Similar Matrices and Diagonalizable Matrices

S. F. Ellermeyer

July 1, 2002

1 Similar Matrices

Definition 1 *If A and B are nxn (square) matrices, then A is said to be similar to B if there exists an invertible nxn matrix, P, such that $A = P^{-1}BP$.*

Example 2 *Let A and B be the matrices*

$$A = \begin{bmatrix} 13 & -8 \\ 25 & -17 \end{bmatrix}, \qquad B = \begin{bmatrix} -4 & 7 \\ 3 & 0 \end{bmatrix}.$$

Then A is similar to B because $A = P^{-1}BP$ where

$$P = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}.$$

Proposition 3 *If A and B are nxn matrices and A is similar to B, then B is similar to A. (Thus, we can just say that A and B are similar to each other.)*

Proof. If A is similar to B, then there exists an invertible nxn matrix, P, such that $A = P^{-1}BP$. Multiplying both sides of this equation on the left by P, we obtain $PA = BP$. Then, multiplying both sides of this equation on the right by P^{-1}, we obtain $PAP^{-1} = B$ or $\left(P^{-1}\right)^{-1} AP^{-1} = B$. This shows that $B = Q^{-1}AQ$ where Q is the matrix $Q = P^{-1}$ which is invertible. Thus, B is similar to A. ■

Exercise 4 *For the matrices A, B, and P of Example 2, verify by direct computation that $A = P^{-1}BP$ and that $B = PAP^{-1}$.*

Theorem 5 *If the matrices A and B are similar to each other, then A and B have the same characteristic equation, and hence have the same eigenvalues.*

Proof. If A and B are similar to each other, then there exists an invertible matrix P such that $A = P^{-1}BP$. The characteristic equation of A is $\det(A - \lambda I) = 0$ and the characteristic equation of B is $\det(B - \lambda I) = 0$. However, note that for any number λ, we have

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(P^{-1}BP - \lambda I\right) \\
&= \det\left(P^{-1}BP - \lambda P^{-1}IP\right) \\
&= \det\left(P^{-1}BP - P^{-1}(\lambda I)P\right) \\
&= \det\left(P^{-1}(BP - (\lambda I)P)\right) \\
&= \det\left(P^{-1}(B - \lambda I)P\right) \\
&= \det\left(P^{-1}\right)\det(B - \lambda I)\det(P) \\
&= \det\left(P^{-1}P\right)\det(B - \lambda I) \\
&= \det(I)\det(B - \lambda I) \\
&= 1 \cdot \det(B - \lambda I) \\
&= \det(B - \lambda I)
\end{aligned}$$

which shows that A and B have the same characteristic equation and hence the same eigenvalues. ■

Exercise 6 *Show by direct computation that the matrices A and B of Example 2 have the same characteristic equation. What are the eigenvalues of A and B?*

2 Diagonalizable Matrices

Definition 7 *A diagonal matrix is a square matrix with all of its off–diagonal entries equal to zero.*

Example 8 *The matrix*

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is a diagonal matrix.

An important property of diagonal matrices is that it is easy to compute their powers. For example, using the matrix B in the above example, we have

$$B^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$B^3 = \left(B^2\right) B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -125 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

and in general

$$B^k = \begin{bmatrix} (1)^k & 0 & 0 \\ 0 & (-5)^k & 0 \\ 0 & 0 & (3)^k \end{bmatrix}.$$

This example illustrates the general idea: If B is any diagonal matrix and k is any positive integer, then B^k is also a diagonal matrix and each diagonal entry of B^k is the corresponding diagonal entry of B raised to the power k.

Definition 9 *An nxn matrix, A, is said to be diagonalizable if it is similar to diagonal matrix.*

If A is diagonalizable, it is also easy to compute powers of A. In particular, if A is similar to a diagonal matrix B, then there exists an invertible matrix P such that $A = P^{-1}BP$. We thus have

$$A^2 = \left(P^{-1}BP\right)\left(P^{-1}BP\right) = \left(P^{-1}B\right)\left(PP^{-1}\right)(BP) = P^{-1}B^2P$$
$$A^3 = \left(A^2\right)A = \left(P^{-1}B^2P\right)\left(P^{-1}BP\right) = P^{-1}B^3P$$

and in general

$$A^k = P^{-1}B^kP.$$

Since B^k is easy to compute, then so is A^k. It only requires two matrix multiplications. (First compute $P^{-1}B^k$ and then multiply the result on the right by P.)

Theorem 10 *If A is an nxn matrix and A has n linearly independent eigenvectors, then A is diagonalizable.*

Proof. Suppose that A has eigenvalues λ_1, λ_2,...,λ_n with corresponding linearly independent eigenvectors $\mathbf{v}_1$,$\mathbf{v}_2$,...,$\mathbf{v}_n$.

Let B be the diagonal matrix

$$B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

and let P be the matrix

$$P = [\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_n] \, .$$

Then P is invertible because its columns form a linearly independent set. Also

$$AP = A\,[\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_n] = [A\mathbf{v}_1\ A\mathbf{v}_2 \cdots A\mathbf{v}_n] = [\lambda_1\mathbf{v}_1\ \lambda_2\mathbf{v}_2 \cdots \lambda_n\mathbf{v}_n]$$

and

$$PB = \left[P\begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\ P\begin{bmatrix} 0 \\ \lambda_2 \\ \vdots \\ 0 \end{bmatrix} \cdots P\begin{bmatrix} 0 \\ 0 \\ \vdots \\ \lambda_n \end{bmatrix} \right] = [\lambda_1\mathbf{v}_1\ \lambda_2\mathbf{v}_2 \cdots \lambda_n\mathbf{v}_n]$$

which shows that $AP = PB$ and hence that $A = PBP^{-1}$. Thus, A is diagonalizable. ■

The proof of the above theorem shows us how, in the case that A has n linearly independent eigenvectors, to find both a diagonal matrix B to which A is similar and an invertible matrix P for which $A = PBP^{-1}$. We state this as a corollary.

Corollary 11 *If A is an nxn matrix and A has n linearly independent eigenvectors $\mathbf{v}_1$,$\mathbf{v}_2$,...,$\mathbf{v}_n$ with corresponding eigenvalues λ_1,λ_2,...,λ_n, then $A = PBP^{-1}$ where B is the diagonal matrix*

$$B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

and P is the invertible matrix $P = [\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_n]$.

Example 12 *Let us show that the matrix*

$$A = \begin{bmatrix} 13 & -8 \\ 25 & -17 \end{bmatrix}$$

is diagonalizable.

First, we study the characteristic equation of A*. Since*

$$A - \lambda I = \begin{bmatrix} 13-\lambda & -8 \\ 25 & -17-\lambda \end{bmatrix},$$

the characteristic equation of A *is*

$$(13-\lambda)(-17-\lambda) - (-8)(25) = 0$$

which can be written as

$$\lambda^2 + 4\lambda - 21 = 0$$

or, in factored form, as

$$(\lambda + 7)(\lambda - 3) = 0.$$

We thus see that the eigenvalues of A *are* $\lambda_1 = -7$ *and* $\lambda_2 = 3$.

To find an eigenvector of A *corresponding to* $\lambda_1 = -7$*, we must find a non–trivial solution of the equation* $(A - (-7)I)\mathbf{x} = \mathbf{0}$*. Since*

$$A - (-7)I = \begin{bmatrix} 20 & -8 \\ 25 & -10 \end{bmatrix} \sim \begin{bmatrix} -5 & 2 \\ 0 & 0 \end{bmatrix},$$

we see that an eigenvector of A *corresponding to* $\lambda_1 = -7$ *is* $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

To find an eigenvector of A *corresponding to* $\lambda_2 = 3$*, we must find a non–trivial solution of the equation* $(A - 3I)\mathbf{x} = \mathbf{0}$*. Since*

$$A - 3I = \begin{bmatrix} 10 & -8 \\ 25 & -20 \end{bmatrix} \sim \begin{bmatrix} -5 & 4 \\ 0 & 0 \end{bmatrix},$$

we see that an eigenvector of A *corresponding to* $\lambda_2 = 3$ *is* $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Since the vectors $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are linearly independent, we conclude that* $A = PBP^{-1}$ *where* B *is the diagonal matrix*

$$B = \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix}$$

and P is the invertible matrix

$$P = \begin{bmatrix} 2 & 4 \\ 5 & 5 \end{bmatrix}.$$

Exercise 13 *For the matrices A, B, and P of the above example, verify by direct computation that $A = PBP^{-1}$.*

Exercise 14 *Show that the matrix*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

is diagonalizable by finding a diagonal matrix B and an invertible matrix P such that $A = PBP^{-1}$.

Exercise 15 *Show that the matrix*

$$A = \begin{bmatrix} 0 & -4 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

is diagonalizable by finding a diagonal matrix B and an invertible matrix P such that $A = PBP^{-1}$.

As it turns out, the converse of Theorem 10 is also true.

Theorem 16 *If A is an nxn matrix and A is diagonalizable, then A has n linearly independent eigenvectors.*

Proof. If A is diagonalizable, then there is a diagonal matrix B and an invertible matrix P such that $A = PBP^{-1}$. Suppose that

$$B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

and $P = [\mathbf{v}_1 \; \mathbf{v}_2 \cdots \mathbf{v}_n]$.

Since P is invertible, then we know that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ form a linearly independent set. Also, since $AP = PB$, we see (as in the proof of Theorem 10) that $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$,…, $A\mathbf{v}_n = \lambda_n\mathbf{v}_n$, which means that $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ are all eigenvectors of A (with corresponding eigenvalues λ_1, λ_2,…, λ_n). ■

Exercise 17 *Show that the matrix*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is not diagonalizable.

Exercise 18 *Show that the matrix*

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

is not diagonalizable.

3 Application: Linear Difference Equations

A linear difference equation is an equation of the form

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \tag{1}$$

where A is a (known) nxn matrix. Given a vector $\mathbf{x}_1 \in \Re^n$, to "solve" the difference equation (1) means to find the entire sequence of vectors $\mathbf{x}_k$, $k = 1, 2, 3, \ldots$ In particular, it is often a problem of interest to know the behavior of the sequence $\mathbf{x}_k$ for large values of k. For instance, a question that might be of interest is "Does $\lim_{k\to\infty} \mathbf{x}_k$ exist or does the sequence $\mathbf{x}_k$ behave in a periodic or perhaps even unpredictable fashion as $k \to \infty$?"

Example 19 *Let A be the matrix*

$$A = \begin{bmatrix} 5 & -4.5 \\ 6 & -5.5 \end{bmatrix}$$

and consider the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$. Given the vector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, what is the behavior of the sequence $\mathbf{x}_k$ as $k \to \infty$?

Solution 1 (informal solution by direct computation): *By direct computation, we see that*

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 5 & -4.5 \\ 6 & -5.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$
$$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 5 & -4.5 \\ 6 & -5.5 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$
$$\mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix} 5 & -4.5 \\ 6 & -5.5 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 4.5 \end{bmatrix}$$
$$x_5 = A\mathbf{x}_4 = \begin{bmatrix} 5 & -4.5 \\ 6 & -5.5 \end{bmatrix} \begin{bmatrix} 3.5 \\ 4.5 \end{bmatrix} = \begin{bmatrix} -2.75 \\ -3.75 \end{bmatrix}$$
$$\mathbf{x}_6 = A\mathbf{x}_5 = \begin{bmatrix} 5 & -4.5 \\ 6 & -5.5 \end{bmatrix} \begin{bmatrix} -2.75 \\ -3.75 \end{bmatrix} = \begin{bmatrix} 3.125 \\ 4.125 \end{bmatrix}$$
$$\mathbf{x}_7 = A\mathbf{x}_6 = \begin{bmatrix} 5 & -4.5 \\ 6 & -5.5 \end{bmatrix} \begin{bmatrix} 3.125 \\ 4.125 \end{bmatrix} = \begin{bmatrix} -2.9375 \\ -3.9375 \end{bmatrix}.$$

After have done the computations of $\mathbf{x}_1$ *through* $\mathbf{x}_7$*, it is easy to see that there is a pattern: It appears that for any odd* k *(except* $k = 1$*), both entries of* $\mathbf{x}_k$ *are negative numbers; whereas for any even* k*, both entries of* $\mathbf{x}_k$ *are positive numbers. Furthermore, the odd iterates appear to be getting closer and closer to the vector* $\begin{bmatrix} -3 \\ -4 \end{bmatrix}$ *as* $k \to \infty$ *and the even iterates appear to be getting closer and closer to the vector* $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ *as* $k \to \infty$*. In fact, we are led by this observation to also observe that*

$$A \begin{bmatrix} -3 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 & -4.5 \\ 6 & -5.5 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

and

$$A \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 & -4.5 \\ 6 & -5.5 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}.$$

Another way to state this is that

$$A^2 \begin{bmatrix} -3 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

and

$$A^2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Using terminology from the subject of difference equations, we would say that the vectors $\begin{bmatrix} -3 \\ -4 \end{bmatrix}$ *and* $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ *are* ***period–two points*** *of the matrix* A *or that the set of vectors* $\left\{ \begin{bmatrix} -3 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ *is a* ***period–two orbit*** *of the matrix* A*. We would also say that the vector* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *is* ***attracted*** *to this period–two orbit as* $k \to \infty$ *(meaning that if* $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$*, then the sequence* $\mathbf{x}_k$ *approaches closer and closer to this period–two orbit as* $k \to \infty$*).*

Solution 2 (more formal solution): *To get a more precise description of the sequence* $\mathbf{x}_k$*, we note that the eigenvalues of the matrix* A *are* $\lambda_1 = 0.5$ *and* $\lambda_2 = -1$*. An eigenvector corresponding to* $\lambda_1 = 0.5$ *is* $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and an eigenvector corresponding to* $\lambda_2 = -1$ *is* $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ *.*

Since the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ *is a linearly independent set, then any vector in* $\Re^2$ *can be expressed as a linear combination of these two vectors. In particular, the vector* $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *can be expressed as a linear combination of* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*. Since*

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix},$$

we in fact see that $\mathbf{x}_1 = 4\mathbf{v}_1 - \mathbf{v}_2$*.*

Since $A\mathbf{v}_1 = 0.5\mathbf{v}_1$ *and* $A\mathbf{v}_2 = -\mathbf{v}_2$*, we have* $A^k\mathbf{v}_1 = (0.5)^k\,\mathbf{v}_1$ *and* $A^k\mathbf{v}_2 =$

$(-1)^k \mathbf{v}_2$ *for all integers* $k = 1, 2, 3, \ldots$. *Thus,*

$$\begin{aligned}
\mathbf{x}_{k+1} &= A\mathbf{x}_k \\
&= A^k \mathbf{x}_1 \\
&= A^k (4\mathbf{v}_1 - \mathbf{v}_2) \\
&= 4A^k \mathbf{v}_1 - A^k \mathbf{v}_2 \\
&= 4(0.5)^k \mathbf{v}_1 - (-1)^k \mathbf{v}_2 \\
&= 4(0.5)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} - (-1)^k \begin{bmatrix} 3 \\ 4 \end{bmatrix}.
\end{aligned}$$

Since $\lim_{k\to\infty} (0.5)^k = 0$ *and* $(-1)^k$ *alternates between* $+1$ *and* -1 *as we increase* k*, we now have a more formal verification of what is happening to the sequence* $\mathbf{x}_k$ *as* $k \to \infty$.

Example 20 *In the previous example, we determined the behavior as* $k \to \infty$ *of the sequence defined recursively by* $\mathbf{x}_{k+1} = A\mathbf{x}_k$ *where*

$$A = \begin{bmatrix} 5 & -4.5 \\ 6 & -5.5 \end{bmatrix}$$

and

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The key fact needed was that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ *are eigenvectors of* A *with corresponding eigenvalues* $\lambda_1 = 0.5$ *and* $\lambda_2 = -1$. *Also needed was the fact that* $\mathbf{x}_1 = 4\mathbf{v}_1 - \mathbf{v}_2$. *If we had been given some other "initial vector"* $\mathbf{x}_1$*, we could use the exact same process to study the problem and the only thing that would be different would be to find scalars* c_1 *and* c_2 *such that* $\mathbf{v}_1 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. *This would be a tedious process if we wanted to study solutions of the difference equation for many different initial vectors* $\mathbf{x}_1$. *However, since the matrix* A *is diagonalizable, there is a much more efficient approach to the general problem. In particular, we know that* $A = PBP^{-1}$ *where*

$$B = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}.$$

Thus, for any positive integer k we have

$$\begin{aligned}
A^k &= P^{-1}B^kP \\
&= \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} (0.5)^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} (0.5)^k & 3(-1)^k \\ (0.5)^k & 4(-1)^k \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 4(0.5)^k - 3(-1)^k & -3(0.5)^k + 3(-1)^k \\ 4(0.5)^k - 4(-1)^k & -3(0.5)^k + 4(-1)^k \end{bmatrix}.
\end{aligned}$$

We can now see that for ***any*** *given initial vector,*

$$\mathbf{x}_1 = \begin{bmatrix} a \\ b \end{bmatrix},$$

we have

$$\begin{aligned}
\mathbf{x}_{k+1} &= A\mathbf{x}_k \\
&= A^k\mathbf{x}_1 \\
&= \begin{bmatrix} 4(0.5)^k - 3(-1)^k & -3(0.5)^k + 3(-1)^k \\ 4(0.5)^k - 4(-1)^k & -3(0.5)^k + 4(-1)^k \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.
\end{aligned}$$

Since $\lim_{k\to\infty}(0.5)^k = 0$*, we observe that if k is very large, then*

$$A^k \approx \begin{bmatrix} -3(-1)^k & 3(-1)^k \\ -4(-1)^k & 4(-1)^k \end{bmatrix}$$

which means that if k is very large, then

$$\begin{aligned}
\mathbf{x}_{k+1} &\approx \begin{bmatrix} -3(-1)^k & 3(-1)^k \\ -4(-1)^k & 4(-1)^k \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
&= \begin{bmatrix} -3a(-1)^k + 3b(-1)^k \\ -4a(-1)^k + 4b(-1)^k \end{bmatrix} \\
&= -(-1)^k a \begin{bmatrix} 3 \\ 4 \end{bmatrix} + (-1)^k b \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\
&= \left(-(-1)^k a + (-1)^k b\right) \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\
&= (-1)^k (b-a) \begin{bmatrix} 3 \\ 4 \end{bmatrix}.
\end{aligned}$$

Thus, the sequence $\mathbf{v}_k$ *tends to jump back and forth between* $(b-a)\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ *and* $(a-b)\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ *as* $k \to \infty$*. This explains what we observed in Example 19 in the case that* $a = 1$ *and* $b = 0$*, and if, for example, we were to use the initial vector*

$$\mathbf{x}_1 = \begin{bmatrix} -4 \\ 7 \end{bmatrix},$$

we would observe that $\mathbf{x}_k$ *would tend to jump back and forth between* $\begin{bmatrix} 33 \\ 44 \end{bmatrix}$ *and* $\begin{bmatrix} -33 \\ -44 \end{bmatrix}$ *as* $k \to \infty$*.*

Finally, note that the behavior is totally difference if $\mathbf{x}_1$ *is a vector for which* $a = b$*, such as for example,* $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$*. In this case, we have* $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{0}$.

Exercise 21 *Consider the difference equation* $\mathbf{x}_{k+1} = A\mathbf{x}_k$ *where*

$$A = \begin{bmatrix} 13 & -8 \\ 25 & -17 \end{bmatrix}.$$

Compute A^k *for any positive integer* k *and describe as fully as possible the behavior of the sequence* $\mathbf{x}_k$ *for all possible choices of initial vector* $\mathbf{x}_1$*.*