Natural Deduction for Propositional Logic

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Outline

Natural Deduction

- In our examples, we (informally) infer new sentences.
- In natural deduction, we have a collection of proof rules.
  - These proof rules allow us to infer new sentences logically followed from existing ones.
- Supose we have a set of sentences: $\phi_1, \phi_2, \ldots, \phi_n$ (called premises), and another sentence ψ (called a conclusion).
- The notation

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

  is called a sequent.
- A sequent is valid if a proof (built by the proof rules) can be found.
- We will try to build a proof for our examples. Namely,

$$p \land \neg q \implies r, \neg r, p \vdash q.$$

Proof Rules for Natural Deduction – Conjunction

- Suppose we want to prove a conclusion $\phi \wedge \psi$. What do we do?
  - Of course, we need to prove both ϕ and ψ so that we can conclude $\phi \wedge \psi$.
- Hence the proof rule for conjunction is

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$$

  - Note that premises are shown above the line and the conclusion is below. Also, $\wedge i$ is the name of the proof rule.
  - This proof rule is called "conjunction-introduction" since we introduce a conjunction ($\wedge$) in the conclusion.

Proof Rules for Natural Deduction – Conjunction

- For each connective, we have introduction proof rule(s) and also elimination proof rule(s).
- Suppose we want to prove a conclusion ϕ from the premise $\phi \wedge \psi$. What do we do?
  - We don't do any thing since we know ϕ already!
- Here are the elimination proof rules:

$$\frac{\phi \wedge \psi}{\phi} \wedge e_1 \qquad\qquad \frac{\phi \wedge \psi}{\psi} \wedge e_2$$

- The rule $\wedge e_1$ says: if you have a proof for $\phi \wedge \psi$, then you have a proof for ϕ by applying this proof rule.
- Why do we need two rules?
  - Because we want to manipulate syntax only.

Examples

Example

Prove $p \wedge q, r \vdash q \wedge r$.

Proof.

We are looking for a proof of the form:

$$\begin{array}{c} p \wedge q \quad r \\ \vdots \\ q \wedge r \end{array}$$

□

Examples

Example

Prove $p \wedge q, r \vdash q \wedge r$.

Proof.

We are looking for a proof of the form:

$$\dfrac{\dfrac{p \wedge q}{q}\ \wedge e_2 \qquad r}{q \wedge r}\ \wedge i$$

We will write proofs in lines:

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge e_2$ 1
4	$q \wedge r$	$\wedge i$ 3, 2

□

Proof Rules for Natural Deduction – Double Negation

- Suppose we want to prove ϕ from a proof for $\neg\neg\phi$. What do we do?
  - There is no difference between ϕ and $\neg\neg\phi$. The same proof suffices!
- Hence we have the following proof rules:

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg i \qquad\qquad \frac{\neg\neg\phi}{\phi}\ \neg\neg e$$

Examples

Example

Prove $p, \neg\neg(q \land r) \vdash \neg\neg p \land r$.

Proof.

We are looking for a proof like:

$$\begin{array}{c} p \quad \neg\neg(q \land r) \\ \vdots \\ \neg\neg p \land r \end{array}$$

□

Examples

Example

Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$.

Proof.

We are looking for a proof like:

$$\dfrac{\dfrac{p}{\neg\neg p}\ \neg\neg i \qquad \dfrac{\dfrac{\neg\neg(q \wedge r)}{q \wedge r}\ \neg\neg e}{r}\ \wedge e_2}{\neg\neg p \wedge r}\ \wedge i$$

□

Examples

Example

Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$.

Proof.

We are looking for a proof like:

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	$\neg\neg i$ 1
4	$q \wedge r$	$\neg\neg e$ 2
5	r	$\wedge e_2$ 4
6	$\neg\neg p \wedge r$	$\wedge i$ 3, 5

□

Proof Rules for Natural Deduction – Implication

- Suppose we want to prove ψ from proofs for ϕ and $\phi \implies \psi$. What do we do?
  - We just put the two proofs for ϕ and $\phi \implies \psi$ together.
- Here is the proof rule:

$$\frac{\phi \quad \phi \implies \psi}{\psi} \implies e$$

- This proof rule is also called *modus ponens*.
- Here is another proof rule related to implication:

$$\frac{\phi \implies \psi \quad \neg\psi}{\neg\phi} \; MT$$

- This proof rule is called *modus tollens*.

Example

Example

Prove $p \implies (q \implies r), p, \neg r \vdash \neg q$.

Proof.

1	$p \implies (q \implies r)$	premise
2	p	premise
3	$\neg r$	premise
4	$q \implies r$	$\implies e$ 2, 1
5	$\neg q$	MT 4, 3

□

Proof Rules for Natural Deduction – Implication

- Suppose we want to prove $\phi \implies \psi$. What do we do?
  - We assume ϕ to prove ψ. If succeed, we conclude $\phi \implies \psi$ without any assumption.
  - Note that ϕ is added as an assumption and then removed so that $\phi \implies \psi$ does not depend on ϕ.
- We use "box" to simulate this strategy.
- Here is the proof rule:

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \implies \psi} \implies i$$

- At any point in a box, you can only use a sentence ϕ before that point. Moreover, no box enclosing the occurrence of ϕ has been closed.

Example

Example

Prove $\neg q \implies \neg p \vdash p \implies \neg\neg q$.

Proof.

$$\dfrac{\boxed{\dfrac{\neg q \implies \neg p \quad \dfrac{p}{\neg\neg p}\,\neg\neg i}{\neg\neg q}\,MT}}{p \implies \neg\neg q}\implies i$$

1	$\neg q \implies \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg i$ 2
4	$\neg\neg q$	MT 1, 3
5	$p \implies \neg\neg q$	$\implies i$ 2-4

□

Theorems

Example

Prove $\vdash p \implies p$.

Proof.

1	p	assumption
2	$p \implies p$	$\implies i$ 1 - 1

□

In the box, we have $\phi \equiv \psi \equiv p$.

Definition

A sentence ϕ such that $\vdash \phi$ is called a theorem.

Examples

Example

Prove $p \wedge q \implies r \vdash p \implies (q \implies r)$.

Proof.

1	$p \wedge q \implies r$	premise	
2	p	assumption	⌉
3	q	assumption	⌉ |
4	$p \wedge q$	$\wedge i$ 2, 3	| |
5	r	$\implies e$ 4, 1	⌋ |
6	$q \implies r$	$\implies i$ 3-5	⌋
7	$p \implies (q \implies r)$	$\implies i$ 2-6	

□

Proof Rules for Natural Deduction – Disjunction

- Suppose we want to prove $\phi \lor \psi$. What do we do?
  - We can either prove ϕ or ψ.
- Here are the proof rules:

$$\frac{\phi}{\phi \lor \psi} \lor i_1 \qquad\qquad \frac{\psi}{\phi \lor \psi} \lor i_2$$

  - Note the symmetry with $\land e_1$ and $\land e_2$.

$$\frac{\phi \land \psi}{\phi} \land e_1 \qquad\qquad \frac{\phi \land \psi}{\psi} \land e_2$$

  - Can we have a corresponding symmetric elimination rule for disjunction? Recall

$$\frac{\phi \quad \psi}{\phi \land \psi} \land i$$

Proof Rules for Natural Deduction – Disjunction

- Suppose we want to prove χ from $\phi \vee \psi$. What do we do?
  - We assume ϕ to prove χ and then assume ψ to prove χ.
  - If both succeed, χ is proved from $\phi \vee \psi$ without assuming ϕ and ψ.
- Here is the proof rule:

$$\frac{\phi \vee \psi \qquad \boxed{\begin{array}{c}\phi \\ \vdots \\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}{\chi} \ \vee e$$

- In addition to nested boxes, we may have parallel boxes in our proofs.

Example

Recall that our syntax does not admit commutativity.

Example

Prove $p \lor q \vdash q \lor p$.

Proof.

$$\dfrac{p \lor q \qquad \boxed{\dfrac{p}{q \lor p}\,\lor i_2} \qquad \boxed{\dfrac{q}{q \lor p}\,\lor i_1}}{q \lor p}\,\lor e$$

1	$p \lor q$	premise	
2	p	assumption	⌉
3	$q \lor p$	$\lor i_2$ 2	⌋
4	q	assumption	⌉
5	$q \lor p$	$\lor i_1$ 4	⌋
6	$q \lor p$	$\lor e$ 1, 2-3, 4-5	

□

Example

Example

Prove $q \implies r \vdash p \lor q \implies p \lor r$.

Proof.

1	$q \implies r$	premise
2	$p \lor q$	assumption
3	p	assumption
4	$p \lor r$	$\lor i_1$ 3
5	q	assumption
6	r	$\implies e$ 5, 1
7	$p \lor r$	$\lor i_2$ 6
8	$p \lor r$	$\lor e$ 2, 3-4, 5-7
9	$p \lor q \implies p \lor r$	$\implies i$ 2-8

□

Example

Example

Prove $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$.

Proof.

1	$p \wedge (q \vee r)$	premise	
2	p	$\wedge e_1$ 1	
3	$q \vee r$	$\wedge e_2$ 1	
4	q	assumption	⎤
5	$p \wedge q$	$\wedge i$ 2, 4	⎥
6	$(p \wedge q) \vee (p \wedge r)$	$\vee i_1$ 5	⎦
7	r	assumption	⎤
8	$p \wedge r$	$\wedge i$ 2, 7	⎥
9	$(p \wedge q) \vee (p \wedge r)$	$\vee i_2$ 8	⎦
10	$(p \wedge q) \vee (p \wedge r)$	$\vee e$ 3, 4-6, 7-9	

□

Example

Example

Prove $(p \land q) \lor (p \land r) \vdash p \land (q \lor r)$.

Proof.

1	$(p \land q) \lor (p \land r)$	premise	
2	$p \land q$	assumption	⎤
3	p	$\land e_1$ 2	⎥
4	q	$\land e_2$ 2	⎥
5	$q \lor r$	$\lor i_1$ 4	⎥
6	$p \land (q \lor r)$	$\land i$ 3, 5	⎦
7	$p \land r$	assumption	⎤
8	p	$\land e_1$ 7	⎥
9	r	$\land e_2$ 7	⎥
10	$q \lor r$	$\lor i_2$ 9	⎥
11	$p \land (q \lor r)$	$\land i$ 8, 10	⎦
12	$p \land (q \lor r)$	$\lor e$ 1, 2-6, 7-11	

Contradiction

Definition

Contradictions are sentences of the form $\phi \wedge \neg\phi$ or $\neg\phi \wedge \phi$.

- Examples:
  - $p \wedge \neg p$, $\neg(p \vee q \implies r) \wedge (p \vee q \implies r)$.
- Logically, any sentence can be proved from a contradiction.
  - If $0 = 1$, then $100 \neq 100$.
- Particularly, if ϕ and ψ are contradictions, we have $\phi \dashv\vdash \psi$.
  - $\phi \dashv\vdash \psi$ means $\phi \vdash \psi$ and $\psi \vdash \phi$ (called provably equivalent).
- Since all contradictions are equivalent, we will use the symbol $\bot$ (called "bottom") for them.
- We are now ready to discuss proof rules for negation.

Proof Rules for Natural Deduction – Negation

- Since any sentence can be proved from a contradiction, we have

$$\frac{\bot}{\phi}\ \bot e$$

- When both ϕ and $\neg\phi$ are proved, we have a contradiction.

$$\frac{\phi \quad \neg\phi}{\bot}\ \neg e$$

  - The proof rule could be called $\bot i$. We use $\neg e$ because it eliminates a negation.

Example

Example

Prove $\neg p \vee q \vdash p \implies q$.

Proof.

1	$\neg p \vee q$	premise	
2	$\neg p$	assumption	⎤
3	p	assumption	⎤ ⎥
4	$\bot$	$\neg e$ 3, 2	⎥ ⎥
5	q	$\bot e$ 4	⎦ ⎥
6	$p \implies q$	$\implies i$ 3-5	⎦
7	q	assumption	⎤
8	p	assumption	⎤ ⎥
9	q	copy 7	⎦ ⎥
10	$p \implies q$	$\implies i$ 8-9	⎦
11	$p \implies q$	$\vee e$ 1, 2-6, 7-10	

□

Proof Rules for Natural Deduction – Negation

- Suppose we want to prove $\neg\phi$. What do we do?
  - We assume ϕ and try to prove a contradiction. If succeed, we prove $\neg\phi$.
- Here is the proof rule:

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}}{\neg\phi}\ \neg i$$

Example

Example

Prove $p \implies q, p \implies \neg q \vdash \neg p$.

Proof.

1	$p \implies q$	premise	
2	$p \implies \neg q$	premise	
3	p	assumption	⌉
4	q	$\implies e$ 3, 1	|
5	$\neg q$	$\implies e$ 3, 2	|
6	$\bot$	$\neg e$ 4, 5	⌋
7	$\neg p$	$\neg i$ 3-6	

□

Example

Example

Prove $p \wedge \neg q \implies r, \neg r, p \vdash q$.

Proof.

1	$p \wedge \neg q \implies r$	premise	
2	$\neg r$	premise	
3	p	premise	
4	$\neg q$	assumption	⌉
5	$p \wedge \neg q$	$\wedge i$ 3, 4	|
6	r	$\implies e$ 5, 1	|
7	$\bot$	$\neg e$ 6, 2	⌋
8	$\neg\neg q$	$\neg i$ 4-7	
9	q	$\neg\neg e$ 8	

□

Derived Rules

- Some rules can actually be derived from others.

Examples

Prove $p \implies q, \neg q \vdash \neg p$ (modus tollens).

Proof.

1	$p \implies q$	premise	
2	$\neg q$	premise	
3	p	assumption	⌉
4	q	$\implies e$ 3, 1	|
5	$\bot$	$\neg e$ 4, 2	⌋
6	$\neg p$	$\neg i$ 3-5	

□

Derived Rules

Examples

Prove $p \vdash \neg\neg p$ $(\neg\neg i)$

Proof.

1	p	premise	
2	$\neg p$	assumption	⌉
3	$\bot$	$\neg e$ 1, 2	⌋
4	$\neg\neg p$	$\neg i$ 2-3	

□

- These rules can be replaced by their proofs and are not necessary.
  - They are just macros to help us write shorter proofs.

Reductio ad absurdum (RAA)

Example

Prove $\neg p \implies \bot \vdash p$ (RAA).

Proof.

1	$\neg p \implies \bot$	premise	
2	$\neg p$	assumption	⌉
3	$\bot$	$\implies e$ 2, 1	⌋
4	$\neg\neg p$	$\neg i$ 2-3	
5	p	$\neg\neg e$ 4	

□

Tertium non datur, Law of the Excluded Middle (LEM)

Example

Prove $\vdash p \lor \neg p$.

Proof.

1	$\neg(p \lor \neg p)$	assumption	
2	p	assumption	
3	$p \lor \neg p$	$\lor i_1$ 2	
4	$\bot$	$\neg e$ 3, 1	
5	$\neg p$	$\neg i$ 2-4	
6	$p \lor \neg p$	$\lor i_2$ 5	
7	$\bot$	$\neg e$ 6, 1	
8	$\neg\neg(p \lor \neg p)$	$\neg i$ 1-7	
9	$p \lor \neg p$	$\neg\neg e$ 8	

□

Proof Rules for Natural Deduction (Summary)

Conjunction ($\wedge$)

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i \qquad \frac{\phi \wedge \psi}{\phi} \wedge e_1 \quad \frac{\phi \wedge \psi}{\psi} \wedge e_2$$

Disjunction ($\vee$)

$$\frac{\phi}{\phi \vee \psi} \vee i_1 \quad \frac{\psi}{\phi \vee \psi} \vee i_2 \qquad \frac{\phi \vee \psi \quad \boxed{\begin{matrix}\phi \\ \vdots \\ \chi\end{matrix}} \quad \boxed{\begin{matrix}\psi \\ \vdots \\ \chi\end{matrix}}}{\chi} \vee e$$

Implication ($\implies$)

$$\frac{\boxed{\begin{matrix}\phi \\ \vdots \\ \psi\end{matrix}}}{\phi \implies \psi} \implies i \qquad \frac{\phi \quad \phi \implies \psi}{\psi} \implies e$$

Proof Rules for Natural Deduction (Summary)

Negation ($\neg$)

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}}{\neg\phi}\ \neg i \qquad\qquad \frac{\phi \quad \neg\phi}{\bot}\ \neg e$$

Contradiction ($\bot$)

(no introduction rule)

$$\frac{\bot}{\phi}\ \bot e$$

Double negation ($\neg\neg$)

(no introduction rule)

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg e$$

Useful Derived Proof Rules

$$\frac{\phi \implies \psi \quad \neg\psi}{\neg\phi}\ MT \qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg i$$

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ RAA \qquad \frac{}{\phi \vee \neg\phi}\ LEM$$

Provable Equivalence

- Recall $p \dashv\vdash q$ means $p \vdash q$ and $q \vdash p$.
- Here are some provably equivalent sentences:

$$
\begin{array}{rcl}
\neg(p \wedge q) & \dashv\vdash & \neg q \vee \neg p \\
\neg(p \vee q) & \dashv\vdash & \neg q \wedge \neg p \\
p \implies q & \dashv\vdash & \neg q \implies \neg p \\
p \implies q & \dashv\vdash & \neg p \vee q \\
p \wedge q \implies p & \dashv\vdash & r \vee \neg r \\
p \wedge q \implies r & \dashv\vdash & p \implies (q \implies r)
\end{array}
$$

- Try to prove them.

Proof by Contradiction

- Although it is very useful, the proof rule RAA is a bit puzzling.

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ RAA$$

- Instead of proving ϕ directly, the proof rule allows indirect proofs.
  - If $\neg\phi$ leads to a contradiction, then ϕ must hold.
- Note that indirect proofs are not "constructive."
  - We do not show why ϕ holds; we only know $\neg\phi$ is impossible.
- In early 20th century, some logicians and mathematicians chose not to prove indirectly. They are intuitionistic logicians or mathematicians.
- For the same reason, intuitionists also reject

$$\frac{}{\phi \lor \neg\phi}\ LEM \qquad\qquad \frac{\neg\neg\phi}{\phi}\ \neg\neg e$$

Proof by Contradiction

Theorem

There are $a, b \in \mathbb{R} \setminus \mathbb{Q}$ such that $a^b \in \mathbb{Q}$.

Proof.

Let $b = \sqrt{2}$. There are two cases:

- If $b^b \in \mathbb{Q}$, we are done since $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.
- If $b^b \notin \mathbb{Q}$, choose $a = b^b = \sqrt{2}^{\sqrt{2}}$. Then $a^b = (b^b)^b = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2$. Since $\sqrt{2}^{\sqrt{2}}, \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, we are done. □

- An intuitionist would criticize the proof since it does not tell us what a, b give $a^b \in \mathbb{Q}$.
  - We know (a, b) is either $(\sqrt{2}, \sqrt{2})$ or $(\sqrt{2}^{\sqrt{2}}, \sqrt{2})$.

Outline

Well-Formedness

Definition

A well-formed formula is constructed by applying the following rules finitely many times:

- atom: Every propositional atom $p, q, r, \ldots$ is a well-formed formula;
- $\neg$: If ϕ is a well-formed formula, so is $(\neg\phi)$;
- $\wedge$: If ϕ and ψ are well-formed formulae, so is $(\phi \wedge \psi)$;
- $\vee$: If ϕ and ψ are well-formed formulae, so is $(\phi \vee \psi)$;
- $\Longrightarrow$: If ϕ and ψ are well-formed formulae, so is $(\phi \Longrightarrow \psi)$.

- More compactly, well-formed formulae are defined by the following grammar in Backus Naur form (BNF):

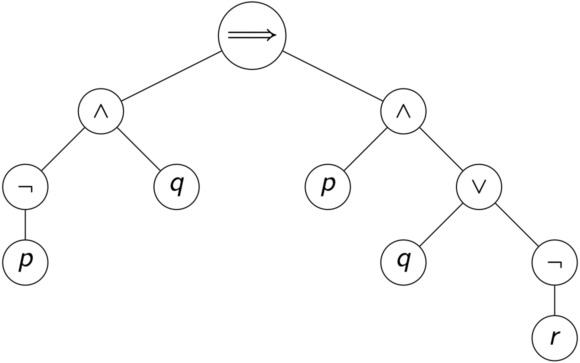
$$\phi ::= p \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \Longrightarrow \phi)$$

Inversion Principle

- How do we check if $(((\neg p) \land q) \implies (p \land (q \lor (\neg r))))$ is well-formed?
- Although a well-formed formula needs five grammar rules to construct, the construction process can always be inverted.
  - This is called inversion principle.
- To show $(((\neg p) \land q) \implies (p \land (q \lor (\neg r))))$ is well-formed, we need to show both $((\neg p) \land q)$ and $(p \land (q \lor (\neg r)))$ are well-formed.
- To show $((\neg p) \land q)$ is well-formed, we need to show both $(\neg p)$ and q are well-formed.
  - q is well-formed since it is an atom.
- To show $(\neg p)$ is well-formed, we need to show p is well-formed.
  - p is well-formed since it is an atom.
- Similarly, we can show $(p \land (q \lor (\neg r)))$ is well-formed.

Parse Tree

- The easiest way to decide whether a formula is well-formed is perhaps by drawing its parse tree.



Subformulae

- Given a well-formed formula, its subformulae are the well-formed formulae corresponding to its parse tree.
- For instance, the subformulae of the well-formed formulae $(((\neg p) \wedge q) \implies (p \wedge (q \vee (\neg r))))$ are

$$p$$
$$q$$
$$r$$
$$(\neg p)$$
$$(\neg r)$$
$$((\neg p) \wedge q)$$
$$(q \vee (\neg r))$$
$$(p \wedge (q \vee (\neg r)))$$
$$(((\neg p) \wedge q) \implies (p \wedge (q \vee (\neg r))))$$

Outline

From $\vdash$ to $\vDash$

- We have developed a calculus to determine whether $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid.
  - That is, from the premises $\phi_1, \phi_2, \ldots, \phi_n$, we can conclude ψ.
  - Our calculus is syntactic. It depends on the syntactic structures of $\phi_1, \phi_2, \ldots, \phi_n$, and ψ.
- We will introduce another relation between premises $\phi_1, \phi_2, \ldots, \phi_n$ and a conclusion ψ.

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi.$$

  - The new relation is defined by 'truth values' of atomic formulae and the semantics of logical connectives.

Truth Values and Models

Definition

The set of <u>truth values</u> is $\{F, T\}$ where F represents 'false' and T represents 'true.'

Definition

A <u>valuation</u> or <u>model</u> of a formula ϕ is an assignment from each proposition atom in ϕ to a truth value.

Truth Values of Formulae

Definition

Given a valuation of a formula ϕ, the truth value of ϕ is defined inductively by the following truth tables:

ϕ	ψ	$\phi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	T

ϕ	ψ	$\phi \implies \psi$
F	F	T
F	T	T
T	F	F
T	T	T

ϕ	$\neg\phi$
F	T
T	F

$\top$
T

$\bot$
F

Example

- $\phi \land \psi$ is T when ϕ and ψ are T.
- $\phi \lor \psi$ is T when ϕ or ψ is T.
- $\bot$ is always F; $\top$ is always T.
- $\phi \implies \psi$ is T when ϕ "implies" ψ.

Example

Consider the valuation $\{q \mapsto T, p \mapsto F, r \mapsto F\}$ of $(q \land p) \implies r$. What is the truth value of $(q \land p) \implies r$?

Proof.

Since the truth values of q and p are T and F respectively, the truth value of $q \land p$ is F. Moreover, the truth value of r is F. The truth value of $(q \land p) \implies r$ is T. □

Truth Tables for Formulae

- Given a formula ϕ with propositional atoms $p_1, p_2, \ldots, p_n$, we can construct a truth table for ϕ by listing 2^n valuations of ϕ.

Example

Find the truth table for $(p \implies \neg q) \implies (q \vee \neg p)$.

Proof.

p	q	$\neg p$	$\neg q$	$p \implies \neg q$	$q \vee \neg p$	$(p \implies \neg q) \implies (q \vee \neg p)$
F	F	T	T	T	T	T
F	T	T	F	T	T	T
T	F	F	T	T	F	F
T	T	F	F	F	T	T

□

Outline

Validity of Sequent Revisited

- Informally $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid if we can derive ψ with assumptions $\phi_1, \phi_2, \ldots, \phi_n$.
  - We have formalized "deriving ψ with assumptions $\phi_1, \phi_2, \ldots, \phi_n$" by "constructing a proof in a formal calculus."
- We can give another interpretation by valuations and truth values.
- Consider a valuation ν over all propositional atoms in $\phi_1, \phi_2, \ldots, \phi_n, \psi$.
  - By "assumptions $\phi_1, \phi_2, \ldots, \phi_n$," we mean "$\phi_1, \phi_2, \ldots, \phi_n$ are T under the valuation ν.
  - By "deriving ψ,", we mean ψ is also T under the valuation ν.
- Hence, "we can derive ψ with assumptions $\phi_1, \phi_2, \ldots, \phi_n$" actually means "if $\phi_1, \phi_2, \ldots, \phi_n$ are T under a valuation, then ψ must be T under the same valuation.

Semantic Entailment

Definition

We say

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds if for every valuations where $\phi_1, \phi_2, \ldots, \phi_n$ are T, ψ is also T. In this case, we also say $\phi_1, \phi_2, \ldots, \phi_n$ semantically entail ψ.

- Examples
  - $p \wedge q \vDash p$. For every valuation where $p \wedge q$ is T, p must be T. Hence $p \wedge q \vDash p$.
  - $p \vee q \nvDash q$. Consider the valuation $\{p \mapsto \mathrm{T}, q \mapsto \mathrm{F}\}$. We have $p \vee q$ is T but q is F. Hence $p \vee q \nvDash q$.
  - $\neg p, p \vee q \vDash q$. Consider any valuation where $\neg p$ and $p \vee q$ are T. Since $\neg p$ is T, p must be F under the valuation. Since p is F and $p \vee q$ is T, q must be T under the valuation. Hence $\neg p, p \vee q \vDash q$.
- The validity of $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is defined by syntactic calculus. $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ is defined by truth tables. Do these two relations coincide?

Soundness Theorem for Propositional Logic

Theorem (Soundness)

Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional logic formulae. If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.

Proof.

Consider the assertion $M(k)$:

"For all sequents $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi (n \geq 0)$ that have a proof of length k, then $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds."

$k = 1$. The only possible proof is of the form

1	ϕ	premise

This is the proof of $\phi \vdash \phi$. For every valuation such that ϕ is T, ϕ must be T. That is, $\phi \vDash \phi$.

Soundness Theorem for Propositional Logic

Proof (cont'd).

Assume $M(i)$ for $i < k$. Consider a proof of the form

1	ϕ_1	premise
2	ϕ_2	premise
	$\vdots$	
n	ϕ_n	premise
	$\vdots$	
k	ψ	justification

We have the following possible cases for justification:

i $\wedge i$. Then ψ is $\psi_1 \wedge \psi_2$. In order to apply $\wedge i$, ψ_1 and ψ_2 must appear in the proof. That is, we have $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1$ and $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_2$. By inductive hypothesis, $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi_1$ and $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi_2$. Hence $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi_1 \wedge \psi_2$ (Why?).

Proof (cont'd).

ii $\vee e$. Recall the proof rule for $\vee e$:

$$\frac{\eta_1 \vee \eta_2 \qquad \boxed{\begin{array}{c}\eta_1\\ \vdots\\ \psi\end{array}} \qquad \boxed{\begin{array}{c}\eta_2\\ \vdots\\ \psi\end{array}}}{\psi}\ \vee e$$

In order to apply $\vee e$, $\eta_1 \vee \eta_2$ must appear in the proof. We have $\phi_1, \phi_2, \ldots, \phi_n \vdash \eta_1 \vee \eta_2$. By turning "assumptions" η_1 and η_2 to "premises," we obtain proofs for $\phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vdash \psi$ and $\phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vdash \psi$. By inductive hypothesis, $\phi_1, \phi_2, \ldots, \phi_n \vDash \eta_1 \vee \eta_2$, $\phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vDash \psi$, and $\phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vDash \psi$. Consider any valuation such that $\phi_1, \phi_2, \ldots, \phi_n$ evaluates to T. $\eta_1 \vee \eta_2$ must be T. If η_1 is T under the valuation, ψ is also T (Why?). Similarly for η_2 is T. Thus $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$.

Soundness Theorem for Propositional Logic

Proof (cont'd).

iii Other cases are similar. Prove the case of $\Longrightarrow e$ to see if you understand the proof.

□

- The soundness theorem shows that our calculus does not go wrong.
- If there is a proof of a sequent, then the conclusion must be true for all valuations where all premises are true.
- The theorem also allows us to show the non-existence of proofs.
- Given a sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$, how do we prove there is no proof for the sequent?
  - Try to find a valuation where $\phi_1, \phi_2, \ldots, \phi_n$ are T but ψ is F.

Outline

Completeness Theorem for Propositional Logic

- "$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid" and "$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds" are very different.
  - "$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid" requires proof search (syntax);
  - "$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds" requires a truth table (semantics).
- If "$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds" implies "$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid," then our natural deduction proof system is complete.
- The natural deduction proof system is both sound and complete. That is

  $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid iff $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.

Completeness Theorem for Propositional Logic

- We will show the natural deduction proof system is complete.
- That is, if $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds, then there is a natural deduction proof for the sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$.
- Assume $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$. We proceed in three steps:
  1. $\vDash \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi)))$ holds;
  2. $\vdash \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi)))$ is valid;
  3. $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid.

Completeness Theorem for Propositional Logic (Step 1)

Lemma

If $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ *holds, then* $\vDash \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi)))$ *holds.*

Proof.

Suppose $\vDash \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi)))$ does not hold. Then there is valuation where $\phi_1, \phi_2, \ldots, \phi_n$ is T but ψ is F. A contradiction to $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$. □

Definition

Let ϕ be a propositional logic formula. We say ϕ is a tautology if $\vDash \phi$.

- A tautology is a propositional logic formula that evaluates to T for all of its valuations.

Completeness Theorem for Propositional Logic (Step 2)

- Our goal is to show the following theorem:

Theorem

If $\vDash \eta$ holds, then $\vdash \eta$ is valid.

- Similar to tautologies, we introduce the following definition:

Definition

Let ϕ be a propositional logic formula. We say ϕ is a <u>theorem</u> if $\vdash \phi$.

- Two types of theorems:
  - If $\vdash \phi$, ϕ is a theorem proved by the natural deduction proof system.
  - The soundness theorem for propositional logic is another type of theorem proved by mathematical reasoning (less formally).

Completeness Theorem for Propositional Logic (Step 2)

Proposition

Let ϕ be a formula with propositional atoms $p_1, p_2, \ldots, p_n$. Let l be a line in ϕ's truth table. For all $1 \leq i \leq n$, let $\hat{p}_i$ be p_i if p_i is T in l; otherwise $\hat{p}_i$ is $\neg p_i$. Then

1. *$\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi$ is valid if the entry for ϕ at l is T;*
2. *$\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg\phi$ is valid if the entry for ϕ at l is F.*

Proof.

We prove by induction on the height of the parse tree of ϕ.

- ϕ is a propositional atom p. Then $p \vdash p$ or $\neg p \vdash \neg p$ have one-line proof.
- ϕ is $\neg\phi_1$.
  - If ϕ is T at l. Then ϕ_1 is F. By IH, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg\phi_1 (\equiv \phi)$.
  - If ϕ is F at l. Then ϕ_1 is T. By IH, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1$. Using $\neg\neg i$, we have $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg\neg\phi_1 (\equiv \neg\phi)$.

Completeness Theorem for Propositional Logic (Step 2)

Proof (cont'd).

- ϕ is $\phi_1 \implies \phi_2$.
  - If ϕ is F at l, then ϕ_1 is T and ϕ_2 is F at l. By IH, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1$ and $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg\phi_2$. Consider

1	$\phi_1 \implies \phi_2$	assumption
	$\vdots$	
i	ϕ_1	IH
i + 1	ϕ_2	$\implies e$ i, 1
	$\vdots$	
j	$\neg\phi_2$	IH
j + 1	$\bot$	¬ e i+1, j
j + 2	$\neg(\phi_1 \implies \phi_2)$	¬ i 1-(j+1)

Completeness Theorem for Propositional Logic (Step 2)

Proof (cont'd).

- ϕ is $\phi_1 \implies \phi_2$.
  - If ϕ is T at l, we have three subcases. Consider the case where ϕ_1 and ϕ_2 are F at l. Then

1	ϕ_1	assumption	⌉
	$\vdots$		|
i	$\neg\phi_1$	IH	|
i + 1	$\bot$	$\neg$ e 1, i	|
i + 2	ϕ_2	$\bot$ e (i+1)	⌋
i + 3	$\phi_1 \implies \phi_2$	$\implies$ i 1-(i+2)	

The other two subcases are simple exercises.

Completeness Theorem for Propositional Logic (Step 2)

Proof (cont'd).

- ϕ is $\phi_1 \wedge \phi_2$.
  - If ϕ is T at l, then ϕ_1 and ϕ_2 are T at l. By IH, we have $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1$ and $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_2$. Using $\wedge$ i, we have $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1 \wedge \phi_2$.
  - If ϕ is F at l, there are three subcases. Consider the subcase where ϕ_1 and ϕ_2 are F at l. Then

1	$\phi_1 \wedge \phi_2$	assumption	⌉
2	ϕ_1	$\wedge$ e$_1$ 1	|
	$\vdots$		|
i	$\neg\phi_1$	IH	|
i + 1	$\bot$	$\neg$ e 2, i	⌋
i + 2	$\neg(\phi_1 \wedge \phi_2)$	$\neg$ i 1-(i+1)	

The other two subcases are simple exercises.

Completeness Theorem for Propositional Logic (Step 2)

Proof.

- ϕ is $\phi_1 \vee \phi_2$.
  - If ϕ is F at l, then ϕ_1 and ϕ_2 are F at l. Then

1	$\phi_1 \vee \phi_2$	assumption
2	ϕ_1	assumption
	$\vdots$	
i	$\neg\phi_1$	IH
i + 1	$\bot$	$\neg$ e 2, i
i + 2	ϕ_2	assumption
	$\vdots$	
j	$\neg\phi_2$	IH
j + 1	$\bot$	$\neg$ e i+2, j
j + 2	$\bot$	$\vee$ e 2-(i+1), (i+2)-(j+1)
j + 3	$\neg(\phi_1 \vee \phi_2)$	$\neg$i 1-(j+2)

  - If ϕ is T at l, there are three subcases. All of them are simple exercises.

□

Completeness Theorem for Propositional Logic (Step 2)

Theorem

If ϕ is a tautology, then ϕ is a theorem.

Proof.

Let ϕ have propositional atoms $p_1, p_2, \ldots, p_n$. Since ϕ is a tautology, each line in ϕ's truth table is T. By the above proposition, we have the following 2^n proofs for ϕ:

$$\begin{array}{rcl} \neg p_1, \neg p_2, \ldots, \neg p_n & \vdash & \phi \\ p_1, \neg p_2, \ldots, \neg p_n & \vdash & \phi \\ \neg p_1, p_2, \ldots, \neg p_n & \vdash & \phi \\ & \vdots & \\ p_1, p_2, \ldots, p_n & \vdash & \phi \end{array}$$

We apply the rule LEM and the $\vee$e rule to obtain a proof for $\vdash \phi$. (See the following example.) □

Completeness Theorem for Propositional Logic (Step 2)

Example

Observe that $\vDash p \implies (q \implies p)$. Prove $\vdash p \implies (q \implies p)$.

Proof.

1	$p \lor \neg p$	LEM
2	p	assumption
3	$q \lor \neg q$	LEM
4	q	assumption
	$\vdots$	
i	$p \implies (q \implies p)$	$p, q \vdash p \implies (q \implies p)$
i + 1	$\neg q$	assumption
	$\vdots$	
j	$p \implies (q \implies p)$	$p, \neg q \vdash p \implies (q \implies p)$
j + 1	$p \implies (q \implies p)$	$\lor$e 3, 4-i, (i+1)-j
j + 2	$\neg p$	assumption
j + 3	$q \lor \neg q$	LEM
j + 4	q	assumption
	$\vdots$	
k	$p \implies (q \implies p)$	$\neg p, q \vdash p \implies (q \implies p)$
k + 1	$\neg q$	assumption
	$\vdots$	
l	$p \implies (q \implies p)$	$\neg p, \neg q \vdash p \implies (q \implies p)$
l + 1	$p \implies (q \implies p)$	$\lor$e (j+3), (j+4)-k, (k+1)-l
l + 2	$p \implies (q \implies p)$	$\lor$e 1, 2-(j+1), (j+2)-(l+1)

□

Completeness Theorem for Propositional Logic (Step 3)

Lemma

If $\phi_1 \implies (\phi_2 \implies (\cdots(\phi_n \implies \psi)))$ *is a theorem, then* $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ *is valid.*

Proof.

Consider

1	ϕ_1	premise
2	ϕ_2	premise
	$\vdots$	
n	ϕ_n	premise
	$\vdots$	
i	$\phi_1 \implies (\phi_2 \implies (\cdots(\phi_n \implies \psi)))$	theorem
i + 1	$\phi_2 \implies (\cdots(\phi_n \implies \psi))$	$\implies$e 1, i
i + 2	$\phi_3 \implies (\cdots(\phi_n \implies \psi))$	$\implies$e 2, (i+1)
	$\vdots$	
i + n - 1	$\phi_n \implies \psi$	$\implies$e (n-1), (i+n-2)
i + n	ψ	$\implies$e n, (i+n-1)

□