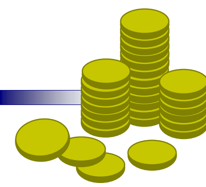
Yield to Maturity



Outline and Suggested Reading

- Outline
  - Yield to maturity on bonds
  - Coupon effects
  - Par rates
  - Yield vs. rate of return
- Buzzwords
  - Internal rate of return,
  - Yield curve
  - Term structure of interest rates
- Suggested reading
  - Tuckman, Chapter 3

General Definition

Suppose a bond (or portfolio of bonds) has price P and positive fixed cash flows K_1, K_2,…, K_n at times t_1, t_2,…, t_n. Its yield to maturity is the single rate y that solves:

$$\frac{K_1}{(1+y/2)^{2t_1}}+\frac{K_2}{(1+y/2)^{2t_2}}+...+\frac{K_n}{(1+y/2)^{2t_n}}=P$$

or

$$\sum_{j=1}^{n}\frac{K_j}{(1+y/2)^{2t_j}}=P$$

Note that the higher the price, the lower the yield.

Example

•Recall the 1.5-year, 8.5%-coupon bond.

•Using the zero rates 5.54%, 5.45%, and 5.47%, the bond price is 1.043066 per dollar par value.

•That implies a yield of 5.4704%:

$$\frac{0.0425}{(1+0.0554/2)^1}+\frac{0.0425}{(1+0.0545/2)^2}+\frac{1.0425}{(1+0.0547/2)^3}$$

$$=1.043066$$

$$=\frac{0.0425}{(1+0.054704/2)^1}+\frac{0.0425}{(1+0.054704/2)^2}+\frac{1.0425}{(1+0.054704/2)^3}$$

Yield of a Bond on a Coupon Date

For an ordinary semi-annual coupon bond on a coupon date, the yield formula is

$$P = \frac{c}{2}\sum_{s=1}^{2T}\frac{1}{(1+y/2)^s} + \frac{1}{(1+y/2)^{2T}}$$

where c is the coupon rate and T is the maturity of the bond in years.

Formula for the Present Value of an Annuity

Math result: $$\sum_{j=1}^{n}\frac{1}{(1+r)^j} = \frac{1}{r}(1-\frac{1}{(1+r)^n})$$

Finance application: This formula gives the present value of an annuity of $1 to be received every period for n periods at a simply compounded rate of *r* per period.

Price-Yield Formula for a Bond on a Coupon Date

Applying the annuity formula to the value of the coupon stream, with $r=y/2$ and $n=2T$:

$$P = \frac{c}{y}[1-(\frac{1}{1+y/2})^{2T}] + \frac{1}{(1+y/2)^{2T}}$$

- The closed-form expression simplifies computation.
- Note that if $c=y$, $P=1$ (the bond is priced at par).
- If $c>y$, $P>1$ (the bond is priced at a *premium* to par).
- If $c<y$, $P<1$ (the bond is priced at a *discount*).
- The yield on a zero is the zero rate: $c=0$; $y=r_T$

Example

- Recall that the 1.5-year 8.5%-coupon bond was priced at 1.043066 per dollar par value.
- In other words, the bond is priced at a premium.
- That implies that the coupon rate must be higher than the yield.
- Indeed, $c= 8.50\% > y= 5.4704\%$.

Bond Yields and Zero Rates

- Recall that we can construct coupon bonds from portfolios of zeroes, and we can construct zeroes from portfolios of coupon bonds.
- This means that, in the absence of arbitrage, the prices of zeroes imply prices for coupon bonds and the prices of coupon bonds imply prices for zeroes.
- Equivalently, **yields on zeroes imply yields on coupon bonds and yields on coupon bonds imply yields on zeroes.**

Valuation Formulas

- We have expressions for the value of a portfolio of fixed cash flows in terms of
  - discount factors (by no arbitrage)
  - discount rates (by the definition of the discount rates)
  - yield (by the definition of yield).

$$P = \sum_{j=1}^{n} K_j \times d_{t_j} = \sum_{j=1}^{n} K_j \times \frac{1}{(1 + r_{t_j}/2)^{2t_j}} = \sum_{j=1}^{n} K_j \times \frac{1}{(1 + y/2)^{2t_j}}$$

Yield and Zero Rates...

Compare the formula with zero rates and the formula with yield:

$$P = \sum_{j=1}^{n} K_j \times \frac{1}{(1+r_{t_j}/2)^{2t_j}} = \sum_{j=1}^{n} K_j \times \frac{1}{(1+y/2)^{2t_j}}$$

Notice that the yield is a blend or *a kind of average* of the different zero rates associated with the cash flows. In other words, the yield must be between the highest and lowest zero rates.

Example

Compare the two formulas for the 1.5-year 8.5%-coupon bond:

$$1.043066 = \frac{0.0425}{(1+0.0554/2)^1} + \frac{0.0425}{(1+0.0545/2)^2} + \frac{1.0425}{(1+0.0547/2)^3}$$

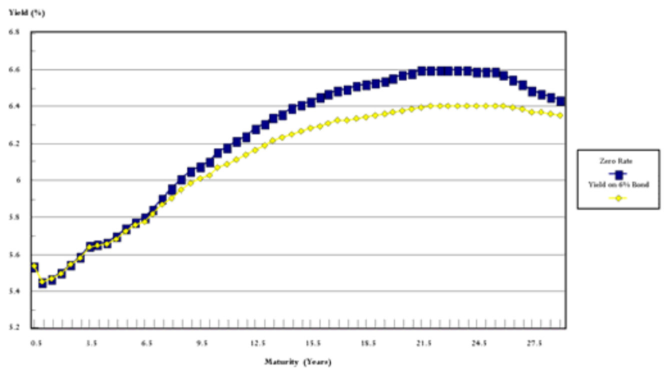
$$1.043066 = \frac{0.0425}{(1+0.054704/2)^1} + \frac{0.0425}{(1+0.054704/2)^2} + \frac{1.0425}{(1+0.054704/2)^3}$$

The yield of 5.4704% is a kind of average of the discount rates 5.54%, 5.45%, and 5.47%.

Term Structure and Yield Curves

- The phrase *term structure of interest rates* refers to the general relation between yield and maturity that exists in a given bond market.
- A *yield curve* is a plot of a specific set of bond yields as a function of their maturity.
- The yield curve for zeroes is typically different than the yield curve for coupon bonds. In principle, we can derive one curve from the other.

Yield Curves for Zeroes and 6% Bonds



Why Does the Coupon Bond Yield Curve Lie Below the Zero Curve?

Because this yield curve is upward-sloping.

More generally…

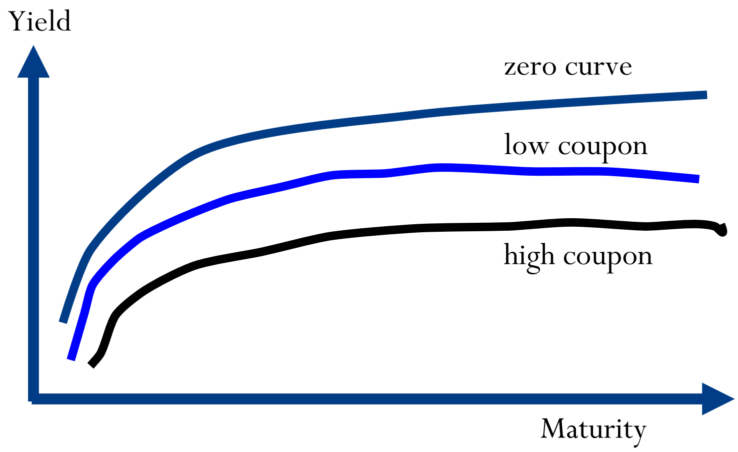
The Coupon Effect

- Consider two bonds with the **same maturity** but **different coupon** rates.
- Their yields are "averages" of the zero rates associated with their cash flows.
- Because they have the same maturity, the two bonds have cash flows at the same times, so their yields are averages of the same set of zero rates,
- But the bond with the <u>higher coupon rate places more weight on the shorter-term rates</u>.

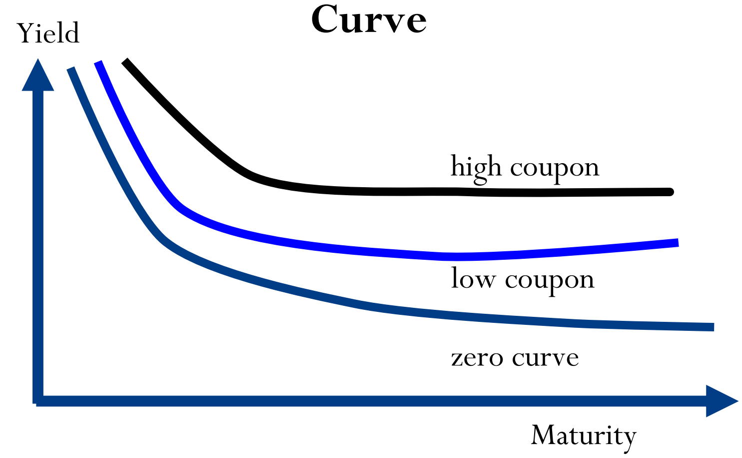
Illustration of the Coupon Effect with Yield Curves

- The next few slides sketch yield curves for bonds with different coupon rates.
- In each slide, a single discount function determines three yield curves:
  - zero rates across different maturities
  - yields on bonds with a constant *low* coupon with different maturities
  - yields on bonds with a constant *high* coupon with different maturities

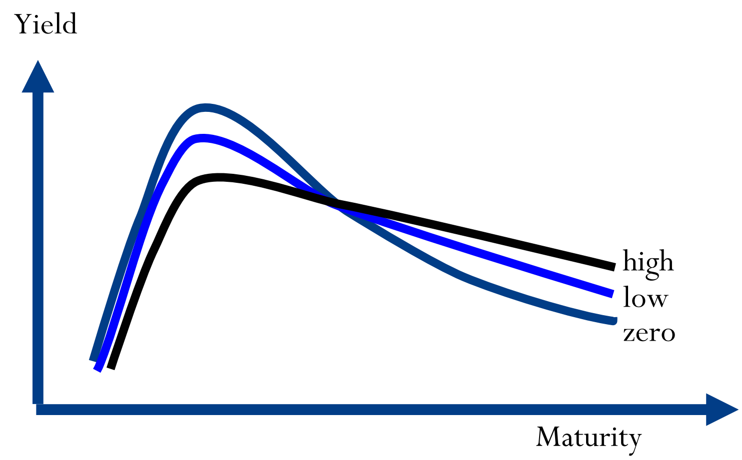
Upward Sloping Yield Curve



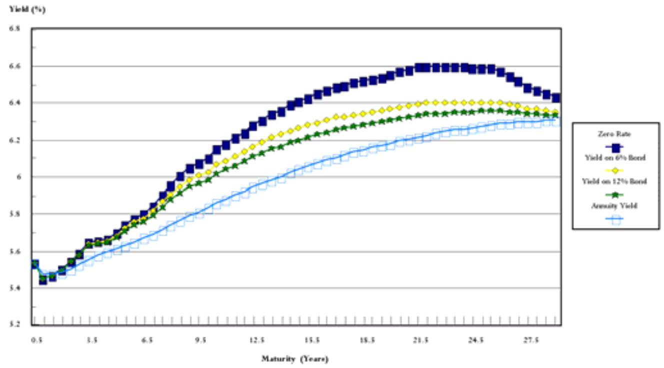
Downward Sloping Yield Curve



Humped Yield Curve



Yield Curves for Zeroes, Bonds, and Annuities



Par Rates

- The *par rate* for a given maturity T is the coupon rate that makes a T-year coupon bond sell for par.
- Of course, the yield on the bond will also be the par rate.
- Since coupon bonds are usually issued at par, par rates are yields on newly issued bonds.

Par Rate Formula

For each maturity T, the par rate c_T is the coupon rate that sets the bond price equal to par.

$$(c_T/2)\times d_{0.5}+(c_T/2)\times d_1+(c_T/2)\times d_{1.5}+\ldots+(c_T/2)\times d_T+1\times d_T=1$$

implies

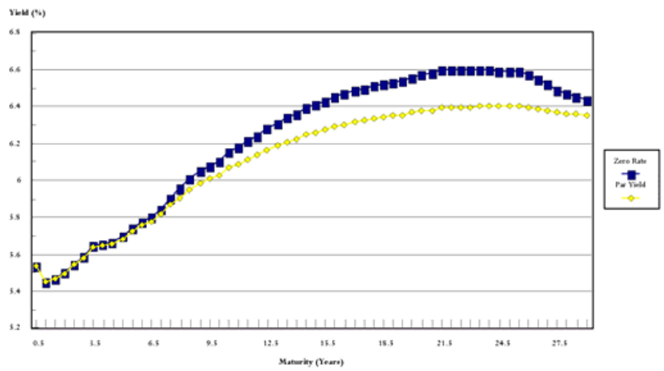
$$c_T=\frac{2(1-d_T)}{d_{0.5}+d_1+d_{1.5}+\ldots d_T}$$

Example

To solve for the 1.5-year par rate, use the discount factors for time 0.5, 1.0, 1.5:

$$d_{0.5}=0.973047, d_1=0.947649, d_{1.5}=0.922242$$
$$\Rightarrow c_{1.5}=\frac{2(1-0.922242)}{0.973047+0.947649+0.922242}$$
$$\Rightarrow c_{1.5}=5.4703\%$$

Yield Curves for Zeroes and Par Bonds



Yield vs. Rate of Return

- Yield and rate of return are different concepts:
- Yield is the single discount rate that sets the present value of a bond's cash flows equal to a given price.
- Yield can only be computed when the asset's cash flows are fixed.
- Given cash flows, yield and price are equivalent information.
- Computing yield requires only current information.

Yield vs. Rate of Return...

- Rate of return describes the growth of the investment value over a given horizon. The horizon need not coincide with the "maturity" of the asset.
- Rate of return is a concept that is meaningful for any asset, not just an asset with fixed cash flows.
- At the time the investment is made, the rate of return is generally unknown--a random number.

Yield is a Not a Measure of Value

- If two assets have *identical fixed cash flows*, the higher yielding one is a better value, because its price is lower.
- Otherwise, yield contains no direct information about value.
- In assessing the value of a bond, the investor needs to consider the possible return over the investment horizon:
  - What is the expected return?
  - What is the variance of that return (risk)?
  - Does the return serve to hedge other risks in the portfolio (diversification by way of co-variance)?