

# Hypothesis Testing for Proportions

## Chapter 8 Tests of Statistical Hypotheses

### 8.1 Tests about Proportions

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## Inference on Proportion

Parameter: Population Proportion  $p$  (or  $\pi$ )  
(Percentage of people has no health insurance)

Statistic: Sample Proportion  $\hat{p} = \frac{x}{n}$

$x$  is number of successes

$n$  is sample size

Data: 1, 0, 1, 0, 0  $\Rightarrow \hat{p} = \frac{2}{5} = .4 \Rightarrow \hat{p} = \bar{x}$

$$\bar{x} = \frac{1+0+1+0+0}{5} = .4$$

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## Sampling Distribution of Sample Proportion

A random sample of size  $n$  from a large population with proportion of successes (usually represented by a value 1)  $p$ , and therefore proportion of failures (usually represented by a value 0)  $1-p$ , the sampling distribution of sample proportion,

$\hat{p} = x/n$ , where  $x$  is the number of successes in the sample, is **asymptotically normal** with a **mean**  $p$

and **standard deviation**  $\sqrt{\frac{p(1-p)}{n}}$ .

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## Confidence Interval

**Confidence interval:** The  $(1-\alpha)\%$  confidence interval estimate for population proportion is

$$\hat{p} \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

**Large Sample Assumption:**

Both  $np$  and  $n(1-p)$  are greater than 5, that is, it is *expected that there at least 5 counts in each category*.

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## Hypothesis Testing

1. State research hypotheses or questions.  $p = 30\% ?$
2. Gather data or evidence (observational or experimental) to answer the question.  $\hat{p} = .25 = 25\%$
3. Summarize data and test the hypothesis.
4. Draw a conclusion.

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## Statistical Hypothesis

**Null hypothesis ( $H_0$ ):**

Hypothesis of no difference or no relation, often has  $=$ ,  $\geq$ , or  $\leq$  notation when testing value of parameters.

Example:

$H_0: p = 30\%$  or

$H_0$ : Percentage of votes for A is 30%.

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# Hypothesis Testing for Proportions

## Statistical Hypothesis

**Alternative hypothesis** ( $H_1$  or  $H_a$ )

Usually corresponds to research hypothesis and opposite to null hypothesis, often has  $>$ ,  $<$  or  $\neq$  notation in testing mean.

Example:

$H_a: p \neq 30\%$       or

$H_a$ : Percentage of votes for A is not 30%.

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## Hypotheses Statements Example

- A researcher is interested in finding out whether percentage of people in favor of policy A is different from 60%.

$H_0: p = 60\%$

$H_a: p \neq 60\%$

[Two-tailed test]

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## Hypotheses Statements Example

- A researcher is interested in finding out whether percentage of people in a community that has health insurance is more than 77%.

$H_0: p = 77\%$       ( or  $p \leq 77\%$  )

$H_a: p > 77$

[Right-tailed test]

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## Hypotheses Statements Example

- A researcher is interested in finding out whether the percentage of bad product is less than 10%.

$H_0: p = 10\%$       ( or  $p \geq 10\%$  )

$H_a: p < 10\%$

[Left-tailed test]

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## Evidence

**Test Statistic (Evidence):** A sample statistic used to decide whether to reject the null hypothesis.

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## Logic Behind Hypothesis Testing

*In testing statistical hypothesis, **the null hypothesis is first assumed to be true.***

*We collect evidence to see if the evidence is strong enough to reject the null hypothesis and support the alternative hypothesis.*

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# Hypothesis Testing for Proportions

**One Sample Z-Test for Proportion**  
**(Large sample test)**

Two-Sided Test

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**I. Hypothesis**

One wishes to test whether the percentage of votes for A is different from 30%

$H_0: p = 30\%$  v.s.  $H_a: p \neq 30\%$

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**Evidence**

What will be the key statistic (evidence) to use for testing the hypothesis about population proportion?

**Sample Proportion:**  $\hat{p}$

**A random sample of 100 subjects is chosen and the sample proportion is 25% or .25.**

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**Sampling Distribution**

If  $H_0: p = 30\%$  is true, sampling distribution of sample proportion will be approximately normally distributed with mean .3 and standard deviation (or standard error)  $\sqrt{\frac{.3 \cdot (1-.3)}{100}} = 0.0458$

$\sigma_{\hat{p}} = 0.0458$

.30

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**II. Test Statistic**

$$z = \frac{\hat{p} - p_0}{\sigma_{\hat{p}}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 \cdot (1 - p_0)}{n}}}$$

$$= \frac{.25 - .3}{\sqrt{\frac{.3 \cdot (1-.3)}{100}}} = -1.09$$

This implies that the statistic is 1.09 standard deviations away from the mean .3 under  $H_0$ , and is to the left of .3 (or less than .3)

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**Level of Significance**

**Level of significance for the test ( $\alpha$ )**  
A probability level selected by the researcher at the beginning of the analysis that defines unlikely values of sample statistic if null hypothesis is true.

c.v. = critical value

Total tail area =  $\alpha$

c.v.    0    c.v.

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# Hypothesis Testing for Proportions

### III. Decision Rule

**Critical value approach:** Compare the test statistic with the **critical values** defined by significance level  $\alpha$ , usually  $\alpha = 0.05$ .

We reject the null hypothesis, if the test statistic  $z < -z_{\alpha/2} = -z_{0.025} = -1.96$ , or  $z > z_{\alpha/2} = z_{0.025} = 1.96$ . (i.e.,  $|z| > z_{\alpha/2}$ )

Two-sided Test

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### III. Decision Rule

**p-value approach:** Compare the probability of the evidence or more extreme evidence to occur when null hypothesis is true. If this probability is less than the level of significance of the test,  $\alpha$ , then we reject the null hypothesis. (Reject  $H_0$  if  $p\text{-value} < \alpha$ )

$p\text{-value} = P(Z \leq -1.09 \text{ or } Z \geq 1.09)$   
 $= 2 \times P(Z \leq -1.09) = 2 \times .1379 = .2758$

Two-sided Test

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### p-value

♥ **p-value** ♥

**The probability of obtaining a test statistic that is as extreme or more extreme than actual sample statistic value given null hypothesis is true.** It is a probability that indicates the extremeness of evidence against  $H_0$ .

**The smaller the p-value, the stronger the evidence for supporting  $H_a$  and rejecting  $H_0$ .**

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### IV. Draw conclusion

Since from either critical value approach  $z = -1.09 > -z_{\alpha/2} = -1.96$  or p-value approach  $p\text{-value} = .2758 > \alpha = .05$ , we do not reject null hypothesis.

Therefore we conclude that **there is no sufficient evidence to support the alternative hypothesis that the percentage of votes would be different from 30%.**

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### Steps in Hypothesis Testing

1. State hypotheses:  $H_0$  and  $H_a$ .
2. Choose a proper **test statistic**, collect data, checking the assumption and compute the value of the statistic.
3. Make decision rule based on **level of significance( $\alpha$ )**.
4. Draw conclusion.  
 (Reject or not reject null hypothesis)  
 (Support or not support alternative hypothesis)

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### When do we use this z-test for testing the proportion of a population?

- Large random sample.

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# Hypothesis Testing for Proportions

## One-Sided Test

### Example with the same data:

A random sample of **100** subjects is chosen and the **sample proportion is 25%**.

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## I. Hypothesis

One wishes to test whether the percentage of votes for A is less than 30%

$$H_0: p = 30\% \text{ v.s. } H_a: p < 30\%$$

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## Evidence

What will be the key statistic (evidence) to use for testing the hypothesis about population proportion?

**Sample Proportion:**  $\hat{p}$

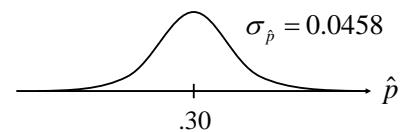
**A random sample of 100** subjects is chosen and the **sample proportion is 25%** or **.25**.

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## Sampling Distribution

If  $H_0: p = 30\%$  is true, sampling distribution of sample proportion will be approximately normally distributed with **mean .3** and **standard**

**deviation** (or standard error)  $\sqrt{\frac{.3 \cdot (1-.3)}{100}} = 0.0458$

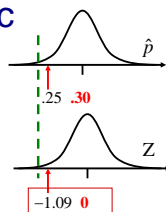


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## II. Test Statistic

$$z = \frac{\hat{p} - p_0}{\sigma_{\hat{p}}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 \cdot (1 - p_0)}{n}}}$$

$$= \frac{.25 - .3}{\sqrt{\frac{.3 \cdot (1-.3)}{100}}} = -1.09$$



This implies that the statistic is **1.09 standard deviations** away from the mean .3 under  $H_0$ , and is to the left of .3 (or less than .3)

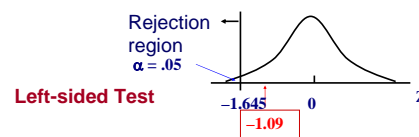
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## III. Decision Rule

**Critical value approach:** Compare the test statistic with the **critical values** defined by significance level  $\alpha$ , usually  $\alpha = 0.05$ .

We reject the null hypothesis, if the test statistic

$$z < -z_{\alpha} = -z_{0.05} = -1.645,$$



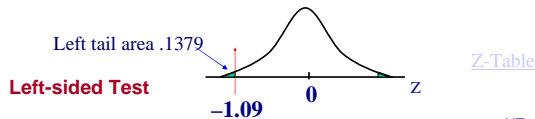
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# Hypothesis Testing for Proportions

## III. Decision Rule

**p-value approach:** Compare the probability of the evidence or more extreme evidence to occur when null hypothesis is true. If this probability is less than the level of significance of the test,  $\alpha$ , then we reject the null hypothesis.

$$p\text{-value} = P(Z \leq -1.09) = P(Z \leq -1.09) = .1379$$



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## IV. Draw conclusion

Since from either critical value approach  $z = -1.09 > -z_{\alpha/2} = -1.645$  or p-value approach  $p\text{-value} = .1379 > \alpha = .05$ , we do not reject null hypothesis.

Therefore we conclude that **there is no sufficient evidence to support the alternative hypothesis that the percentage of votes is less than 30%.**

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Can we see data and then make hypothesis?

1. Choose a **test statistic**, compute it, and check the p-value. **data snooping!**
2. State hypotheses:  $H_0$  and  $H_A$ .
3. Make decision rule based on **level of significance ( $\alpha$ )**.
4. Draw conclusion. (Reject null hypothesis or not)

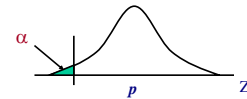
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## Errors in Hypothesis Testing

**Possible statistical errors:**

- **Type I error.** The null hypothesis is true, but we reject it.
- **Type II error.** The null hypothesis is false, but we don't reject it.

" $\alpha$ " is the probability of committing Type I Error.



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## One-Sample z-test for a population proportion

**z-test:**

**Step 1:** State Hypotheses (choose one of the three hypotheses below)

- i)  $H_0 : p = p_0$  v.s.  $H_A : p \neq p_0$  (Two-sided test)
- ii)  $H_0 : p = p_0$  v.s.  $H_A : p > p_0$  (Right-sided test)
- iii)  $H_0 : p = p_0$  v.s.  $H_A : p < p_0$  (Left-sided test)

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## Test Statistic

**Step 2:** Compute z test statistic:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

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# Hypothesis Testing for Proportions

## Step 3: Decision Rule:

**p-value approach:** Compute  $p$ -value,  
 if  $H_A: p \neq p_0$ ,  $p\text{-value} = 2 \cdot P(Z \geq |z|)$   
 if  $H_A: p > p_0$ ,  $p\text{-value} = P(Z \geq z)$   
 if  $H_A: p < p_0$ ,  $p\text{-value} = P(Z \leq z)$   
reject  $H_0$  if  $p\text{-value} < \alpha$

**Critical value approach:** Determine critical value(s) using  $\alpha$ , reject  $H_0$  against  
 i)  $H_A: p \neq p_0$ , if  $|z| > z_{\alpha/2}$   
 ii)  $H_A: p > p_0$ , if  $z > z_\alpha$   
 iii)  $H_A: p < p_0$ , if  $z < -z_\alpha$

## Step 4: Draw Conclusion.

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**Example:** A researcher hypothesized that the percentage of the people living in a community who has no insurance coverage during the past 12 months is **not 10%**. In his study, 1000 individuals from the community were randomly surveyed and checked whether they were covered by any health insurance during the 12 months. Among them, 122 answered that they did not have any health insurance coverage during the last 12 months. Test the researcher's hypothesis at the level of significance of 0.05.

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**Hypothesis:**  $H_0: p = .10$  v.s.  $H_A: p \neq .10$  (Two-sided test)

**Test Statistic:** 
$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{.122 - .10}{\sqrt{\frac{.10(1-.10)}{1000}}} = 2.32$$
  
 $p\text{-value} = 2 \times .0102 = .0204$

**Decision Rule:** Reject null hypothesis if  $p\text{-value} < .05$ .

**Conclusion:**  $p\text{-value} = .0204 < .05$ . There is sufficient evidence to support the alternative hypothesis that the percentage is statistically significantly different from 10%.

Ex. 8.10

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## Two Independent Samples $z$ -test for Two Proportions

**Purpose:** Compare proportions of two populations  
**Assumption:** Two independent large random samples.

### Step 1: Hypothesis:

- 1)  $H_0: p_1 = p_2$  v.s.  $H_A: p_1 \neq p_2$
- 2)  $H_0: p_1 = p_2$  v.s.  $H_A: p_1 > p_2$
- 3)  $H_0: p_1 = p_2$  v.s.  $H_A: p_1 < p_2$

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If a random sample of size  $n_1$  from population 1 has  $x_1$  successes, and a random sample of size  $n_2$  from population 2 has  $x_2$  successes, the sample proportions of these two samples are

$$\hat{p}_1 = \frac{x_1}{n_1} \quad (\text{proportion of successes in sample 1})$$

$$\hat{p}_2 = \frac{x_2}{n_2} \quad (\text{proportion of successes in sample 2})$$

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} \quad (\text{overall sample proportion of successes})$$

**Step 2: Test Statistic:**

$$z = \frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2)}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

(If  $H_0: p_1 = p_2$ , then  $p_1 - p_2 = 0$ )  
 $z$  has a standard normal distribution if  $n_1$  and  $n_2$  are large.

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### Step 3: Decision Rule:

**p-value approach:** Compute  $p$ -value,  
 if  $H_A: p_1 \neq p_2$ ,  $p\text{-value} = 2 \cdot P(Z \geq |z|)$   
 if  $H_A: p_1 > p_2$ ,  $p\text{-value} = P(Z \geq z)$   
 if  $H_A: p_1 < p_2$ ,  $p\text{-value} = P(Z \leq z)$   
reject  $H_0$  if  $p\text{-value} < \alpha$

**Critical value approach:** Determine critical value(s) using  $\alpha$ , reject  $H_0$  against  
 i)  $H_A: p_1 \neq p_2$  if  $|z| > z_{\alpha/2}$   
 ii)  $H_A: p_1 > p_2$  if  $z > z_\alpha$   
 iii)  $H_A: p_1 < p_2$  if  $z < -z_\alpha$

### Step 4: Conclusion

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# Hypothesis Testing for Proportions

**Example:** Test to see if the percentage of smokers in country A is significant different from country B, at 5% level of significance?  
 For country A, 1500 adults were randomly selected and 551 of them were smokers.  
 For country B, 2000 adults were randomly selected and 652 of them were smokers.

$$\hat{p}_1 = 551/1500 = .367 \text{ (Country A)}$$

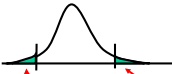
$$\hat{p}_2 = 652/2000 = .326 \text{ (Country B)}$$

$$\hat{p} = (551+652)/(1500+2000) = .344 \text{ (overall percentage of smokers)}$$

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**Step 1:**  
**Hypothesis:**  
 $H_0: p_1 = p_2$  v.s.  $H_A: p_1 \neq p_2$

**Step 2:**  
**Test Statistic:**

$$z = \frac{.367 - .326 - 0}{\sqrt{.344(1 - .344)\left(\frac{1}{1500} + \frac{1}{2000}\right)}} = 2.53$$


$p\text{-value} = .0057 \times 2 = 0.0114$

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**Step 3:**  
**Decision Rule:** Using the level of significance at 0.05, the null hypothesis would be rejected if  $p\text{-value}$  is less than 0.05.

**Step 4:**  
**Conclusion:**  
 Since  $p\text{-value} = 0.0114 < 0.05$ , the null hypothesis is rejected. There is sufficient evidence to support the alternative hypothesis that there is a statistically significant difference in the percentages of smokers in country A and country B.

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**Confidence interval:** The  $(1 - \alpha)\%$  confidence interval estimate for the difference of two population proportions is

$$\hat{p}_1 - \hat{p}_2 \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}$$

**The 95% confidence interval estimate for the difference of the two population proportions is:**

$$.367 - .326 \pm 1.96 \cdot \sqrt{\frac{.367(1 - .367)}{1500} + \frac{.326(1 - .326)}{2000}}$$

$$.041 \pm .032 \Rightarrow 4.1\% \pm 3.2\%$$

(0.9%, 7.3%) → CI does not cover 0 implies significant difference.

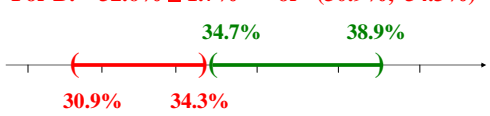
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**Confidence Interval Estimate of One Proportion**

$$\hat{p}_1 = 551/1500 = .367 = 36.7\% \text{ (from A)}$$

$$\hat{p}_2 = 652/2000 = .326 = 32.6\% \text{ (from B)}$$

**For A:** 36.7% ± 2% or (34.7%, 38.9%)  
**For B:** 32.6% ± 1.7% or (30.9%, 34.3%)



Two CI's do not overlap implies significant difference.

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**Methods of Testing Hypotheses**

- Traditional Critical Value Method
- P-value Method
- Confidence Interval Method

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