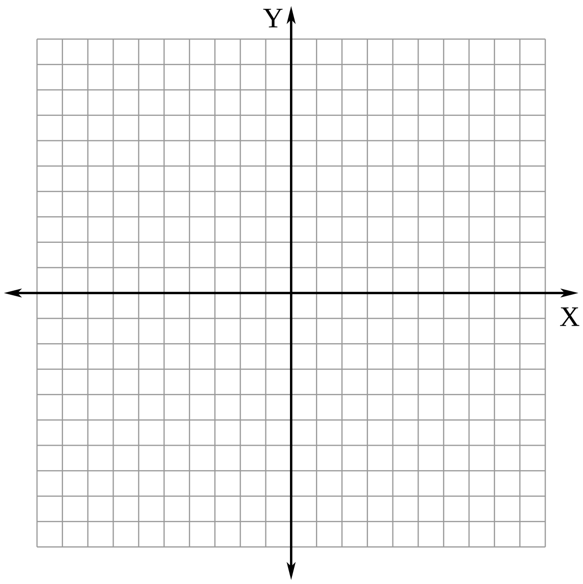
2. $(3, 2)$ and $(5, 2)$.

3. $(1, 3)$ and $(3, 1)$.

4. $(2, 1)$ and $(-2, -1)$.

5. $(\frac{3}{2}, 2)$ and $(\frac{1}{2}, -\frac{5}{2})$.

Problem 6. Can you describe how the taxicab distance is computed in words?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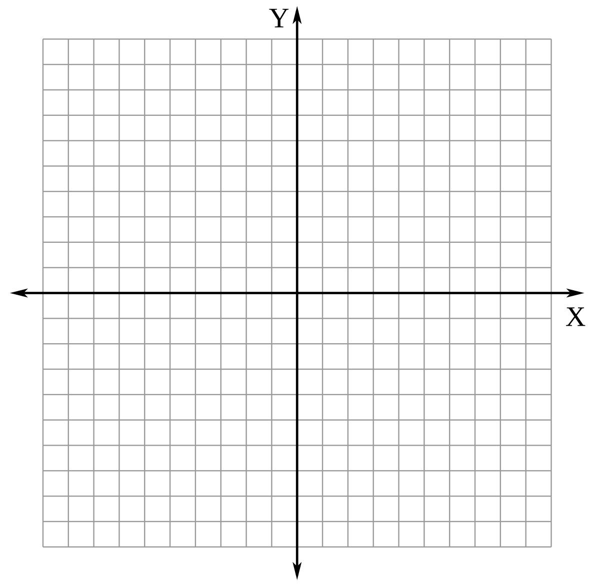
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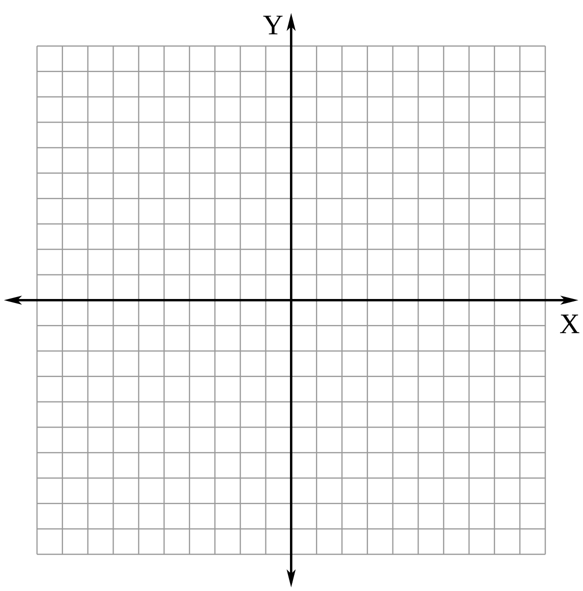
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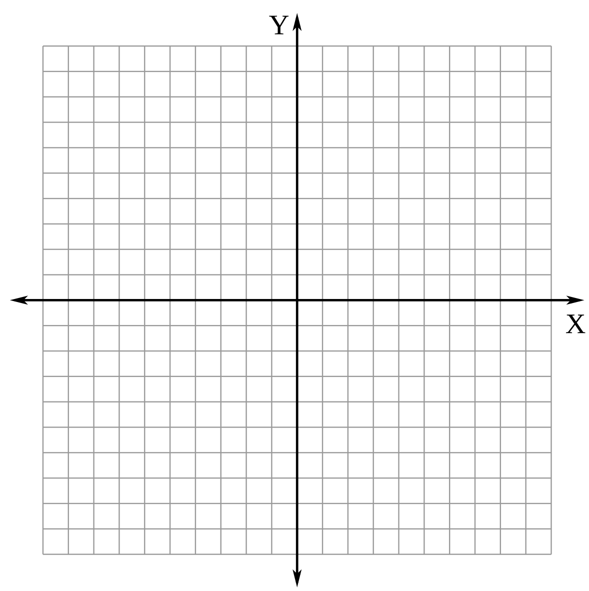
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Taxicab distance *v.s.* Euclidean distance

Taxicab distance.
For points $A = (a, b)$ and $B = (c, d)$, the *taxicab distance* is given by

$$d_{taxi}(A, B) = |a - c| + |b - d|.$$

Here $|a - c|$ is the absolute value of the difference between a and c and $|b - d|$ is the absolute value of the difference between b and d.

Euclidean Distance
Let $A = (a, b)$, $B = (c, d)$, and $C = (c, b)$ be three points on the coordinate plane. Can you see that ΔABC is a right triangle? The lengths of the two shorter sides of the triangle are

$$\begin{aligned} |AC| &= |a - c| \, , \\ |BC| &= |b - d| \, , \end{aligned}$$

where $|...|$ denotes the absolute value. The distance between A and B is the length of the third side, which can be calculated using the *Pythagorean* theorem:

$$\begin{aligned} |AB|^2 &= |AC|^2 + |BC|^2 \\ |AB|^2 &= (a - c)^2 + (b - d)^2 \\ d(A, B) = |AB| &= \sqrt{(a - c)^2 + (b - d)^2}. \end{aligned}$$

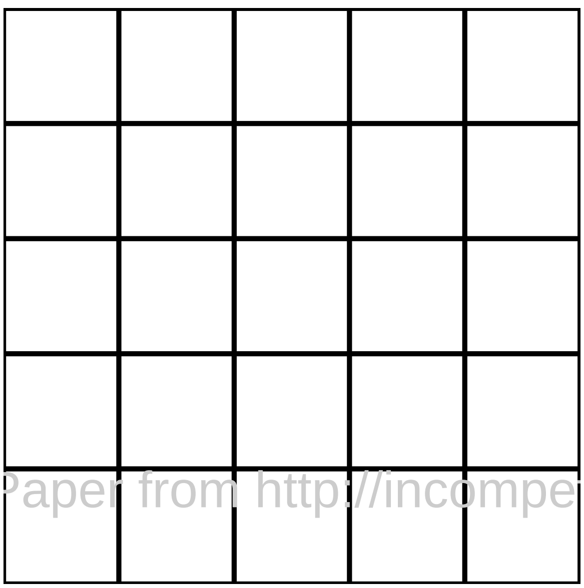
Example. The distance between the points $(4, 0)$ and $(0, 3)$ is

$$\sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5 \, .$$

Problem 7. Let us compare the usual Euclidean distance and the taxicab distance.

1. Give an example of two points such that the Euclidean (usual) distance and the taxicab distances between them are equal to each other.

2. Give an example of two points such that the taxicab distance is bigger than the Euclidean distance.



3. Can you find a pair of points for which the Euclidean distance is bigger?

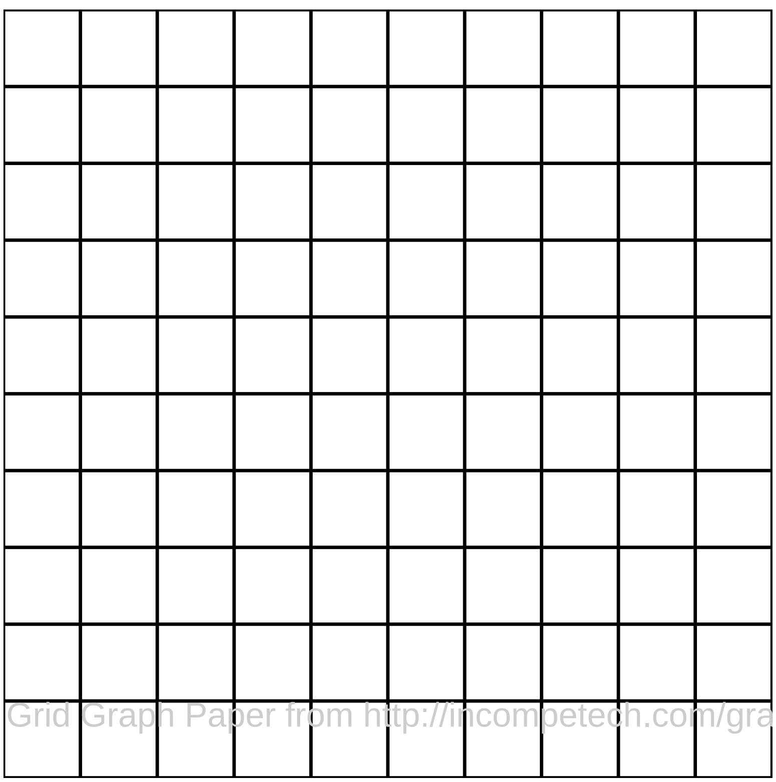
Circles and π in Taxicab Geometry

In plane Euclidean geometry, a *circle* can be defined as the set of all points which are at a fixed distance from a given point. The given point is the *center* of the circle. The fixed distance is the *radius* of the circle. *Diameter* is the longest possible distance between two points on the circle and equals twice the radius. *Circumference* is the length of the circle.

The same definitions of the circle, radius, diameter and circumference make sense in the taxicab geometry (using the taxicab distance, of course). However, taxicab circles look very different.

Problem 8.

1. The taxicab circle centered at the point $(0, 0)$ of radius 2 is the set of all points for which the taxicab distance to $(0, 0)$ equals to 2. Draw the taxicab circle centered at (0, 0) with radius 2.

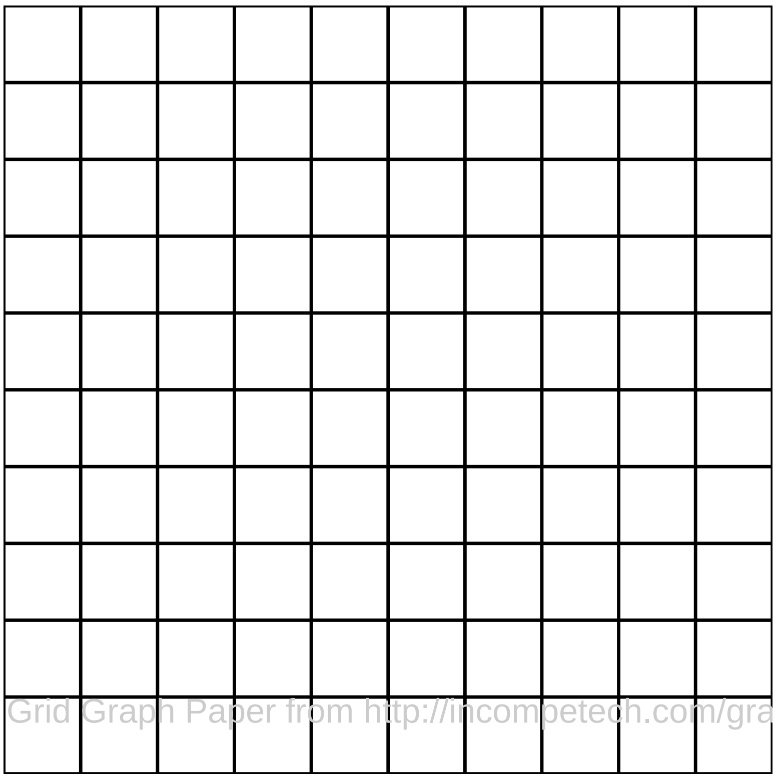


2. What is the shape of this taxicab circle? Do other taxicab circles have the same shape?

In Euclidean geometry, π is defined as the ratio of the circumference to the diameter:

$$\pi = \frac{\text{Circumference}}{\text{Diameter}} = \frac{\text{Circumference}}{2 \times \text{Radius}} = 3.141592\ldots$$

Problem 9. What is the value of π (the ratio of the circumference to the diameter) in the taxicab geometry? To find out, consider a taxicab circle centered at $(0, 0)$. Draw it on the coordinate plane below.



Find the diameter and the taxicab circumference of this circle:

$$\text{Taxicab Diameter} =$$

$$\text{Taxicab Circumference} =$$

Then compute the ratio,

$$\pi_{taxi} = \frac{\text{Taxicab Circumference}}{\text{Taxicab Diameter}} =$$

and compare with the value of π in the usual Euclidean geometry.

Any other interesting distance?

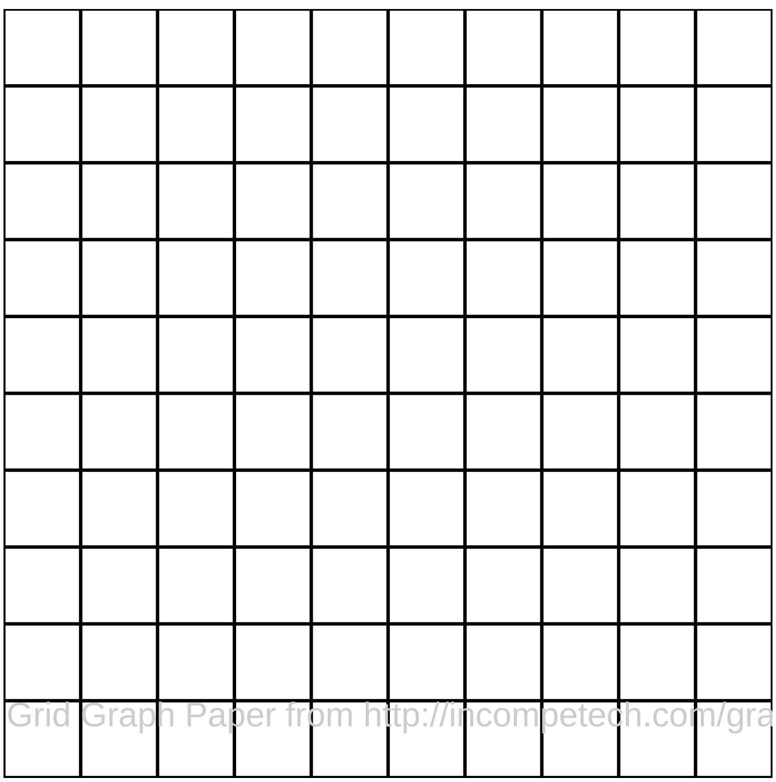
Chebyshev Distance

For points $A = (a, b)$ and $B = (c, d)$, the *taxicab distance* is given by

$$d_{ch}(A, B) = \max\{|a - c|, |b - d|\}.$$

Here $|a - c|$ is the absolute value of the difference between a and c and $|b - d|$ is the absolute value of the difference between b and d.

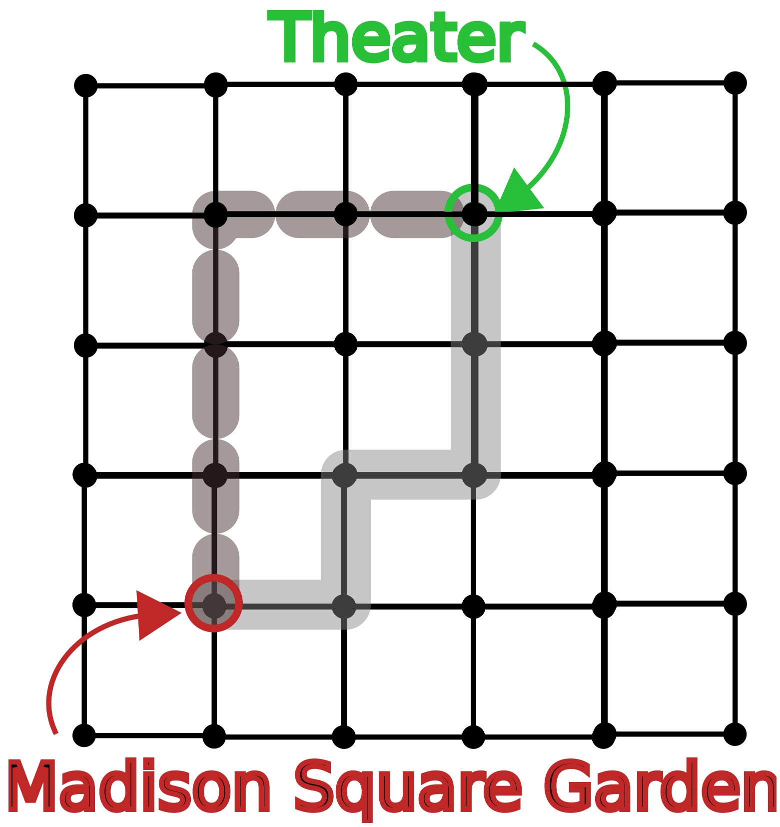
Problem 10. Draw the Chebyshev circle centered at $(0, 0)$ with radius 2. What is the shape of the circles in Chebyshev geometry?



Problem 11. What is the value of π (the ratio of the circumference to the diameter) in the Chebyshev geometry?

Shortest paths in Taxicab Geometry

A cab driver in New York picks up a passenger at Madison Square Garden and asks to travel to a theater which is four blocks north and two blocks east. He is forced to follow the roads, which are laid out on a grid. Note that he will have more than one option for how to travel. Two paths are shown below:



Problem 12. What is the length of the shortest path he can take? How many different paths can you give which have that length?

Problem 13. Express each path you found above in as a series of directions, writing N for moving north, E for moving east. In particular, the dotted path above is NNNEE, the bold path is ENENN.

Problem 14. Now suppose on his next trip the cab driver picks up someone at Madison Square Garden and is asked to take a visitor X blocks east and Y blocks north. What is the shortest length of a path he can take? How many such shortest paths are there? (**Hint:** Think about the process of writing down all paths as in the previous exercise? How many such ways are there?)

Problem 15. Now the cab driver is tasked with driving 5 blocks east and 8 blocks north. However, the intersection 2 blocks east and 3 blocks north is blocked by construction. Now how many paths are available to the driver?