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The Fibonacci Word Fractal

Alexis Monnerot-Dumaine*

February 8, 2009

Abstract

The Fibonacci Word Fractal is a self-similar fractal curve based on the Fibonacci word through a simple and interesting drawing rule. This fractal reveals three types of patterns and a great number of self-similarities. We show a strong link with the Fibonacci numbers, prove several properties and conjecture others, we calculate its Hausdorff Dimension. Among various modes of construction, we define a word over a 3-letter alphabet that can generate a whole family of curves converging to the Fibonacci Word Fractal. We investigate the sturmian words that produce variants of such a pattern. We describe an interesting dynamical process that, also, creates that pattern. Finally, we generalize to any angle.

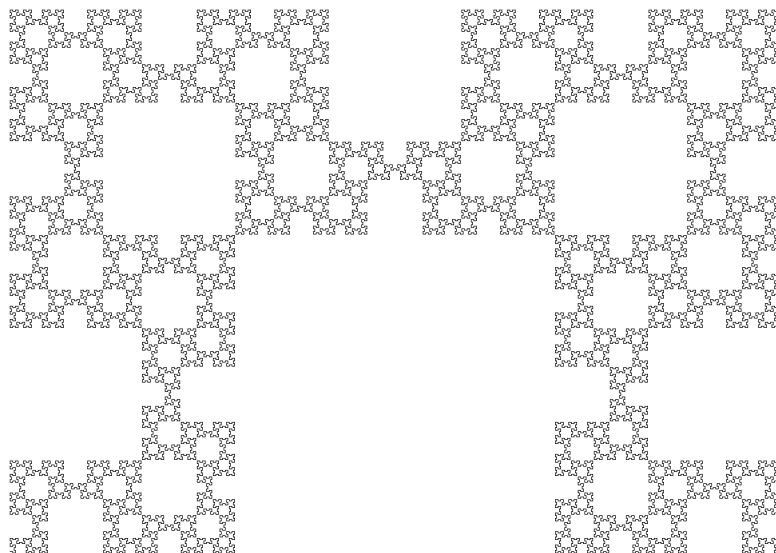


Fig.1: \mathcal{F}_{23} Fibonacci word curve. Illustrates the \mathcal{F}_{3k+2} Fibonacci word fractal

1 The Fibonacci word

The infinite Fibonacci word is a specific infinite sequence in a two-letter alphabet. Let f_1 be "1" and f_2 be "0". Now $f_n = f_{n-1}f_{n-2}$, the contatenation of the two previous terms. It is also defined by the following morphism $\sigma : 0 \rightarrow 01, 1 \rightarrow 0$ with $f_1 = 1$.

The successive Fibonacci words are:

$f_1 : 1$
 $f_2 : 0$
 $f_3 : 01$
 $f_4 : 010$
 $f_5 : 01001$

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$f_6 : 01001010$

$f_7 : 0100101001001$

The infinite Fibonacci word is the limit f_∞ .

If is referenced A003849 in the On-line Encyclopedia of Integer Sequences[15]. its binary complement (sometimes called the "rabbit sequence") is referenced A005614.

2 Construction of the fractal

Take the n^{th} digit of a Fibonacci word,

- draw a segment forward
- If the digit is "0" then :
 - . turn left if "n" is even
 - . turn right if "n" is odd

and iterate

In the rest of this paper, we will call this rule the "odd-even drawing rule".

The first segments are drawn this way: The first digit is "0", so draw a vertical segment and turn right. The second digit is "1" so draw a horizontal segment, the third digit is "0" so continue drawing a horizontal segment and turn right. The fourth digit is "0" so draw a vertical segment and turn left. Continue inductively.

The curve unravels in a fractal pattern. Figure 2 shows the construction of the fractal curves, when the rule is applied to the Fibonacci word f_3 up to the Fibonacci word f_{14} . We will name each curve \mathcal{F}_n from its Fibonacci word f_n . We will keep this notation through the article. Note that the number of segments in \mathcal{F}_n equals Fibonacci number F_n .

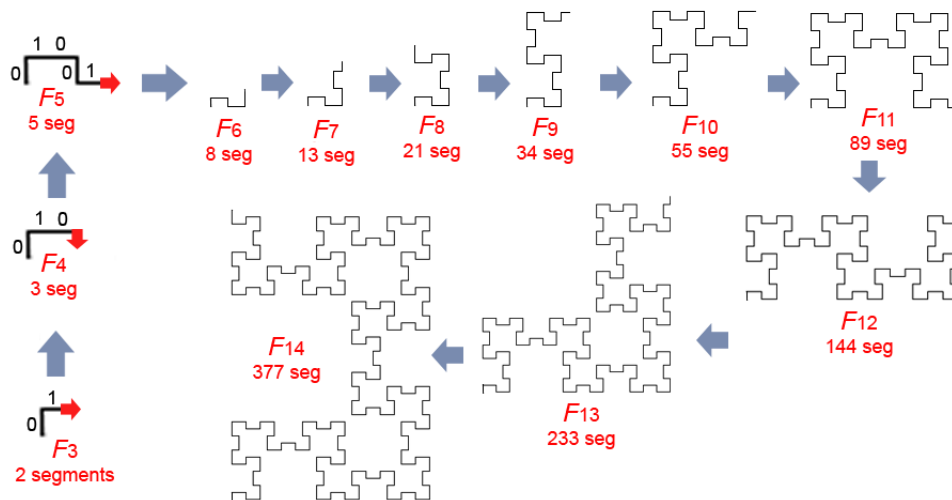


Fig.2:Construction, Fibonacci word after Fibonacci word

Each curve shows characteristic points, and corners, that can be expressed in terms of Fibonacci numbers. For instance, in the \mathcal{F}_{23} , the curve obtained from f_{23} , the upper-left corner is reached after drawing $F_{20} + \frac{F_{19}-1}{2} = 8855$ segments. See figure 3.

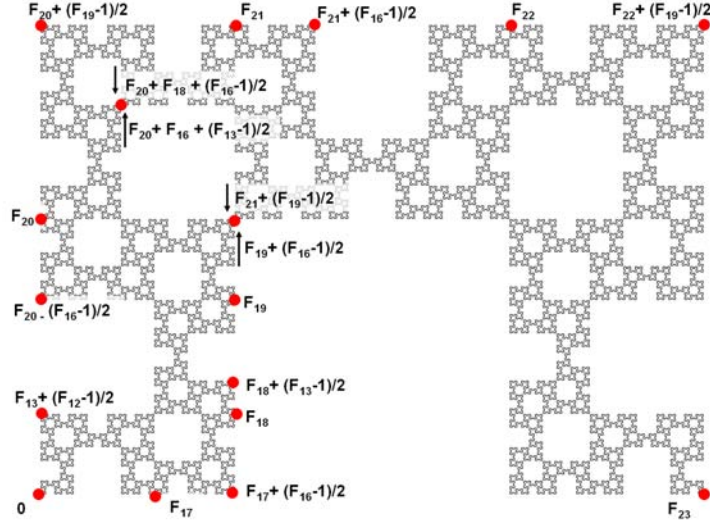


Fig.3:Characteristic points and Fibonacci numbers in the \mathcal{F}_{23} curve

The Fibonacci Word Fractal is defined from the \mathcal{F}_k curve by $\lim_{k \rightarrow \infty} (\mathcal{F}_k)$

3 Useful results about the Fibonacci word

Before presenting some of the properties of the fractal, let us recall or establish some useful results about the Fibonacci word. Most of these properties can be found in [13] or [1], so they are given without any proof.

Property WP1 : The subwords 11 or 000 cannot be found in any Fibonacci word.

Property WP2: Let ab be the last two letters of f_n . For $n \geq 3$, $ab = 01$ if n is odd and $ab = 10$ if n is even. This is proven easily by induction starting from $f_3 = 01$ and $f_4 = 010$ and $f_n = f_{n-1}f_{n-2}$.

Property WP3: If $f_n = p_nab$ with $n > 1$, where the suffix ab represents the last two letters of f_n , then p_n is a palindrome[13]. In other words, removing from f_n its 2 last letters makes it a palindrome. So, for every $n > 1$;

$$f_n = p_nab \quad (1)$$

Property WP4: If $f_n = p_nab$, then $ba\overline{p}_nab$ is also a palindrome, and we write it \overline{p}'_n . this means that adding the last two letters (reversed) of f_n before f_n , makes it a palindrome.

$$\overline{p}'_n = ba\overline{p}_nab = \overline{p}_nab \quad (2)$$

Property WP5: $|f_n|$, the number of letters in f_n , is the Fibonacci number F_n .

Theorem WT1 : let $f_n = p_nab$. We define $t_n = p_nba$ (the two last letters exchange places). Then $f_n = f_{n-2}t_{n-1}$ and $t_n = f_{n-2}f_{n-1}$. This means that when inverting the order of concatenation of f_{n-1} and f_{n-2} , one creates another string similar to f_n , but with the two last letters exchanged. We could say that the concatenation is "almost commutative" for two consecutive Fibonacci words.

Proof : by induction starting by $n=4$.

we have $f_5 = 01001, t_5 = 01010, f_4 = 010, t_4 = 001$ and $f_3 = 01$

$$f_5 = 01001 = (01)(001) = f_3t_4.$$

$$t_5 = 01010 = (01)(010) = f_3f_4$$

Assume $f_n = f_{n-2}t_{n-1}$ and $t_n = f_{n-2}f_{n-1}$ for $n \geq 5$, then

$$f_{n+1} = f_n f_{n-1} = (f_{n-1}f_{n-2})f_{n-1} = f_{n-1}(f_{n-2}f_{n-1}) = f_{n-1}t_n$$

$$\text{and } f_{n-1}f_n = f_{n-1}(f_{n-2}t_{n-1}) = (f_{n-1}f_{n-2})t_{n-1} = f_n t_{n-1} = t_{n+1} \blacksquare$$

Theorem WT2 : For every $n \geq 6$, $f_n = f_{n-3}f_{n-3}f_{n-6}t_{n-3}t_{n-3}$

and, if we consider the palindromes, $p_n = p_{n-3}abp_{n-3}p'_{n-6}p_{n-3}bap_{n-3}$.
 In other words, the f_n word is composed of the concatenation of two p_{n-3} palindromes separated by ab , one p_{n-6} palindrome then two p_{n-3} palindromes separated by ba . This result will help us prove the structure of the Fibonacci word fractal and its inner-symmetries.

Proof:

$$\begin{aligned}
 f_n &= f_{n-1}f_{n-2} \\
 &= (f_{n-2}f_{n-3})(f_{n-3}f_{n-4}) \\
 &= (f_{n-3}f_{n-4})(f_{n-4}f_{n-5})f_{n-3}f_{n-4} \\
 &= f_{n-3}f_{n-4}(f_{n-5}f_{n-6})f_{n-5}(f_{n-4}f_{n-5})f_{n-4} \\
 &= f_{n-3}(f_{n-4}f_{n-5})f_{n-6}(f_{n-5}f_{n-4})(f_{n-5}f_{n-4}) \\
 &= f_{n-3}f_{n-3}f_{n-6}t_{n-3}t_{n-3}. \text{ using theorem WT1. } \blacksquare
 \end{aligned}$$

If we look for the palindromes, then we can write:

$$\begin{aligned}
 f_n &= (p_{n-3}ab)(p_{n-3}ab)f_{n-6}(p_{n-3}ba)(p_{n-3}ba) \text{ using property WP3} \\
 &= p_{n-3}abp_{n-3}(abf_{n-6})p_{n-3}bap_{n-3}ba \\
 &= p_{n-3}abp_{n-3}p'_{n-6}p_{n-3}bap_{n-3}ba \text{ using property WP4} \\
 \text{So } p_n &= p_{n-3}abp_{n-3}p'_{n-6}p_{n-3}bap_{n-3} \text{ using property WP3. } \blacksquare
 \end{aligned}$$

4 Properties of the fractal

Most of these properties are suggested by the figures and need proof. Others come directly from the properties of the Fibonacci word.

1. These properties below are given without any proof, they come trivially from the properties of the Fibonacci Word.
 - There are only "single" or "double" segments, since there cannot be two "1" in a row in the Fibonacci word
 - The number of turns in the \mathcal{F}_n curve is Fibonacci number F_{n-1} , and the number of 'double segments' is Fibonacci number F_{n-2} .
 - Repetitions in the Fibonacci word can be found simply by observing the repetitions of similar patterns in the Fibonacci word curve. For instance, the "left pillar" of the \mathcal{F}_{23} curve is similar to the "right pillar". This implies that the first 6765 (F_{20}) digits of f_{23} can also be found from position 21892 ($=28657-6765$).

2. **Theorem :** The \mathcal{F}_n curve is similar to the \mathcal{F}_{n-3} curve.

Figures suggest self-similarities between \mathcal{F}_n and \mathcal{F}_{n-3} curves:

- In the figure below, \mathcal{F}_{23} looks similar to $\mathcal{F}_{20}, \mathcal{F}_{17}, \mathcal{F}_{14}, \mathcal{F}_{11}$ and so on.

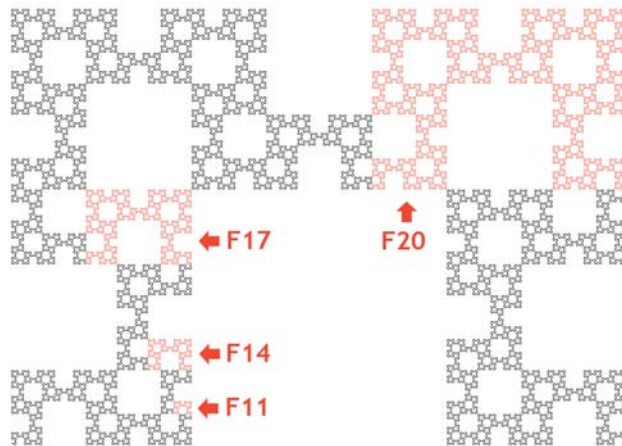


Fig.4:Self-similarities in the \mathcal{F}_{23} fractal curve

- but also \mathcal{F}_{22} looks similar to $\mathcal{F}_{19}, \mathcal{F}_{16}, \mathcal{F}_{13} \dots$
- and \mathcal{F}_{21} looks similar to $\mathcal{F}_{18}, \mathcal{F}_{15}, \mathcal{F}_{12} \dots$

This leads us to define three types of Fibonacci word fractal patterns: $\mathcal{F}_{3k}, \mathcal{F}_{3k+1}$ and \mathcal{F}_{3k+2} :



Fig.5:The three types of Fibonacci word fractals

Proof : The \mathcal{F}_n curve can be constructed iteratively as the Fibonacci word can be constructed iteratively. Can we find a substitution ρ that transforms the f_n word into another f_n word and guaranties the odd-even alternation required by the odd-even drawing rule ?

First of all, we must consider grouping the letters two by two, to handle more conveniently the odd-even alternation. In that respect, the substitution σ , that generates the infinite Fibonacci Word, can be written : $\sigma(00) = 0101$, $\sigma(01) = 010$ and $\sigma(10) = 001$. (Note that the factor '11' cannot be encountered in the Fibonacci word. A factor is a sequence of letters in a word. It is sometimes called "subword").

But σ cannot be the substitution we are looking for because, if w is a word of even length, $|\sigma(w)|$ can be odd. We can make the same observation for σ^2 .

The smallest substitution that guaranties the odd-even alternation is actually σ^3 as we can see:

$$\sigma^3(00) = 0100101001, \sigma^3(01) = 01001010 \text{ and } \sigma^3(10) = 01001001.$$

So, for every word w of even length, $|\sigma^3(w)|$ is always even. And, since we still have $\sigma^3(0) = 01001$, $\sigma^3(1) = 010$, for every word w of odd length, $|\sigma^3(w)|$ is always odd. Since σ^3 preserves the parity of length then, as a corollary, for any factor (subword) in the Fibonacci word, it preserves the parity of position.

But that is not enough. To have a similarity, the resulting angle of a pattern must be preserved by this substitution or, at least, inverted.

The resulting angle of a word w can also be defined as the change in angle from the first to last angle of the curve generated from the word w . Let's define $a(w)$ the function that gives the resulting angle of a word w through the odd-even drawing rule of angle α . For example, $a(00) = 0$ because it is drawn : draw segment, turn right, draw segment and turn left.

So we can write:

$$\begin{aligned} a(00) = 0 & \quad \text{and} \quad a(\sigma^3(00)) = a(0100101001) = 0 \\ a(01) = -\alpha & \quad \text{and} \quad a(\sigma^3(01)) = a(01001010) = \alpha \\ a(10) = \alpha & \quad \text{and} \quad a(\sigma^3(10)) = a(01001001) = -\alpha \end{aligned}$$

σ^3 inverts the resulting angle, as it is straightforward that, for any word w , $a(w) = -a(\sigma^3(w))$. This implies that the image of a pattern by σ^3 is the symmetrical of this pattern by an axial symmetry. Since f_n is the image of f_{n-3} through σ^3 , the \mathcal{F}_n curve is similar to the \mathcal{F}_{n-3} curve. ■

3. **The structure of the curve** : Let's establish some results about the structure of the fractal. See theorem WT2 above which states that:

$$p_n = p_{n-3} a b p_{n-3} p'_{n-6} p_{n-3} b a p_{n-3} \tag{3}$$

with $f_{n-3} = p_{n-3}ab$ and $p'_{n-6} = abf_{n-6}$.

We'll examine the structure for the F_{23} curve, we can extend this easily to other values of n . So we have $p_{23} = p_{20}10p_{20}p'_{17}p_{20}01p_{20}$

The following table shows the link between each factor and the corresponding pattern in the curve:

		factorization of f_{23}							
Word	$f_{23} =$	f_{20}		f_{20}	f_{17}	t_{20}		t_{20}	
	starts at position	p_{20} odd	10 even	p_{20} even	p'_{17} odd	p_{20} even	01 odd	p_{20} odd	01 even
Geometry	draw pattern...	P_{20}	2 seg. turn right	sym. of P_{20}	P'_{17}	sym. of P_{20}	2 seg. turn right	P_{20}	2 seg. turn left
	...and turn	0	0	0	0	0	0	0	0

Fig.6:Structure of the \mathcal{F}_{23} curve

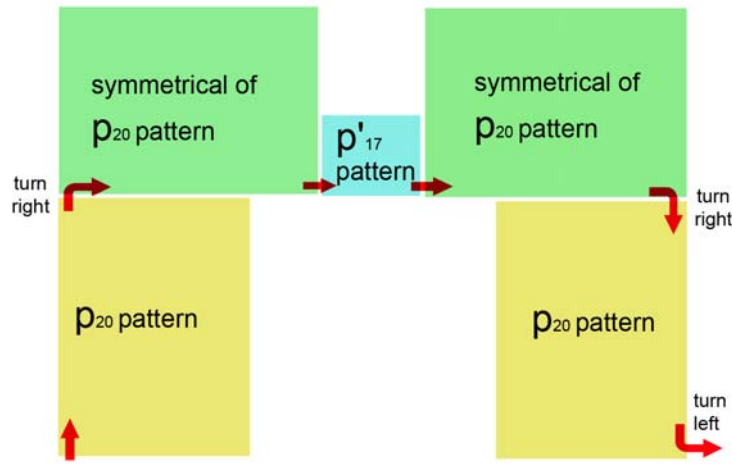


Fig.7:Structure of the \mathcal{F}_{23} curve

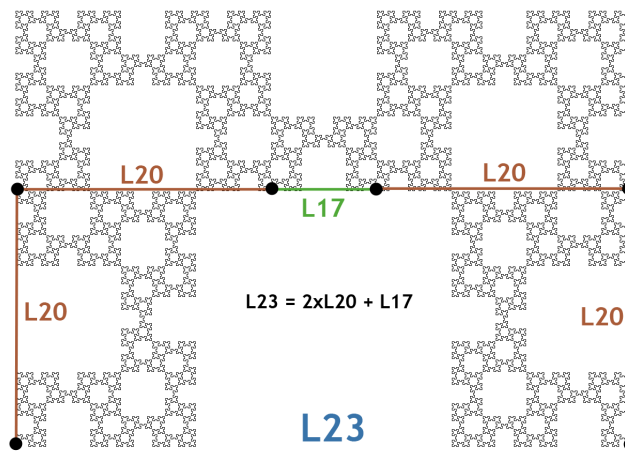
Note that the resulting angle of p_{20} and p'_{17} is zero because they are all palindroms of even length.

4. **Theorem :** The scale factor between \mathcal{F}_n and \mathcal{F}_{n-3} is $1 + \sqrt{2}$.

Proof : We show that \mathcal{F}_n is the concatenation of 4 copies of \mathcal{F}_{n-3} and one copy of \mathcal{F}_{n-6} .

Let us define L_n the length of fractal \mathcal{F}_n from first to last point drawn.

Since ab is either "01" or "10", we deduce that first two copies of \mathcal{F}_{n-3} are always orthogonal. Same deduction for the last two copies. see figure below.



$$\text{Fig.8: } L_n = 2L_{n-3} + 2L_{n-3}\cos(90) + L_{n-6} = 2L_{n-3} + L_{n-6}$$

so :

$$L_n = 2L_{n-3} + 2L_{n-3}\cos(90) + L_{n-6} = 2L_{n-3} + L_{n-6} \quad (4)$$

On the other hand, by definition, the scale factor B is :

$$B = \frac{L_n}{L_{n-3}} = \frac{L_{n-3}}{L_{n-6}} \quad (5)$$

Using the two previous identities, B must satisfy :

$$BL_{n-3} = 2L_{n-3} + \frac{L_{n-3}}{B} \quad (6)$$

then the scale factor B must be a solution of the equation :

$$B^2 = 2B + 1 \quad (7)$$

$$B = 1 + \sqrt{2}. \quad (8)$$

■

5. **Theorem :** The ratio length / width of the \mathcal{F}_{3k+2} rectangle, as k tends to infinity, equals $\sqrt{2}$.

Proof : The theorem WT2 above establishes that, for the Fibonacci word:

$$p_n = p_{n-3}abp_{n-3}p'_{n-6}p_{n-3}bap_{n-3} \quad (9)$$

so the first self-similarities (P_{n-3} palindromes) are separated by a square angle since ab means either 01 or 10 (see property WP2). so the height of the \mathcal{F}_n fractal (H_n) equals the height of a \mathcal{F}_{n-3} (H_{n-3}) + the length of another \mathcal{F}_{n-3} (L_{n-3}).

$$H_n = H_{n-3} + L_{n-3} \quad (10)$$

On the other hand, the reduction ratio, or scale factor between \mathcal{F}_n and \mathcal{F}_{n-3} equals $1 + \sqrt{2}$. (See theorem above). So:

$$H_n = \frac{H_n}{1 + \sqrt{2}} + \frac{L_n}{1 + \sqrt{2}} \quad (11)$$

This leads to :

$$\frac{L_n}{H_n} = \sqrt{2} \quad (12)$$

■

The other dimensions below derive from the scale factor and the self-similarities.

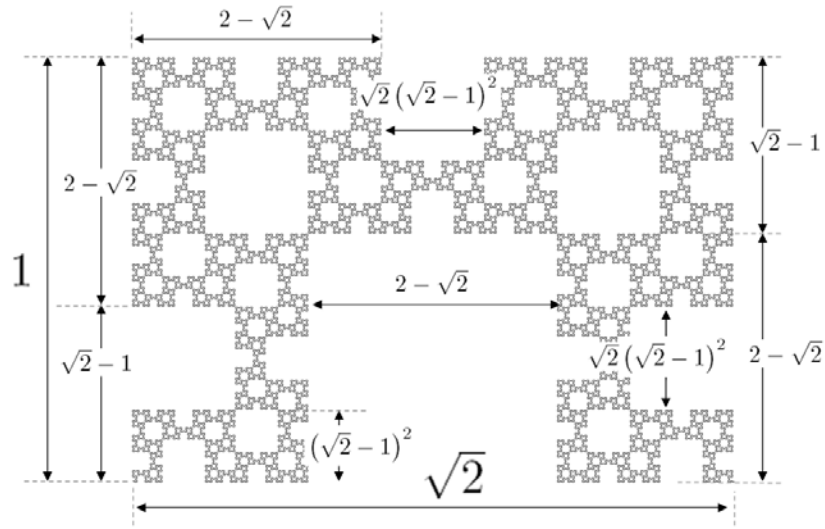


Fig.9: The dimensions of the fractal, at the limit

Note that $\sqrt{2} - 1 = \frac{1}{\sqrt{2} + 1}$, inverse of the scale factor.

6. **Theorem** : The \mathcal{F}_n fractal has a central symmetry if Fibonacci number F_n is even, and an axial symmetry if it is odd. More precisely :

- (a) The \mathcal{F}_{3k} shows a central symmetry,
- (b) The \mathcal{F}_{3k+1} a diagonal axial symmetry,
- (c) and the \mathcal{F}_{3k+2} an orthogonal axial symmetry.

Proof : Those symmetries come from the p_n palindrome : $f_n = p_n ab$ then p_n is a palindrome (see property WP3 above).

The last two letters ab represent the last segment of \mathcal{F}_n . At the limit, this last segment becomes negligible.

If Fibonacci number F_n is even, then $|f_n|$ is even and $|p_n|$ is even. Then, for a zero placed in an odd position $2i + 1$ in p_n we can find the symmetrical zero in position in $F_n - 2 - (2i + 1)$ which is also even. If Fibonacci number F_n is odd, then the symmetrical zero is placed in an odd position. ■

7. We encounter the Fibonacci numbers even outside the fractal. The \mathcal{F}_{3k+2} fractal shows white squares at variable scales (ratio $1 + \sqrt{2}$). The first one is obvious, between the 'pillars'. Its side equals $\sqrt{2}(\sqrt{2} - 1)$. Then we count 5 smaller ones, then 21 even smaller, then 89 and so on ... These are Fibonacci numbers $F_2, F_5, F_8, F_{11} \dots$

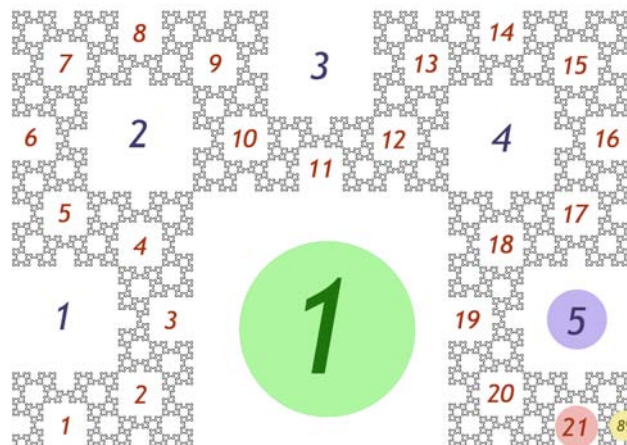


Fig.10: The white squares follow the Fibonacci numbers F_{3k+2}

We can make a similar observation on the \mathcal{F}_{3k} and \mathcal{F}_{3k+1} fractal.

8. **Open problem** : The curve never seems to intersect itself.
Does it have a connexion with the fact that the Fibonacci word is $(1 + \phi^2)^-$ -free [10] ?
9. The number of self-similarities is linked to the Fibonacci numbers. Let us examine the \mathcal{F}_{23} curve, we can count:
 - 2 copies of \mathcal{F}_{22} ,
 - or 3 copies of \mathcal{F}_{21} ,
 - or 7 copies of \mathcal{F}_{20} (self-similar to \mathcal{F}_{23}),
 - or 12 copies of \mathcal{F}_{19} ,
 - or 20 copies of \mathcal{F}_{18} ,
 - or 33 copies of \mathcal{F}_{17} (self-similar to \mathcal{F}_{23}), ...

Note that some copies overlap each other.

We can generalize and conjecture that, in the \mathcal{F}_n curve, the number of copies of the \mathcal{F}_{n-k} curve equals $F_{k+3} - 1$.

At the limit, we can conjecture that, at the k^{th} reduction level (scale ratio k), we find $F_{3k+3} - 1$ self-similarities. That is verified for the three types of fractals.

5 Fractal dimension

5.1 Hausdorff dimension of the fractal

Theorem : The Hausdorff dimension of the Fibonacci word fractal is $3 \frac{\log \phi}{\log(1+\sqrt{2})} = 1.6379$

Proof: The self-similarities between \mathcal{F}_n and \mathcal{F}_{n-3} allow a calculation of the Hausdorff dimension. The Hausdorff dimension is defined by[9]

$$D_H = \lim_{\epsilon \rightarrow 0} \frac{\log N\epsilon}{\log \frac{1}{\epsilon}}. \quad (13)$$

For our case, we can use the self-similarities and write:

$$D_H = \frac{\log A}{\log B}. \quad (14)$$

Where A represents the ratio indicating how the number of segments increase between \mathcal{F}_{n-3} and \mathcal{F}_n and B, the scale factor or reduction ratio between \mathcal{F}_n and \mathcal{F}_{n-3} .

- Calculation of the numerator A:

Since $\frac{F_n}{F_{n-1}}$ tends to ϕ , the golden number $\frac{(1+\sqrt{5})}{2}$, then

$$A = \lim_{n \rightarrow \infty} \frac{F_n F_{n-3}}{=} \phi^3 \quad (15)$$

- Calculation of the the denominator B:

We have proved in a theorem above that the scale factor equals:

$$B = 1 + \sqrt{2}. \quad (16)$$

The Hausdorff dimension of the Fibonacci word fractal is:

$$D_H = \frac{\log A}{\log B} = 3 \frac{\log \phi}{\log(1 + \sqrt{2})} = 1.6379 \quad (17)$$

■

We notice we have here both the golden and the silver ratios[17] . So we can write the following remarkable expression:

$$D_H = 3 \frac{\log \left(1 + \frac{1}{1+\dots} \right)}{\log \left(2 + \frac{1}{2+\dots} \right)} \quad (18)$$

5.2 Hausdorff dimension of the boundary of the fractal

The contour of the fractal is a fractal too.

Theorem : The Hausdorff dimension of the boundary of the Fibonacci word fractal is $\frac{\log 3}{\log(1+\sqrt{2})} = 1.2465$

Proof: As one examines it, it displays a recurrent pattern with auto-similarities at different scales all around the contour. This pattern can be constructed this way:

Start with a segment of length 1. Transform this segment into two segments of length $\sqrt{2} - 1$ on both sides of three segments placed with squared angles of length $(\sqrt{2} - 1)^2$ (see figure below).

Iterate this rule for every long segment.

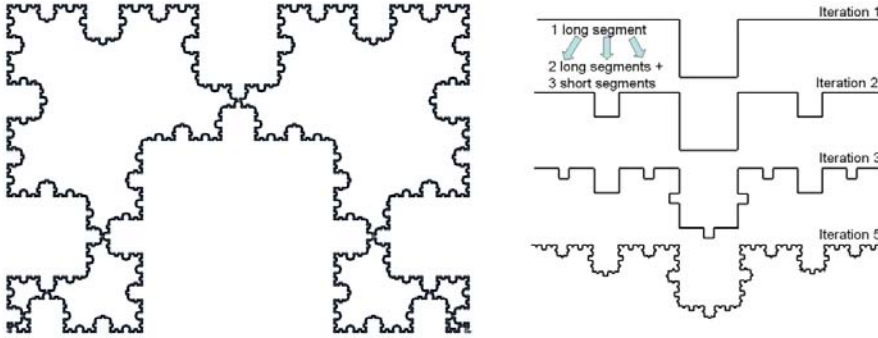


Fig.11:The pattern of the boundary and its construction

Let $a(n)$ be the number of long segments at iteration n

Let $b(n)$ be the number of short segments at iteration n

Then:

$$a(n + 1) = 2a(n) + b(n) \quad (19)$$

and

$$b(n + 1) = 3a(n) \quad (20)$$

We notice that:

$$a(n) = 2a(n - 1) + 3a(n - 2) \quad (21)$$

with $a(0)=0$ and $a(1)=1$, so :

$$a(n) = \lfloor \frac{3^{n+1}}{4} + \frac{1}{2} \rfloor \quad (22)$$

and

$$b(n) = 3a(n - 1) = 3 \lfloor \frac{3^n}{4} + \frac{1}{2} \rfloor. \quad (23)$$

The size of the small segments is reduced by $1 + \sqrt{2}$ at each iteration. That is the scale factor B .

So how does the number of segments increase as their length is reduced ?

As the segments don't have the same length, the long segments must be divided by $1 + \sqrt{2}$ so that all segments have the size of the small segments. The number of segments at iteration n is then

calculated : $s(n) = a(n)(1 + \sqrt{2}) + b(n)$

So:

$$\frac{s(n+1)}{s(n)} = \frac{\lfloor \frac{3^{n+1}}{4} + \frac{1}{2} \rfloor (1 + \sqrt{2}) + 3 \lfloor \frac{3^n}{4} + \frac{1}{2} \rfloor}{\lfloor \frac{3^n}{4} + \frac{1}{2} \rfloor (1 + \sqrt{2}) + 3 \lfloor \frac{3^{n-1}}{4} + \frac{1}{2} \rfloor} \quad (24)$$

Note that the term $\frac{1}{2}$ becomes negligible as n tends to infinity. So:

$$A = \lim_{n \rightarrow \infty} \frac{s(n+1)}{s(n)} = \frac{\frac{3^{n+1}}{4}(1 + \sqrt{2}) + 3 \frac{3^n}{4}}{\frac{3^n}{4}(1 + \sqrt{2}) + 3 \frac{3^{n-1}}{4}} = 3 \quad (25)$$

At each iteration, that number of segments tends to be multiplied by 3.

So, the Hausdorff dimension of the boundary of the Fibonacci word fractal is:

$$D_H = \frac{\ln A}{\ln B} = \frac{\log 3}{\log (1 + \sqrt{2})} = 1.2465 \blacksquare \quad (26)$$

6 Other construction modes

The Fibonacci word fractal is the limit set of an infinity of different curves. We define below a L-system to build it, but also a method that proceeds by removing white squares to a black rectangle, or different constructions using morphisms, Rauzy rules, or a dynamical process. We also show that other characteristic words display similar patterns. This multiplicity of variants shows that the Fibonacci word fractal pattern goes far beyond the Fibonacci word itself.

6.1 by joining curves

This construction mode is the more obvious and comes from the most known property of the Fibonacci word : for every $n \geq 1$, $f_{n+1} = f_n f_{n-1}$.

Since $|f_n| =$ Fibonacci number F_n , if n is a multiple of 3, then $|f_n|$ is even, otherwise, it is odd. So the \mathcal{F}_{n+1} curve can be constructed iteratively by joining together the \mathcal{F}_n curve and:

- the \mathcal{F}_{n-1} curve, if n is a multiple of 3.
- the symmetrical of the \mathcal{F}_{n-1} curve if n is not a multiple of 3.

The figures below illustrate how \mathcal{F}_{22} is build from \mathcal{F}_{21} and \mathcal{F}_{20} and how \mathcal{F}_{23} is build from \mathcal{F}_{22} and the symmetrical of \mathcal{F}_{21} .

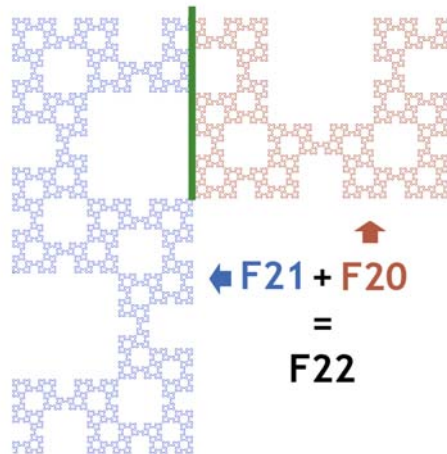


Fig.12: The \mathcal{F}_{22} curve is constructed from \mathcal{F}_{21} and \mathcal{F}_{20}

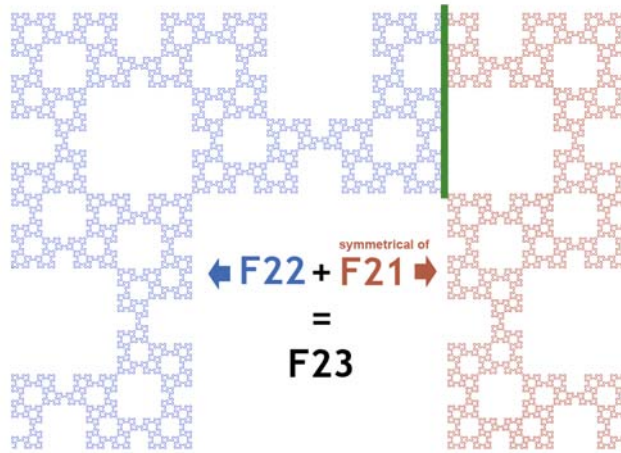


Fig.13: The \mathcal{F}_{23} curve is constructed from \mathcal{F}_{22} and the symmetrical of \mathcal{F}_{21}

6.2 with construction rules involving the golden ratio

The Fibonacci word fractal can exactly be drawn with the following drawing rule:

Take the n^{th} digit of the Fibonacci word,

- If it is 0 then draw a segment forward
- If it is 1 then:
 - . turn left if $\lfloor (n\phi^2) \rfloor$ is odd
 - . turn right if $\lfloor (n\phi^2) \rfloor$ is even

and iterate

ϕ is the golden ratio $(1 + \sqrt{5})/2$.

The sequence generated by $\lfloor (n\phi^2) \rfloor$ is the Upper Wythoff sequence (a Beatty sequence). It is referenced A001950 in the OEIS. This sequence gives the successive positions of the "1" in the Fibonacci word.

A version of the Fibonacci word fractal, oriented 45° , can be obtained by iterating the following drawing rule:

Take the n^{th} digit of the Fibonacci word,

- If it is 0 then draw a segment forward
- If it is 1 then:
 - . turn left if $\lfloor (n\phi^3) \rfloor$ is odd
 - . turn right if $\lfloor (n\phi^3) \rfloor$ is even

and iterate

The sequence generated by $\lfloor (n\phi^3) \rfloor$ is referenced A004976 in the OEIS.

6.3 with a Lindermayer system

The self-similarities between \mathcal{F}_n and \mathcal{F}_{n-3} suggest again another mode of construction based on a Lindermayer system (or L-System)[14].

First of all, we remember that if $F_n = F_{n-1} + F_{n-2}$ then $F_n = 4F_{n-3} + F_{n-6}$. By examining the \mathcal{F}_n fractal, we notice that it is made of two \mathcal{F}_{n-3} separated by a square angle, a \mathcal{F}_{n-6} , then, again, two \mathcal{F}_{n-3} separated by a square angle.

The can define, the following parametric L-system:

Constants :

+ (= turn left 90°)
 - (= turn right 90°)

Axiom :

L

Rules:

$L(x) \rightarrow + R(x) - L(x) K(\frac{x}{1+\sqrt{2}}) L(x) - R(x)$
 $R(x) \rightarrow - L(x) + R(x) Q(\frac{x}{1+\sqrt{2}}) R(x) + L(x)$
 $K(x) \rightarrow L(x)$
 $Q(x) \rightarrow R(x)$

L and R mean : 'draw a segment'. L will allow, at the next iteration, the construction of the curve to its left, R to its right.

K et Q also mean 'draw a segment'. but notice that the length of that segment is divided by $(1 + \sqrt{2})$. We know (see section 'Fractal dimension') that it is the scale factor of the fractal. K and Q delay the construction of L and R by one iteration.

The first iterations are illustrated in the figure below :

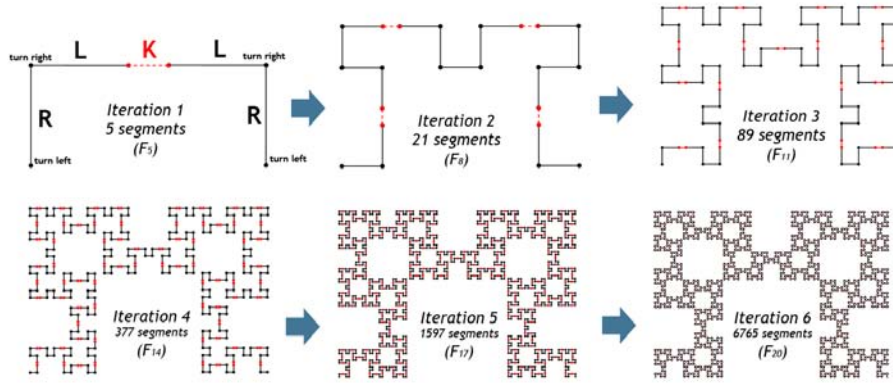


Fig.14: The first iterations of the L-system

The curve obtained has a different aspect than the standard one but, at the limit, the pattern seems to converge to the same fractal.

Let's define :

$B(n)$, the number of black segments (L or R) at iteration n,
 $D(n)$, the number of red dotted segments (K or Q) at iteration n,
 $T(n)$, the total number of segments at iteration n, $T(n) = B(n) + D(n)$
 then, $B(n) = 4*B(n-1) + D(n-1)$ and $D(n) = B(n-1)$
 Starting from $B(0)=1$ and $D(0)=0$ we have:

iter	B(n)	D(n)	T(n)	Fibo.numb.
0	1	0		
1	4	1	5	F_5
2	17	4	21	F_8
3	72	17	89	F_{11}
4	305	72	377	F_{14}
5	1292	305	1597	F_{17}
6	5473	1292	6765	F_{20}

Yet, again, the Fibonacci numbers appear.

$$T(n) = (4B(n-1) + D(n-1)) + (4B(n-2) + D(n-2)) \quad (27)$$

$$T(n) = 4B(n-1) + B(n-2) + 4D(n-1) + D(n-2) \quad (28)$$

$$T(n) = 4(B(n-1) + D(n-1)) + B(n-2) + D(n-2) \quad (29)$$

$$T(n) = 4T(n-1) + T(n-2) \quad (30)$$

Which is a Fibonacci identity. with $T(0)=1$, the number of segments at iteration n is $T(n) = F_{3n+2}$. It's exactly the number of segments necessary for the \mathcal{F}_{3k+2} Fibonacci word curve. We can define similar L-system rules for the \mathcal{F}_{3k} and the \mathcal{F}_{3k+1} fractal.

6.4 by removing white squares

This mode of construction is suggested by one of the properties listed above (see chapter "properties"). This rule is similar to a well-known construction rule for the Sierpinski gasket which consists in removing white triangles to an original black one. From a $1 \times \sqrt{2}$ black rectangle, remove iteratively F_{3k-1} white squares. Squares whose size is reduced by $1 + \sqrt{2}$ at each iteration, and whose center is placed at a distance equal to $2 - \sqrt{2}$ and distributed with an angle of $\pi/4$ around the center of the previous iteration squares.

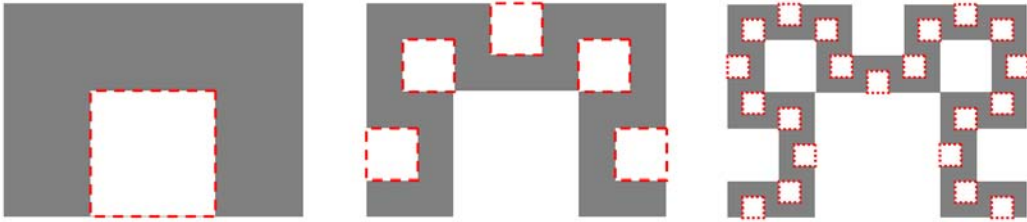


Fig.15:Construction of the fractal with white squares

We can also show that the area of the fractal equals 0, if we assume the above construction mode is true. The area S covered by the white squares is the area of the original rectangle:

$$S = \sum_{k=1}^{\infty} \left(F_{3k-1} \left(\sqrt{2} (\sqrt{2}-1)^k \right)^2 \right) = \sqrt{2} \sum_{k=1}^{\infty} \left(F_{3k-1} \sqrt{2} (\sqrt{2}-1)^{2k} \right) \quad (31)$$

After testing it numerically, we conjecture that the following unreferenced Fibonacci identity is true :

$$\sum_{k=1}^{\infty} \left(F_{3k-1} \sqrt{2} (\sqrt{2}-1)^{2k} \right) = 1. \quad (32)$$

so $S = \sqrt{2}$. The white squares tend to cover the entire black rectangle.

6.5 by changing directions

This construction mode was found by chance by Ron Knott, as he was trying to reproduce the fractal on his computer.

Iterate the following rule on the Fibonacci word:

- turn and move
- if the digit equals "1" then change the direction of turns.

The curve is different from the Fibonacci word curve but at the limit, they seem to converge to the same fractal. See figure below.

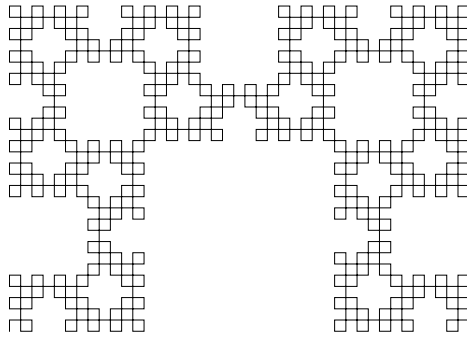


Fig.16: The curve after F_{16} iterations

A similar curve, but oriented 45° , can be obtained when changing the rule slightly:

- turn and move
- if the digit equals "0" then change the direction of turns.

6.6 The Dense Fibonacci Word : a whole family of curves

The odd-even drawing rule is difficult to handle easily and we can change for a more convenient one. As suggested to me by Jean-Paul Allouche, we should find it by defining an adequate word on a 3-letters alphabet.

We can create a 3-letter word in $\{0; 1; 2\}$ that can draw the Fibonacci fractal with the following more simple drawing rule:

- if "0" then draw segment
- if "1" then draw segment and turn right
- if "2" then draw segment and turn left

We will call this drawing rule the "Natural drawing rule".

Then this 3-letter word must be 1012021100200.... It can be created from the infinite Fibonacci Word by grouping the letters 2 by 2 and applying the following map: $01 \rightarrow 10$, $00 \rightarrow 12$ and $10 \rightarrow 02$. Note that grouping the letters 2 by 2 allows to get rid of the odd-even alternation. ...but we can do better, as we will see next.

Definition : Consider applying the "binary-to-decimal" conversion to each pair of letters of the infinite Fibonacci Word: $00 \rightarrow 0$, $01 \rightarrow 1$ and $10 \rightarrow 2$. We define the resulting word as the "Dense Fibonacci Word" and write it DFW, for short:

Dense Fibonacci Word : 102210221102110211022102211021102110221022102211021...

The DFW word is now referenced as A143667 in the OEIS [15].

Theorem : The DFW is generated by iterating the following morphism we call ρ : $\rho(0) = 10221$, $\rho(1) = 1022$, $\rho(2) = 1021$, starting from the word "1".

Proof. Let us transform ρ through the "decimal-to-binary" conversion. This gives $00 \rightarrow 0100101001$, $01 \rightarrow 01001010$, $10 \rightarrow 01001001$, starting from the word "01". We clearly recognise here that this morphism equals σ^3 , which is the Fibonacci word morphism applied three times, from "01". $\sigma^3(0) = 01001$ and $\sigma^3(1) = 010$. Since σ^3 generates the Fibonacci word, ρ generates the DFW. ■

Definition: We call the resulting angle of a word w , through a drawing rule, the function α that gives the global angle when drawn with that drawing rule. The global angle is the sum of the successive angles generated by the word through the rule.

Examples : with the natural drawing rule, $\alpha(1) = -\pi/2$, $\alpha(0) = 0$, $\alpha(2) = \pi/2$, $\alpha(012) = \alpha(0) + \alpha(1) + \alpha(2) = 0$.

Definition: We call a morphism that preserves the resulting angle a morphism m so that, for any word w , $\alpha(m(w)) = \alpha(w)$,

We call a morphism that inverts the resulting angle a morphism n so that, for any word w , $\alpha(n(w)) = -\alpha(w)$,

We can prove easily that ρ is a morphism that inverts the angles.

The DFW is strongly linked to the Fibonacci word fractal because it can generate a whole family of curves whose limit is the Fibonacci word fractal. We just need to apply to it a morphism that preserves or inverts the resulting angle. For example:

- the morphism $\mu_1 : 1 \rightarrow 10; 0 \rightarrow 12; 2 \rightarrow 02$ creates the "standard" Fibonacci word curve, as it appears on the first page of this paper.
- the morphism "identity" draws a similar curve with a rotation of $\pi/8$.
- the morphism $\mu_2 : 1 \rightarrow 1, 0 \rightarrow \epsilon, 2 \rightarrow 2$ creates a version with a rotation of $\pi/4$. ϵ is the empty word.
- the morphism $\mu_3 : 1 \rightarrow 1, 0 \rightarrow 12, 2 \rightarrow 2$ "compacts" the standard curve, like Ron Knott's curve. The resulting word '1122211222...' has been already referenced in 2003 by Benoit Cloitre as A085002[15] in the OEIS (in a different form). It is also the sequence of "rights" an "lefts" of the standard Fibonacci word Fractal (We notice that μ_3 equals μ_1 without the letter "0").
- the morphism $\mu_4 : 1 \rightarrow 010, 0 \rightarrow 0102, 2 \rightarrow 002$ creates an "expanded" version of the standard curve.
- the morphism $\mu_5 : 1 \rightarrow 02, 0 \rightarrow 21, 2 \rightarrow 10$ creates a version with a rotation of $\pi/5$ that includes "svastika-like" patterns

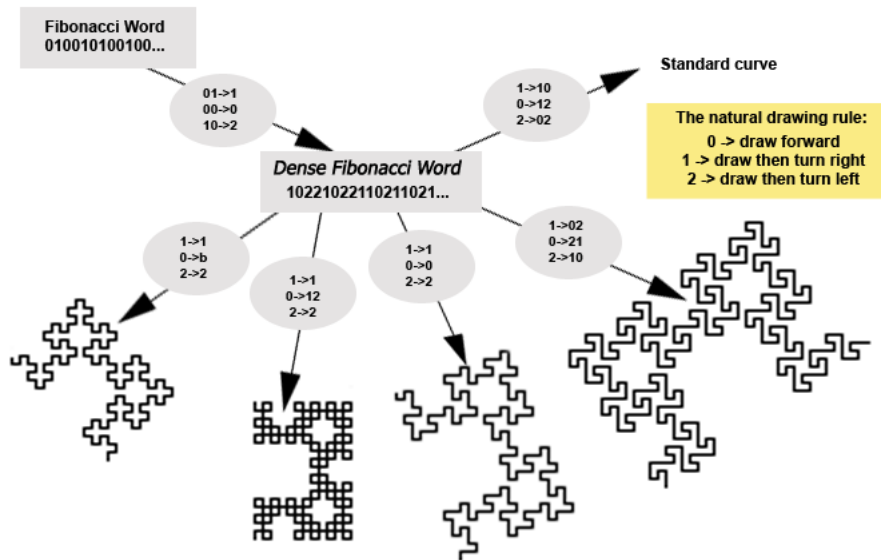


Fig.17: some of the various curves generated from the Dense Fibonacci Word

- the morphism $\mu_6 : 1 \rightarrow 02, 0 \rightarrow 00, 2 \rightarrow 10$ creates an expanded version with a rotation of $\pi/4$. The resulting word 020010100200... can also be obtained from the Fibonacci word with the morphism : $0 \rightarrow 0$ and $1 \rightarrow 1$ if in odd position or $1 \rightarrow 2$ if in even position.

6.7 by inverting the role of 0 and 1

We can also draw the expanded version with a rotation of $\pi/4$ described just above (see morphism μ_6) from the Fibonacci word and the odd-even drawing rule: We just invert the roles of "0" and "1" in the rule, or in the word.

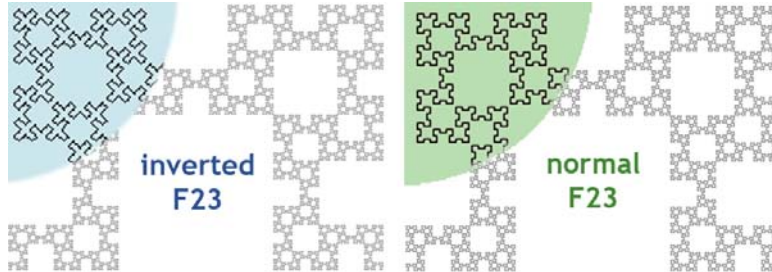


Fig.18: Inverting the roles of 1 and 0 in the odd-even drawing rule. The curve is identical (but tilted 45 degrees)

6.8 by the most simple drawing rule

The Dense Fibonacci Word and the μ_4 morphism described above is also the base of another mode of construction.

Take the DFW and apply the μ_4 morphism : $1 \rightarrow 010$, $0 \rightarrow 0102$, $2 \rightarrow 002$. The resulting word is: 0100102002002010010200200201001001020020100100102...

Now apply this simple drawing rule, where each elementary action is described by a letter:

Take the n^{th} digit of this word,

- If it is 0 then draw a segment forward
- If it is 1 then turn right
- If it is 2 then turn left

and iterate

This rule is the one used commonly for L-Systems, if we write "F" for "0", "+" for "1" and "-" for "2". The curve drawn is the standard Fibonacci word fractal curve.

Proof : If, from the DFW, the morphism $\mu_1 : 1 \rightarrow 10$; $0 \rightarrow 12$; $2 \rightarrow 02$ creates, through the "Natural Drawing rule", the standard Fibonacci word curve (see above), then, for the same result, and with the drawing rule above, one must add a 0 ("draw forward") before 1 and before 2 in the morphism.

6.9 by applying the Rauzy Rules

The Rauzy rules are described in [11] : We define two sequences in A^∞ , A_n and B_n , where $A_0 = 0$ and $B_0 = 1$ and two rules D and G :

$D : A_{n+1} = A_n B_n$ and $B_{n+1} = A_n$

$G : A_{n+1} = B_n A_n$ and $B_{n+1} = B_n$

We define by T , a chosen sequence of G and D .

It is straightforward to see that the infinite Fibonacci word is generated by applying the sequence $T = GGGG\dots$

Now, we can choose other sequences T that create other interesting words A_n that can generate variants of the Fibonacci word fractal through the odd-even drawing rule. The figure below shows some of them: $T = DGGG\dots$, or $T = GDGGG\dots$. The sequence $T = GGGGGGGGGGGDDDD\dots$ shows a "closed" variant of the Fibonacci word fractal pattern. Not all T sequences generate such variants.

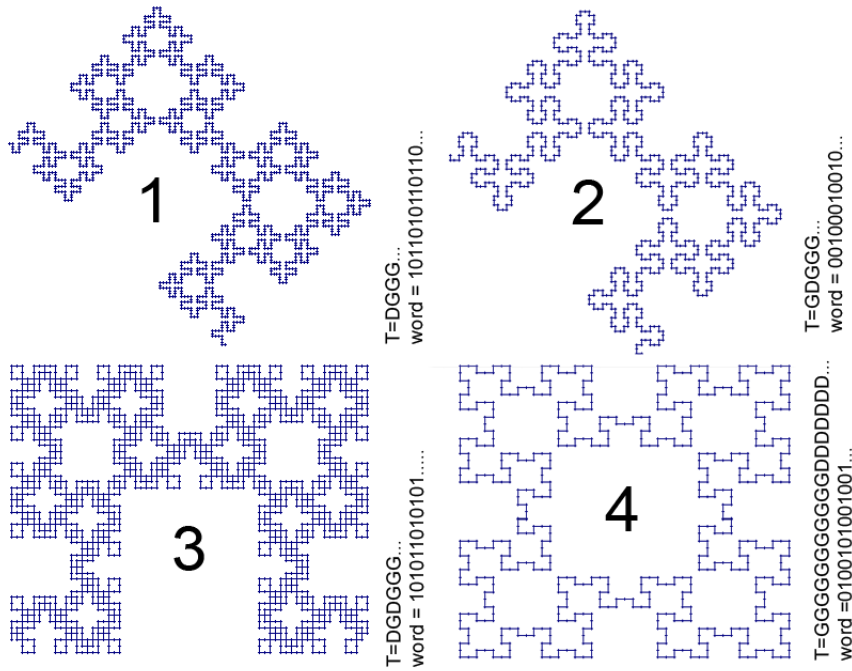


Fig.19:Variants generated by applying the Rauzy Rules

6.10 with other Sturmian words

An infinite word $w = w_0w_1w_2\dots$, with $w_n \in \{0, 1\}$ for all n , is a sturmian word if and only if there exists two real numbers α and ρ , with α irrationnal, such that:

$$w_n = \lfloor \alpha(n + 1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor - \lfloor \alpha \rfloor \quad (33)$$

α is called the "slope" and ρ the "intercept". A characteristic word is a sturmian word with an intercept value of zero. All sturmian words with the same slope have the same set of factors and share the same characteristic word. We will use here a more useful definition of characteristic words taken from [4]. We define a directive sequence as an infinite sequence of integers $d = (d_0, d_1, d_2, \dots)$ with $d_n > 0$. Then, we define the sequence $w_{n+1} = w_n^{d_n} w_{n-1}$ with $n \geq 0$ and $w_1 = 0$ and $w_{-1} = 1$. It is straighforward to see, from its definition, that the Fibonacci word is the characteristic word generated by the directive sequence $(1, 1, 1\dots)$.

Now we can choose other directive sequences d that create other characteristic words w_n that generate variants of the Fibonacci word fractal through the odd-even drawing rule. The figure below show some of them.

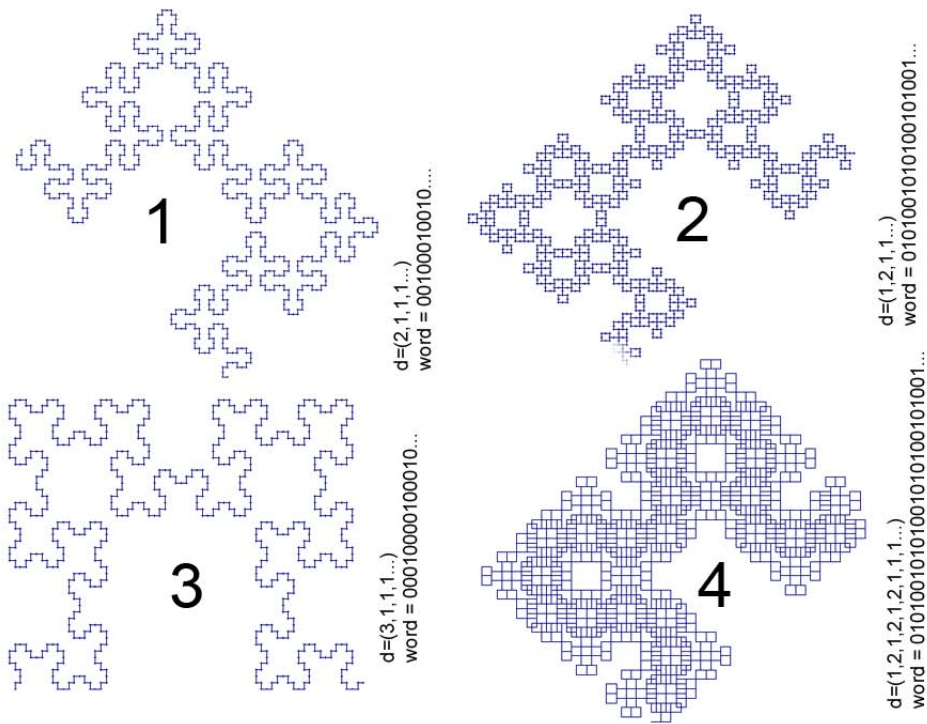


Fig.20:Variants generated by other characteristic words

The following table presents some examples illustrated on the figures above and more. It displays also the slope associated to the word, its continued fraction expansion and its repetition index. The characteristic words that have the smallest index draw the most simple patterns. For a definition of the index, see [5]

See fig.	direct. seq.	word	slope	cont. fraction	repet. index
Fig.1	(1, 1, 1, 1...)	Fib. word	$2 - \phi = 0, 3819660$	[0; 2, 1, 1, 1...]	$2 + \phi$
Fig.20-1	(2, 1, 1, 1...)	00100010010...	$(3 - \phi)/5 = 0, 276393$	[0; 3, 1, 1, ...]	$2 + \phi$
Fig.20-2	(1, 2, 1, 1...)	01010010101...	$(3 + \phi)/11 = 0, 4198212$	[0; 2, 2, 1, 1...]	$2 + \phi$
Fig.20-3	(3, 1, 1, 1...)	00010000100...	$(4 - \phi)/11 = 0, 2165423$	[0; 4, 1, 1, 1...]	$> 2 + \phi$
Fig.20-4	(1, 2, 1, 2, 1, 2, 1, 1...)	0101001010100...	0, 422635186	[0; 2, 2, 1, 2, 1, 2, 1...]	$> 2 + \phi$
...	(1, 1, 1, 1...)	010010100...	$(2 + \phi)/5 = 0, 7236067$	[0; 1, 2, 1, 1...]	$2 + \phi$
Fig.19-1	(0, 1, 1, 1...)	101101011...	$\phi - 1 = 0, 6180339$	[0; 1, 1, 1...]	$2 + \phi$
Fig.19-3	(0, 1, 2, 2, 1, 1...)	101011010...	$(8 - \phi)/11 = 0, 5801787$	[0; 1, 1, 2, 1, 1...]	$2 + \phi$

Proposition : All sturmian words display the Fibonacci word fractal pattern, providing their directive sequence ends with an infinite sequence of "ones" (i.e. their slope have a continued fraction expansion ending with an infinite sequence of "ones").

7 An interesting dynamical process

The construction we will describe now is apparently disconnected from the Fibonacci word. It is an interesting view towards understanding the geometry of the Fibonacci word fractal curve in terms of dynamics. We will call this process the "corner extension" process. We start with a simple "corner" made by two segments. We then "expand" the corner in the direction it is "pointing at", in a similar way a river tends to extend its meanders. New corners are created and we expand these corners the same way. Gradually the Fibonacci word fractal pattern appears, or, at least a part of it.

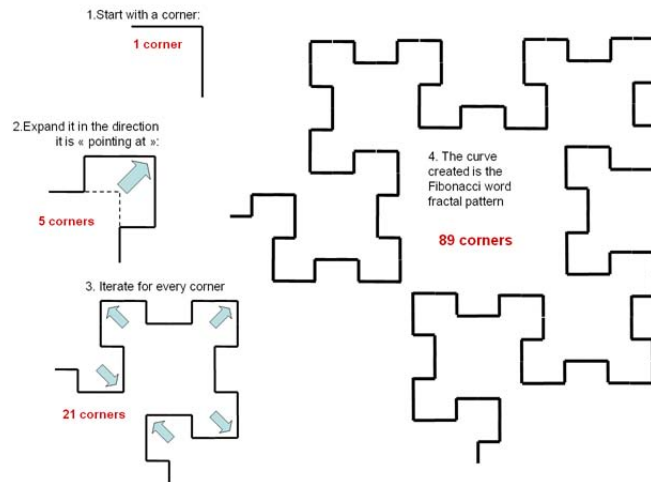


Fig.21: The "corner extension" process

Note that the number of corners and the number of segments at each iteration seem to be Fibonacci numbers.

That process can be described by two different maps applied iteratively to a word on a two-letter alphabet. See figure below.

Map δ	Isolated « 0 » 	Double « 0 »
	Map γ 	

Fig.22: map of the "corner extension" process

The first map is the more obvious to find: Interpreting the odd-even rule, we consider an "isolated corner" as an "isolated zero" and "double corner" as "double zeroes". We then build the map δ as follows, and iterate it:

- $1 \longrightarrow 1$
- isolated 0 $\longrightarrow 0010100$
- double 0 $\longrightarrow 001010010100$

Starting from the word 01, the initial corner made of two segments, that map creates the successive words w : 00101001, 0010100101001001001001001001001001001 and so on. One can verify that this map cannot generate triple "0"s nor double "1"s. We also verify that, for all such words w , the parity of $|w|$ is the same as the parity of $|\delta(w)|$, so δ is compatible with the odd-even drawing rule. We can observe that an "isolated" creates 1 "isolated" and 2 "doubles", and that a "double" creates 2 "isolated" and 3 "doubles". From that we can prove that the number of corners is always a Fibonacci number. We can also prove that the number of segments is also a Fibonacci number. We observe also that this map preserves the resulting angle. We leave these easy proofs to the reader.

The second map is more difficult to find but more convenient to work with : Interpreting the odd-even rule, we consider a "corner" as a "zero" and a flat angle as a "1". We then build the map γ as follows:

Drawn with the odd-even drawing rule, ab gives a resulting angle of α . So does ba . Let us define L_n the length of fractal \mathcal{F}_n from first to last point drawn. It equals :

$$L_n = 2L_{n-3} + 2L_{n-3}\cos(\alpha) + L_{n-6} = 2L_{n-3} + L_{n-6} \quad (35)$$

On the other hand, by definition, the scale factor B is :

$$B = \frac{L_n}{L_{n-3}} = \frac{L_{n-3}}{L_{n-6}} \quad (36)$$

Using the two previous identities, B must satisfy :

$$BL_{n-3} = 2L_{n-3} + 2L_{n-3}\cos(\alpha) + \frac{L_{n-3}}{B} \quad (37)$$

then the scale factor B must be a solution of the equation :

$$B^2 = 2B(1 + \cos(\alpha)) + 1 \quad (38)$$

The scale factor of the generalized Fibonacci word fractal is:

$$B = 1 + \cos(\alpha) + \sqrt{(1 + \cos(\alpha))^2 + 1}. \quad (39)$$

■

Theorem : For the generalized Fibonacci word fractal, with angle α , the Hausdorff Dimension equals $3 \frac{\log \phi}{\log(1 + \cos(\alpha) + \sqrt{(1 + \cos(\alpha))^2 + 1})}$.

Proof : Straightforward, since the Hausdorff dimension of the generalized Fibonacci word fractal is: $D_H = \frac{\log A}{\log B}$, with $A = \phi^3$ and B is the scale factor seen above.

8.1 the 60 degrees variant

When choosing an angle of 60° instead of 90° , the Fibonacci word fractal displays a different pattern.

It must not be confused with the von Koch curve : building the Von Koch curve requires two 60° turns in a row, which is impossible with the odd-even drawing rule.

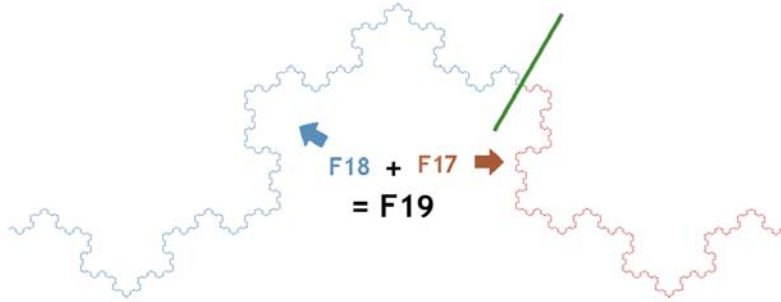


Fig.24: The \mathcal{F}_{19} curve, constructed from \mathcal{F}_{18} and \mathcal{F}_{17}

From the theorem above, we can write that the Hausdorff dimension of the 60° variant is :

$$D_H = 3 \frac{\log \phi}{\log \frac{3 + \sqrt{13}}{2}} = 1.2083 \quad (40)$$

Which can also be written in this remarkable way:

$$D_H = 3 \frac{\log \left(1 + \frac{1}{1 + \dots} \right)}{\log \left(3 + \frac{1}{3 + \dots} \right)} \quad (41)$$

8.2 the 72 degrees variant

Now, it is tempting to choose the 72° angle of the pentagon and its link with the golden number, so there it is:

An interesting observation : The fractal seem to follow the contour of an iterated construction of pentagons, as show bellow.

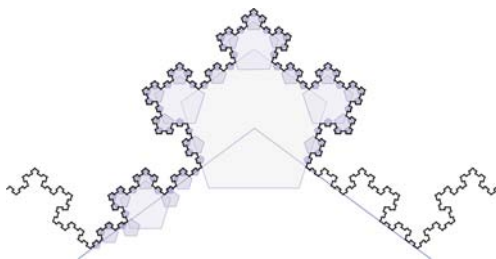


Fig.25: The \mathcal{F}_{3k+1} 72° Fibonacci word fractal, and an iterated construction of pentagons.

This suggests yet another mode of construction of the Fibonacci word fractal. Note that this mode also works for the 60 degrees variant, with hexagons.

9 Conclusion

What the Fibonacci Word Fractal deserves, now, is that its properties be studied further and find rigorous proofs for the conjectures listed in this paper. Apart from its particular beauty, the Fibonacci word fractal could offer a new view on the Fibonacci word, and on the Fibonacci numbers. The study of the odd-even drawing rule can be extended to other words, such as this other sturmian word generated by $0 \rightarrow 001$ and $1 \rightarrow 0$. The resulting curve, using an angle of $2\pi/3$, displays a different and interesting fractal pattern. We even encounter the Sierpinski gasket with the word generated by the uniform morphism : $0 \rightarrow 011$ and $1 \rightarrow 010$.

10 Acknowledgements

I really must thank Ron Knott for the interest he showed to my work as I was looking for help. He encouraged me to write this paper for publication and kindly proposed his help and gave suggestions. I also thank Jean-Paul Allouche and Filippo Mignosi for giving me news insights and hints to go further on this study.

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AMS classification:

- 05B30 Combinatorics - Other designs, configurations
- 11B39 Fibonacci and Lucas numbers and polynomials and generalizations
- 28A80 Fractals
- 68R15 Combinatorics on words