

## Section 6.5, Trigonometric Form of a Complex Number

Homework: 6.5 #1, 3, 5, 11–17 odds, 21, 31–37 odds, 45–57 odds, 71, 77, 87, 89, 91, 105, 107

### 1 Review of Complex Numbers

Complex numbers can be written as  $z = a + bi$ , where  $a$  and  $b$  are real numbers, and  $i = \sqrt{-1}$ . This form,  $a + bi$ , is called the **standard form** of a complex number.

When graphing these, we can represent them on a coordinate plane called the **complex plane**. It is a lot like the  $x$ - $y$ -plane, but the horizontal axis represents the real coordinate of the number, and the vertical axis represents the imaginary coordinate.

#### Examples

Graph each of the following numbers on the complex plane:

$2 + 3i$ ,  $-1 + 4i$ ,  $-3 - 2i$ ,  $4$ ,  $-i$  (Graph sketched in class)

The **absolute value** of a complex number is its distance from the origin. If  $z = a + bi$ , then

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

#### Example

Find  $|-1 + 4i|$ .

$$|-1 + 4i| = \sqrt{1 + 16} = \sqrt{17}$$

### 2 Trigonometric Form of a Complex Number

The **trigonometric form** of a complex number  $z = a + bi$  is

$$z = r(\cos \theta + i \sin \theta),$$

where  $r = |a + bi|$  is the **modulus** of  $z$ , and  $\tan \theta = \frac{b}{a}$ .  $\theta$  is called the **argument** of  $z$ . Normally, we will require  $0 \leq \theta < 2\pi$ .

#### Examples

- Write the following complex numbers in trigonometric form:

(a)  $-4 + 4i$

To write the number in trigonometric form, we need  $r$  and  $\theta$ .

$$r = \sqrt{16 + 16} = \sqrt{32} = 4\sqrt{2}$$

$$\tan \theta = \frac{4}{-4} = -1$$

$$\theta = \frac{3\pi}{4},$$

since we need an angle in quadrant II (we can see this by graphing the complex number). Then,

$$-4 + 4i = 4\sqrt{2} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

Note: You want to leave the angle  $\theta$  in your answer instead of simplifying. There are several reasons for this. First, we worked hard to get the angle. Second, it will be easier to do certain mathematical operations if we have the angle, as we'll see later in this section.

(b)  $2 - \frac{2\sqrt{3}}{3}i$

$$r = \sqrt{4 + \frac{12}{9}} = \sqrt{\frac{48}{9}} = \frac{4\sqrt{3}}{3}$$

$$\tan \theta = \frac{-2\sqrt{3}}{3 \cdot 2} = -\frac{\sqrt{3}}{3}$$

$$\theta = \frac{11\pi}{6},$$

since we need an angle in quadrant IV. Then, the trigonometric form is

$$\frac{4\sqrt{3}}{3} \left( \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} \right)$$

2. Write the complex number  $4(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3})$  in standard form.

To go from trigonometric form to standard form, we only need to simplify:

$$4 \left( \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right) = 4 \left( -\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) = -2 - 2\sqrt{3}i$$

### 3 Products and Quotients of Two Complex Numbers

If  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$  and  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$  are two complex numbers in trigonometric form, then

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \text{ and}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

#### Proof of Multiplication Formula

We use “FOIL” to multiply the two trigonometric forms, noting that  $i^2 = -1$ :

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 (\cos \theta_1 \cos \theta_2 + i \sin \theta_2 \cos \theta_1 + i \sin \theta_1 \cos \theta_2 + i^2 \sin \theta_1 \sin \theta_2) \\ &= r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\sin \theta_2 \cos \theta_1 + \sin \theta_1 \cos \theta_2)) \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)], \end{aligned}$$

where we used the sum formulas in Section 5.4 in the last line.

To show that the division formula holds, you can use the multiplication formula and that  $z_1 = \frac{z_1}{z_2} \cdot z_2$ .

#### Examples

Carry out each of the following operations:

$$\begin{aligned} 1. \quad & 3 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \cdot 4 \left( \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right) \\ & 3 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \cdot 4 \left( \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right) = 3 \cdot 4 \left( \cos \left( \frac{\pi}{3} + \frac{7\pi}{4} \right) + i \sin \left( \frac{\pi}{3} + \frac{7\pi}{4} \right) \right) \\ & = 12 \left( \cos \frac{25\pi}{12} + i \sin \frac{25\pi}{12} \right) \\ & = 12 \left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right), \end{aligned}$$

since  $\pi/12$  is coterminal with  $25\pi/12$ .

$$\begin{aligned}
2. \quad & \frac{\sqrt{2}(\cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3})}{\frac{\sqrt{2}}{2}(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})} \\
& \frac{\sqrt{2}(\cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3})}{\frac{\sqrt{2}}{2}(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})} = \sqrt{2} \div \frac{\sqrt{2}}{2} \left[ \cos \left( \frac{8\pi}{3} - \frac{\pi}{2} \right) + i \sin \left( \frac{8\pi}{3} - \frac{\pi}{2} \right) \right] \\
& = 2 \left[ \cos \frac{13\pi}{6} + i \sin \frac{13\pi}{6} \right] \\
& = 2 \left[ \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right],
\end{aligned}$$

since  $13\pi/6$  and  $\pi/6$  are coterminal angles.

## 4 DeMoivre's Theorem

**DeMoivre's Theorem** says that if  $n$  is a positive integer and  $z = r(\cos \theta + i \sin \theta)$  is a complex number, then

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

We can show this by using the multiplication formula from above  $n$  times.

### Example

Calculate  $(3 + 3i)^4$ .

Since this complex number  $3 + 3i$  is not in trigonometric form, we need to first convert it to trigonometric form to use the above formula:

$$3 + 3i = 3\sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

Then,

$$\begin{aligned}
(3 + 3i)^4 &= (3\sqrt{2})^4 \left( \cos 4 \frac{\pi}{4} + i \sin 4 \frac{\pi}{4} \right) \\
&= 324(\cos \pi + i \sin \pi) = -324
\end{aligned}$$

(Depending on the directions, you may not need to convert your answer to standard form, as we did here.)

## 5 Roots of Complex Numbers

The complex number  $z = r(\cos \theta + i \sin \theta)$  has exactly  $n$  distinct  $n^{\text{th}}$  roots. They are:

$$\sqrt[n]{r} \left( \cos \frac{\theta + 2\pi k}{n} + i \sin \frac{\theta + 2\pi k}{n} \right),$$

where  $k = 0, 1, \dots, n - 1$ .

### Examples

1. Find all square roots of  $i$ .

We can write  $i$  in trigonometric form as  $i = 1(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$ . Then, we use the formula with  $r = 1$ ,  $\theta = \frac{\pi}{2}$ ,  $n = 2$ , and  $k = 0$  and  $k = 1$  to see that the two second roots of  $i$  are

$$\begin{aligned}
\sqrt{1} \left( \cos \frac{\frac{\pi}{2}}{2} + i \sin \frac{\frac{\pi}{2}}{2} \right) &= \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \\
\sqrt{1} \left( \cos \frac{\frac{\pi}{2} + 2\pi}{2} + i \sin \frac{\frac{\pi}{2} + 2\pi}{2} \right) &= \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}
\end{aligned}$$

2. Find all sixth roots of 64.

We can write 64 in trigonometric form as  $64(\cos 0 + i \sin 0)$ , so  $r = 64$ ,  $\theta = 0$ , and  $n = 0$ . Then, we will use  $k = 0, 1, 2, 3, 4, 5$  to get the six sixth roots of 64:

$$\sqrt[6]{64} \left( \cos \frac{0}{6} + i \sin \frac{0}{6} \right) = 2$$

$$\sqrt[6]{64} \left( \cos \frac{0+2\pi}{6} + i \sin \frac{0+2\pi}{6} \right) = 2 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) = 2 \left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \right) = 1 + i\sqrt{3}$$

$$\sqrt[6]{64} \left( \cos \frac{0+4\pi}{6} + i \sin \frac{0+4\pi}{6} \right) = 2 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) = 2 \left( -\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) = -1 + i\sqrt{3}$$

$$\sqrt[6]{64} \left( \cos \frac{0+6\pi}{6} + i \sin \frac{0+6\pi}{6} \right) = 2 \left( \cos \pi + i \sin \pi \right) = -2$$

$$\sqrt[6]{64} \left( \cos \frac{0+8\pi}{6} + i \sin \frac{0+8\pi}{6} \right) = 2 \left( \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right) = 2 \left( -\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) = -1 - i\sqrt{3}$$

$$\sqrt[6]{64} \left( \cos \frac{0+10\pi}{6} + i \sin \frac{0+10\pi}{6} \right) = 2 \left( \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right) = 2 \left( \frac{1}{2} - i \frac{\sqrt{3}}{2} \right) = 1 - i\sqrt{3}$$