Solution to Homework 2

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Section 1.4: 1(a)(b)(i)(k), 4, 5, 14; Section 1.5: 1(a)(b)(c)(d)(e)(n), 2(a)(c), 13, 16, 17, 18, 27

Section 1.4

1. Compute the following, if possible, for the matrices:

$$A = \begin{bmatrix} -4 & 2 & 3 \\ 0 & 5 & -1 \\ 6 & 1 & -2 \end{bmatrix} \; B = \begin{bmatrix} 6 & -1 & 0 \\ 2 & 2 & -4 \\ 3 & -1 & 1 \end{bmatrix} \; C = \begin{bmatrix} 5 & -1 \\ -3 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} -7 & 1 & -4 \\ 3 & -2 & 8 \end{bmatrix} \; E = \begin{bmatrix} 3 & -3 & 5 \\ 1 & 0 & -2 \\ 6 & 7 & -2 \end{bmatrix} \; F = \begin{bmatrix} 8 & -1 \\ 2 & 0 \\ 5 & -3 \end{bmatrix}$$

(a) $A + B$

Solution: Adding entry by entry,

$$A + B = \begin{bmatrix} -4 & 2 & 3 \\ 0 & 5 & -1 \\ 6 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 6 & -1 & 0 \\ 2 & 2 & -4 \\ 3 & -1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 1 & 3 \\ 2 & 7 & -5 \\ 9 & 0 & -1 \end{bmatrix}}$$

(b) $C + D$

Solution: C is 2×2, while D is 2×3, and only matrices of the same dimensions can be added. Therefore, this is $\boxed{\text{impossible.}}$

(i) $A^T + E^T$

Solution: Taking the transpose of a matrix is the same as just 'reflecting' it along the main diagonal (or swapping its rows for its columns.) Therefore,

$$
\begin{aligned}
A^T + E^T &= \begin{bmatrix} -4 & 2 & 3 \\ 0 & 5 & -1 \\ 6 & 1 & -2 \end{bmatrix}^T + \begin{bmatrix} 3 & -3 & 5 \\ 1 & 0 & -2 \\ 6 & 7 & -2 \end{bmatrix}^T \\
&= \begin{bmatrix} -4 & 0 & 6 \\ 2 & 5 & 1 \\ 3 & -1 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 6 \\ -3 & 0 & 7 \\ 5 & -2 & -2 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 1 & 12 \\ -1 & 5 & 8 \\ 8 & -3 & -4 \end{bmatrix}}
\end{aligned}
$$

(k) $4D + 2F^T$

Solution: Since multiplying a matrix by a scalar just multiplies each entry by that scalar,

$$
\begin{aligned}
4D + 2F^T &= 4\begin{bmatrix} -7 & 1 & -4 \\ 3 & -2 & 8 \end{bmatrix} + 2\begin{bmatrix} 8 & -1 \\ 2 & 0 \\ 5 & -3 \end{bmatrix}^T \\
&= \begin{bmatrix} -28 & 4 & -16 \\ 12 & -8 & 32 \end{bmatrix} + \begin{bmatrix} 16 & 4 & 10 \\ -2 & 0 & -6 \end{bmatrix} = \boxed{\begin{bmatrix} -12 & 8 & -6 \\ 10 & -8 & 26 \end{bmatrix}}
\end{aligned}
$$

4. Prove that if $A^T = B^T$, then $A = B$.

Proof:
Assumptions: $A^T = B^T$.
Need to show: $A = B$.

If two matrices are equal, then clearly their transposes are equal as well. Therefore, using Theorem 1.12(a),

$$A = (A^T)^T = (B^T)^T = B$$

so we're done. □

Note: This could also be done by considering the (i, j) entry of A and showing it to be equal to the (i, j) entry of B.

5. (a) Prove that any symmetric or skew-symmetric matrix is square.

Solution: This is really two proof questions: show that a symmetric matrix must be square, and show that a skew-symmetric matrix must be square. We will do these separately. Recall that a matrix A is symmetric if $A^T = A$, and is skew-symmetric if $A^T = -A$.

Proof:
Assumptions: A is symmetric: that is, $A^T = A$.
Need to show: A is a square matrix.

Let A be an $m \times n$ matrix. Then, A^T is by definition an $n \times m$ matrix. Since $A = A^T$, the dimensions of A^T must be the same as the dimensions of A. Therefore, $m \times n$ must be the same as $n \times m$, and so we can conclude that $m = n$. This means that A is $n \times n$, which means that A is a square matrix. $\square$

The next proof is almost identical:

Proof:
Assumptions: A is skew-symmetric: that is, $A^T = -A$.
Need to show: A is a square matrix.
Let A be an $m \times n$ matrix. Then, A^T is by definition an $n \times m$ matrix, and therefore $-A^T$ is $n \times m$ as well. Since $A = -A^T$, the dimensions of $-A^T$ must be the same as the dimensions of A. Therefore, $m \times n$ must be the same as $n \times m$, and so we can conclude that $m = n$. This means that A is $n \times n$, which means that A is a square matrix. $\square$

(b) Prove that any diagonal matrix is symmetric.

Proof:
Assumptions: A is diagonal.
Need to show: A is symmetric: that is, $A^T = A$.

This should be fairly intuitively clear, it just needs to be written down. Let A be an $n \times n$ matrix whose (i, j) entry is a_{ij}. Then, since A is diagonal,

$$i \neq j \text{ implies } a_{ij} = 0$$

To show that $A^T = A$, we need to show that the (i, j) entry of A^T is the same as the (i, j) entry of A. Consider two cases:
Case 1: If $i \neq j$ then

$$(i, j) \text{ entry of } A^T = (j, i) \text{ entry of } A = 0 = (i, j) \text{ entry of } A$$

Case 2: If $i = j$, then clearly,

$$(i, i) \text{ entry of } A^T = a_{ii} = (i, i) \text{ entry of } A$$

Therefore, the (i, j) entry of A and A^T transpose coincide, so we're done. $\square$

(c) Show that $(I_n)^T = I_n$. (Hint: Use part (b))

Proof:
Assumptions: None
Need to show: $I_n^T = I_n$.
This follows immediately from (b) – since I_n is by definition a diagonal matrix, and diagonal matrices are symmetric, I_n must be symmetric. Therefore $I_n^T = I_n$. $\square$

(d) Describe completely every matrix that is both diagonal and skew-symmetric.

Solution: Assume that A is diagonal and A is skew-symmetric: that is, $A^T = -A$. Since A is diagonal, we know that its entries off the main diagonal are 0. Since A is skew-symmetric, we know that all the entries on its main diagonal are 0 as well. Therefore, we see that

$$\boxed{A \text{ must be a square 0-matrix}}$$

14. The **trace** of a square matrix A is the sum of the elements along the main diagonal.

(a) Find the trace of each square matrix in Exercise 2.

Solution:

$$\begin{aligned}
&\text{trace}(A) = \text{not defined (not square)}, \text{trace}(B) = 1, \text{trace}(C) = 0,\\
&\text{trace}(D) = \text{not defined}, \text{trace}(E) = -6, \text{trace}(F) = 1, \text{trace}(G) = 18\\
&\text{trace}(H) = 0, \text{trace}(J) = 1, \text{trace}(K) = 4, \text{trace}(L) = 3, \text{trace}(M) = 0,\\
&\text{trace}(N) = 3, \text{trace}(P) = 0, \text{trace}(Q) = 1, \text{trace}(R) = \text{not defined}
\end{aligned}$$

(b) If A and B are both $n \times n$ matrices, prove the following:

Solution: For the remainder of these proofs, assume that A has the (i, j) entry a_{ij} and B has the (i, j) entry b_{ij}.

i. $\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$

Proof:
Assumptions: See above.
Need to show: $\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$.

Note that the (i, i) entry of $A + B$ is $a_{ii} + b_{ii}$ by definition. Since the trace is just the sum of all the (i, i) entries, we see that

$$\begin{aligned}
\text{trace}(A + B) &= (a_{11} + b_{11}) + (a_{22} + b_{22}) + \cdots + (a_{nn} + b_{nn})\\
&= (a_{11} + a_{22} + \cdots + a_{nn}) + (b_{11} + b_{22} + \cdots + b_{nn})\\
&= \text{trace}(A) + \text{trace}(B)
\end{aligned}$$

by definition, so we're done. □

ii. $\text{trace}(cA) = c\text{trace}(A)$

Proof:
Assumptions: See above.
Need to show: $\text{trace}(cA) = c(\text{trace}(A))$.

Note that the (i,i) entry of cA is ca_{ii}. Therefore, by definition

$$\begin{aligned}\operatorname{trace}(cA) &= ca_{11} + ca_{22} + \cdots + ca_{nn} \\ &= c(a_{11} + a_{22} + \cdots + a_{nn}) \\ &= c(\operatorname{trace}(A))\end{aligned}$$

like before. □

iii. $\operatorname{trace}(A) = \operatorname{trace}(A^T)$

Proof:
Assumptions: See above.
Need to show: $\operatorname{trace}(A) = \operatorname{trace}(A^T)$.

Note that the (i,i) entry of A^T is a_{ii} (taking the transpose doesn't change the elements on the diagonal.) Therefore, by definition

$$\operatorname{trace}(A^T) = a_{11} + a_{22} + \cdots + a_{nn} = \operatorname{trace}(A)$$

as required. □

Section 1.5

1. Exercises 1 and 2 refer to the following matrices:

$$A = \begin{bmatrix} -2 & 3 \\ 6 & 5 \\ 1 & -4 \end{bmatrix} B = \begin{bmatrix} -5 & 3 & 6 \\ 3 & 8 & 0 \\ -2 & 0 & 4 \end{bmatrix} C = \begin{bmatrix} 11 & -2 \\ -4 & -2 \\ 3 & -1 \end{bmatrix} D = \begin{bmatrix} -1 & 4 & 3 & 7 \\ 2 & 1 & 7 & 5 \\ 0 & 5 & 5 & -2 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} F = \begin{bmatrix} 9 & -3 \\ 5 & -4 \\ 2 & 0 \\ 8 & -3 \end{bmatrix} G = \begin{bmatrix} 5 & 1 & 0 \\ 0 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix} H = \begin{bmatrix} 6 & 3 & 1 \\ 1 & -15 & -5 \\ -2 & -1 & 10 \end{bmatrix}$$

$$J = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix} K = \begin{bmatrix} 2 & 1 & -5 \\ 0 & 2 & 7 \end{bmatrix} L = \begin{bmatrix} 10 & 9 \\ 8 & 7 \end{bmatrix} M = \begin{bmatrix} 7 & -1 \\ 11 & 3 \end{bmatrix} N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 3 & -1 \\ 4 & 7 \end{bmatrix} Q = \begin{bmatrix} 1 & 4 & -1 & 6 \\ 8 & 7 & -3 & 3 \end{bmatrix} R = \begin{bmatrix} -3 & 6 & -2 \end{bmatrix} S = \begin{bmatrix} 6 & -4 & 3 & 2 \end{bmatrix}$$

$$T = \begin{bmatrix} 4 & -1 & 7 \end{bmatrix}$$

(a)

$$AB = \begin{bmatrix} -2 & 3 \\ 6 & 5 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} -5 & 3 & 6 \\ 3 & 8 & 0 \\ -2 & 0 & 4 \end{bmatrix} = \boxed{\text{not possible}}$$

(b)

$$BA = \begin{bmatrix} -5 & 3 & 6 \\ 3 & 8 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 6 & 5 \\ 1 & -4 \end{bmatrix} = \boxed{\begin{bmatrix} 34 & -24 \\ 42 & 49 \\ 8 & -22 \end{bmatrix}}$$

(c)

$$JM = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ 11 & 3 \end{bmatrix} = \boxed{\text{not possible}}$$

(d)

$$DF = \begin{bmatrix} -1 & 4 & 3 & 7 \\ 2 & 1 & 7 & 5 \\ 0 & 5 & 5 & -2 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ 5 & -4 \\ 2 & 0 \\ 8 & -3 \end{bmatrix} = \boxed{\begin{bmatrix} 73 & -34 \\ 77 & -25 \\ 19 & -14 \end{bmatrix}}$$

(e)

$$RJ = \begin{bmatrix} -3 & 6 & -2 \end{bmatrix} \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} -38 \end{bmatrix}}$$

(n)

$$\begin{aligned} D(FK) &= \begin{bmatrix} -1 & 4 & 3 & 7 \\ 2 & 1 & 7 & 5 \\ 0 & 5 & 5 & -2 \end{bmatrix} \left(\begin{bmatrix} 9 & -3 \\ 5 & -4 \\ 2 & 0 \\ 8 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -5 \\ 0 & 2 & 7 \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} -1 & 4 & 3 & 7 \\ 2 & 1 & 7 & 5 \\ 0 & 5 & 5 & -2 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ 5 & -4 \\ 2 & 0 \\ 8 & -3 \end{bmatrix} \right) \begin{bmatrix} 2 & 1 & -5 \\ 0 & 2 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 73 & -34 \\ 77 & -25 \\ 19 & -14 \end{bmatrix} \begin{bmatrix} 2 & 1 & -5 \\ 0 & 2 & 7 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 146 & 5 & -603 \\ 154 & 27 & -560 \\ 38 & -9 & -193 \end{bmatrix}} \end{aligned}$$

2. Determine whether these pairs of matrices commute.

   (a) L and M

   Solution: To check whether L and M commute, we check whether

$LM = ML$. Accordingly, let's calculate LM and ML:

$$LM = \begin{bmatrix} 10 & 9 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ 11 & 3 \end{bmatrix} = \begin{bmatrix} 169 & 17 \\ 133 & 13 \end{bmatrix}$$

$$ML = \begin{bmatrix} 7 & -1 \\ 11 & 3 \end{bmatrix} \begin{bmatrix} 10 & 9 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 62 & 56 \\ 134 & 120 \end{bmatrix}$$

Clearly, $LM \neq ML$, so they don't commute.

(c) A and K Here, we don't even need to multiple it out. Since A is 3×2 and K is 2×3, AK is 3×3 and KA is 2×2. These can't possibly be the same matrix, so A and K don't commute.

13. For the matrix

$$A = \begin{bmatrix} 7 & -3 & -4 & 1 \\ -5 & 6 & 2 & -3 \\ -1 & 9 & 3 & -8 \end{bmatrix}$$

use matrix multiplication (as in Example 4)to find the following linear combinations:

(a) $-5\vec{v}_1 + 6\vec{v}_2 - 4\vec{v}_3$ where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are the rows of A.

Solution: As we've learned earlier, to get a linear combination of the rows with coefficients $[c_1, c_2, \ldots, c_n]$, we multiply the matrix on the left by the row vector $[c_1, c_2, \ldots, c_n]$. Therefore, we get

$$\begin{bmatrix} -5 & 6 & -4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -4 & 1 \\ -5 & 6 & 2 & -3 \\ -1 & 9 & 3 & -8 \end{bmatrix} = \begin{bmatrix} -61 & 15 & 20 & 9 \end{bmatrix}$$

(b) $6\vec{w}_1 - 4\vec{w}_2 + 2\vec{w}_3 - 3\vec{w}_4$, where $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4$ are the columns of A.

Solution: Similarly to part (a), to get a linear combination of the columns with coefficients $[c_1, c_2, \ldots, c_n]$, we multiply the matrix on the left by the column vector with the same coordinates. Therefore, we get

$$\begin{bmatrix} 7 & -3 & -4 & 1 \\ -5 & 6 & 2 & -3 \\ -1 & 9 & 3 & -8 \end{bmatrix} \begin{bmatrix} 6 \\ -4 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 43 \\ -41 \\ -12 \end{bmatrix}$$

16. Let A be an $m \times n$ matrix. Prove that $AO_{np} = O_{mp}$.

Proof:
Assumptions: A is $m \times n$.
Need to show: $AO_{np} = O_{mp}$.

O_{np} is by definition an $n \times p$ matrix whose every entry is 0. The product of an $m \times n$ matrix with a $n \times p$ matrix is indeed $m \times p$, so AO_{np} is certainly the right size. We now just need to check that every entry of it is 0. By definition,

$$(i, j) \text{ entry of } AO_{np} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } O_{np})$$

where the $\cdot$ indicates dot product. But since O_{np} has every entry equal to 0, column j of this matrix is just $\vec{0}$. Therefore,

$$(i, j) \text{ entry of } AO_{np} = (\text{row } i \text{ of } A) \cdot \vec{0} = 0$$

since any vector dotted with the zero vector results in 0. Thus, we're done. $\square$

17. Let A be an $m \times n$ matrix. Prove that $AI_n = I_m A = A$.

Solution: This is easiest to do by breaking it up into two proofs: $A = AI_n$ and $A = I_m A$.

Proof:
Assumptions: A is $m \times n$.
Need to show: $A = AI_n$

Let the (i, j) entry of A be a_{ij}. Then we need to show that the (i, j) entry of AI_n is a_{ij}. Proceeding like before,

$$\begin{aligned}(i, j) \text{ entry of } AI_n &= (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } I_n) \\ &= [a_{i1}, a_{i2}, \ldots, a_{in}] \cdot [0, \ldots, 0, 1, 0, \ldots, 0]\end{aligned}$$

where the 1 in the vector on the right is in the jth place. Therefore,

$$(i, j) \text{ entry of } AI_n = a_{i1} \cdot 0 + \cdots + a_{ij} \cdot 1 + \cdots + a_{in} \cdot 0 = a_{ij}$$

so we're done. $\square$

The second proof is almost identical:

Proof:
Assumptions: A is $m \times n$.
Need to show: $A = I_m A$

Let the (i, j) entry of A be a_{ij}. Then we need to show that the (i, j) entry of $I_m A$ is a_{ij}. Proceeding like before,

$$\begin{aligned}(i, j) \text{ entry of } I_m A &= (\text{row } i \text{ of } I_m) \cdot (\text{column } j \text{ of } A) \\ &= [0, \ldots, 0, 1, 0, \ldots, 0] \cdot [a_{1j}, a_{2j}, \ldots, a_{mj}]\end{aligned}$$

where the 1 in the vector on the left is in the ith place. Therefore,

$$(i,j) \text{ entry of } AI_n = 0 \cdot a_{1j} + \cdots + 1 \cdot a_{ij} + \cdots + 0 \cdot a_{mj} = a_{ij}$$

so we're done. $\square$

18. (a) Prove that the product of two diagonal matrices is diagonal.

Proof:
Assumptions: A and B are two diagonal matrices.
Need to show: AB is diagonal.

First of all, it is implicit in this proof that A and B have to be the same size, since diagonal matrices are square, and square matrices of different size can't be multiplied at all. So assume that A and B are both $n \times n$ matrices. Let the (i,j) entry of A be a_{ij} and the (i,j) entry of B be b_{ij}.
This musing is not part of the proof, but is a useful way to think: if you try any examples at all, you will soon convince yourself that diagonal matrix multiplication works like this:

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}b_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn}b_{nn} \end{bmatrix}$$

Of course, this isn't quite a proof! What we need is a way to demonstrate the above in general. A good idea is to come up with a formula for the (i,j) entry of the product AB. Since this can be done for every choice of i and j, this will be completely general.
Continuing with the proof:

$$\begin{aligned} (i,j) \text{ entry of } AB &= (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B) \\ &= [0, \ldots, 0, a_{ii}, 0, \ldots, 0] \cdot [0, \ldots, 0, b_{jj}, 0, \ldots, 0] \end{aligned}$$

As you can probably see, different things happen depending on whether $i = j$ or $i \neq j$. If $i = j$, then the only non-zero entries of the vectors above 'match up,' so that the dot product is $a_{ii}b_{ii}$. However, if $j \neq i$, then

$$\begin{aligned} &[0, \ldots, 0, a_{ii}, 0, \ldots, 0] \cdot [0, \ldots, 0, b_{jj}, 0, \ldots, 0] \\ &\qquad = 0 \cdot 0 + \cdots + a_{ii} \cdot 0 + \cdots + 0 \cdot b_{jj} + \cdots 0 \cdot 0 = 0 \end{aligned}$$

since the a_{ii} and the b_{jj} don't multiply together in the dot product. Hence, we see that if $i \neq j$, then the (i, j) entry of AB is 0, and therefore AB is diagonal. □

Note: The above way of writing it out mightn't be completely rigorous (it relies on visuals), but it's clear. In the next proof, I'll be completely rigorous, which unfortunately can be harder to read!

(b) Prove that the product of two upper triangular matrices is upper triangular.

Proof:
Assumptions: A and B are two upper triangular matrices.
Need to show: AB is upper triangular.

It is again implicit in this proof that A and B have to be the same size, since square matrices of different size can't be multiplied. Assume that A and B are both $n \times n$ matrices. Let the (i, j) entry of A be a_{ij} and the (i, j) entry of B be b_{ij}.
Since A and B are upper triangular, all the entries of A and B below the diagonal are 0. An entry a_{ij} is below the diagonal *precisely* if $i > j$. (If you don't believe me, check by labelling every entry of a 3×3 matrix!) Therefore, what we're really given is that:

$$a_{ij} = 0 = b_{ij} \text{ if } i > j$$

What we need to show is clearly that the (i, j) entry of AB is equal to 0 as long as $i > j$. By definition,

$$\begin{aligned}(i, j) \text{ entry of } AB &= (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B) \\ &= [a_{i1}, a_{i2}, \ldots, a_{in}] \cdot [b_{1j}, b_{2j}, \ldots, b_{nj}] \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{in}b_{nj}\end{aligned}$$

Musing: this is clearly a sum of terms of the form $a_{ik}b_{kj}$ – the i and j are as before, and the k just denotes which coordinates we're currently multiplying. If you try actually multiplying upper diagonal matrices, you will see that this sum turns out to be zero as long as (i, j) is an entry below the diagonal, because every single summand in it is zero. As examples aren't proof, let's show this rigorously.
Assume that $i > j$, and let us show that the (i, j) entry of AB is 0. It would clearly suffice to show that $a_{ik}b_{kj} = 0$ for every single value of k. Therefore, we need to show that either a_{ik} or b_{kj} is 0 for all k.
Consider two possibilities $k > j$ and $k \leq j$.

i. If $k > j$, then b_{kj} represent an entry below the main diagonal of B. Therefore, $b_{kj} = 0$ and so the summand $a_{ik}b_{kj} = 0$.

ii. If $k \leq j$, then since $j < i$, $i > k$. Therefore, a_{ik} represents an entry below the main diagonal of A, and $a_{ik} = 0$. Thus, $a_{ik}b_{kj} = 0$.

Since $a_{ik}b_{kj} = 0$ for all k, we see that

$$(i, j) \text{ entry of } AB = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{in}b_{nj} = 0$$

and so AB is upper triamgular, as required. □

Note: The above is a fully rigorous proof. Personally, I'd say it's less illuminating than actually multiplying a pair of upper triangular matrices and seeing what happens (although once you do it, you can see that's exactly what the proof is expressing.)

What is the benefit of something less visual but more 'rigorous'? It turns out to be easier to manipulate when it's not so straightforward to check what exactly is going on, and leads one to be able to prove considerably harder results. It is also less error-prone. For proofs like this, complete rigorous is in some sense just practice!

(c) Prove that the product of two lower triangular matrices is lower triangular.

Proof:
Assumptions: A and B are two lower triangular matrices.
Need to show: AB is lower triangular.

We could prove this result analogously to the result in (b). Instead, we will use a shortcut. Clearly, a matrix C is lower triangular precisely when C^T is upper triangular. Therefore, A^T and B^T are upper triangular, and so by part (b), B^TA^T is upper triangular. And therefore, $AB = (B^TA^T)^T$ is lower triangular. Hence we're done! □

Note: This illustrates the utility of using previous results, doesn't it?

27. An **idempotent** matrix is a square matrix A for which $A^2 = A$.

(a) Find a 2×2 idempotent matrices (besides I_n and O_n.)

Solution: Clearly, if we had one in mind it would suffice to just write it down. Let's instead talk about how to find one.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If we have that $A^2 = A$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

Therefore,

$$\begin{aligned}
a &= a^2 + bc \\
b &= ab + bd = b(a+d) \\
c &= ac + cd = c(a+d) \\
d &= bc + d^2
\end{aligned}$$

Note that if $a + d = 1$, then both the second and third equation will be satisfied. Since we don't have to find all solutions (just one), let's assume for now that $a + d = 1$: that is, $d = 1 - a$. Plugging that into the fourth equation yields:

$$\begin{aligned}
1 - a &= (1-a)^2 + bc = 1 - 2a + a^2 + bc \\
\Leftrightarrow a &= a^2 + bc
\end{aligned}$$

and therefore this assumption also reduces the fourth equation to the first. So the only thing we need to worry about is that $a = a^2 + bc$. At this point, we should just plug in some values. Let's assume that $a = 0$, and therefore that $d = 1$. Then we need $bc = a - a^2 = 0$: picking values randomly, let $b = 0$ and $c = 2$. We have come up with the matrix:

$$A = \boxed{\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}}$$

To make sure that we didn't forget anything, let's check that this is indeed idempotent:

$$A^2 = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} = A$$

so we have found an idempotent and are done. (As you may note, we've made a number of choices above: in fact, the above arguments would produce a whole family of idempotents.)

(b) Show that

$$\begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

is idempotent.

Solution: This just require squaring the matrix and checking that we get the matrix back. Indeed,

$$\begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

so the given matrix is indeed idempotent.

(c) If A is an $n \times n$ idempotent matrix, show that $I_n - A$ is also idempotent.

Proof:
Assumptions: A is an $n \times n$ idempotent matrix.
Need to show: $I_n - A$ is also idempotent.

To check whether a matrix is idempotent, it suffices to square it and see if you get the same thing back. Therefore,

$$(I_n - A)^2 = (I_n - A)(I_n - A) = I_n^2 - AI_n - I_nA + A^2 = I_n - 2A + A^2$$

since multiplying a matrix by I_n results in the original matrix. Now, since we're given that A is idempotent, we know that $A^2 = A$. Therefore,

$$(I_n - A)^2 = I_n - 2A + A^2 = I_n - 2A + A = I_n - A$$

and therefore we have shown that $I_n - A$ is idempotent. $\square$

(d) Use parts (b) and (c) to get another example of an idempotent matrix.

Solution: Clearly, parts (b) and (c) imply that the matrix

$$\begin{aligned} A = I_3 - \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}} \end{aligned}$$

is idempotent, so we're done.

(e) Let A and B be $n \times n$ matrices. Show that A is idempotent if both $AB = A$ and $BA = B$.

Proof:
Assumptions: $AB = A$ and $BA = B$.
Need to prove: A is idempotent: that is, $A^2 = A$.

It's not quite clear where to start here, so obviously one just tries some algebraic manipulation. It'd be good to get an A^2 in there somewhere, so let's proceed! Since $A = AB$, multiplying by A on the right:

$$A^2 = ABA$$

Now, $BA = B$, so

$$A^2 = AB$$

But we also know that $AB = A$! Therefore,

$$A^2 = A$$

By definition, that means that A is idempotent, so I guess we're done! All the Bs disappeared – how did *that* happen? $\square$