First Order Logic

Propositional Logic

- A **proposition** is a declarative sentence (a sentence that declares a fact) that is either true or false, but not both.

- Are the following sentences propositions?
  - Toronto is the capital of Canada. (Yes)
  - Read this carefully. (No)
  - 1+2=3 (Yes)
  - x+1=2 (No)
  - What time is it? (No)

- Propositional Logic – the area of logic that deals with propositions

Propositional Variables

- Propositional Variables – variables that represent propositions: p, q, r, s
  - E.g. Proposition p – "Today is Friday."
- Truth values – T, F

Negation

DEFINITION 1

Let *p* be a proposition. The negation of *p*, denoted by ¬*p*, is the statement "It is not the case that *p*."

The proposition ¬*p* is read "not *p*." The truth value of the negation of *p*, ¬*p* is the opposite of the truth value of *p*.

- Examples
  - Find the negation of the proposition "Today is Friday." and express this in simple English.

    Solution: The negation is "It is not the case that *today is Friday*. In simple English, "Today is not Friday." or "It is not Friday today."

  - Find the negation of the proposition "At least 10 inches of rain fell today in Miami." and express this in simple English.

    Solution: The negation is "It is not the case that *at least 10 inches of rain fell today in Miami*." In simple English, "Less than 10 inches of rain fell today in Miami."

Truth Table

- Truth table:

The Truth Table for the Negation of a Proposition.	
p	¬*p*
T	F
F	T

- Logical operators are used to form new propositions from two or more existing propositions. The logical operators are also called connectives.

Conjunction

DEFINITION 2

Let *p* and *q* be propositions. The *conjunction* of *p* and *q*, denoted by $p \wedge q$, is the proposition "*p* and *q*". The conjunction $p \wedge q$ is true when both *p* and *q* are true and is false otherwise.

- Examples
  - Find the conjunction of the propositions *p* and *q* where *p* is the proposition "Today is Friday." and *q* is the proposition "It is raining today.", and the truth value of the conjunction.

    Solution: The conjunction is the proposition "Today is Friday and it is raining today." The proposition is true on rainy Fridays.

Disjunction

DEFINITION 3

Let *p* and *q* be propositions. The *disjunction* of *p* and *q*, denoted by $p \vee q$, is the proposition "*p* or *q*". The conjunction $p \vee q$ is false when both *p* and *q* are false and is true otherwise.

- Note:

  inclusive or : The disjunction is true when at least one of the two propositions is true.
  - E.g. "Students who have taken calculus or computer science can take this class." – those who take one or both classes.

  exclusive or : The disjunction is true only when one of the proposition is true.
  - E.g. "Students who have taken calculus or computer science, but not both, can take this class." – only those who take one of them.
- Definition 3 uses *inclusive or*.

Exclusive

DEFINITION 4

Let *p* and *q* be propositions. The *exclusive or* of *p* and *q*, denoted by $p \oplus q$, is the proposition that is true when exactly one of *p* and *q* is true and is false otherwise.

The Truth Table for the Conjunction of Two Propositions.

p	q	$p \land q$
T	T	T
T	F	F
F	T	F
F	F	F

The Truth Table for the Disjunction of Two Propositions.

p	q	$p \lor q$
T	T	T
T	F	T
F	T	T
F	F	F

The Truth Table for the Exclusive Or (XOR) of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statements

DEFINITION 5

Let p and q be propositions. The *conditional statement* $p \to q$, is the proposition "if p, then q." The conditional statement is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

- A conditional statement is also called an implication.
- Example: "If I am elected, then I will lower taxes." $p \to q$

implication:

elected, lower taxes.	T	T	T
not elected, lower taxes.	F	T	T
not elected, not lower taxes.	F	F	T
elected, not lower taxes.	T	F	F

Conditional Statement (Cont')

- Example:
  - Let p be the statement "Maria learns discrete mathematics." and q the statement "Maria will find a good job." Express the statement $p \longrightarrow q$ as a statement in English.

Solution: Any of the following -

"If Maria learns discrete mathematics, then she will find a good job.

"Maria will find a good job when she learns discrete mathematics."

"For Maria to get a good job, it is sufficient for her to learn discrete mathematics."

Conditional Statement (Cont')

- Other conditional statements:
  - Converse of $p \longrightarrow q$: $q \longrightarrow p$
  - Contrapositive of $p \longrightarrow q$: $\neg q \longrightarrow \neg p$
  - Inverse of $p \longrightarrow q$: $\neg p \longrightarrow \neg q$

Biconditional Statement

DEFINITION 6

Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

- $p \leftrightarrow q$ has the same truth value as $(p \rightarrow q) \land (q \rightarrow p)$
- "*if and only if*" can be expressed by "*iff*"
- Example:
  - Let p be the statement "You can take the flight" and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement

    "You can take the flight if and only if you buy a ticket."

    Implication:

    If you buy a ticket you can take the flight.

    If you don't buy a ticket you cannot take the flight.

Biconditional Statement (Cont')

The Truth Table for the Biconditional $p \leftrightarrow q$.		
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Tables of Compound Propositions

- We can use connectives to build up complicated compound propositions involving any number of propositional variables, then use truth tables to determine the truth value of these compound propositions.
- Example: Construct the truth table of the compound proposition

$$(p \vee \neg q) \longrightarrow (p \wedge q).$$

The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of Logical Operators

- We can use parentheses to specify the order in which logical operators in a compound proposition are to be applied.
- To reduce the number of parentheses, the precedence order is defined for logical operators.

Precedence of Logical Operators.	
Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

E.g. $\neg p \wedge q = (\neg p) \wedge q$

$p \wedge q \vee r = (p \wedge q) \vee r$

$p \vee q \wedge r = p \vee (q \wedge r)$

Translating English Sentences

- English (and every other human language) is often ambiguous. Translating sentences into compound statements removes the ambiguity.
- Example: How can this English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Solution: Let q, r, and s represent "You can ride the roller coaster," "You are under 4 feet tall," and "You are older than 16 years old." The sentence can be translated into:

$$(r \wedge \neg s) \rightarrow \neg q.$$

Translating English Sentences

- Example: How can this English sentence be translated into a logical expression?

  "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution: Let a, c, and f represent "You can access the Internet from campus," "You are a computer science major," and "You are a freshman." The sentence can be translated into:

$$a \rightarrow (c \vee \neg f).$$

Logic and Bit Operations

- Computers represent information using bits.
- A **bit** is a symbol with two possible values, 0 and 1.
- By convention, 1 represents T (true) and 0 represents F (false).
- A variable is called a Boolean variable if its value is either true or false.
- Bit operation – replace true by 1 and false by 0 in logical operations.

Table for the Bit Operators *OR*, *AND*, and *XOR*.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Logic and Bit Operations

DEFINITION 7

A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

- Example: Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit string 01 1011 0110 and 11 0001 1101.

Solution:

```
01 1011 0110
11 0001 1101
-------------------
11 1011 1111     bitwise OR
01 0001 0100     bitwise AND
10 1010 1011     bitwise XOR
```

Propositional Equivalences

DEFINITION 1

A compound proposition that is always true, no matter what the truth values of the propositions that occurs in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology or a contradiction is called a *contingency*.

Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical Equivalences

> DEFINITION 2
>
> The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.
- Example: Show that $\neg p \vee q$ and $p \rightarrow q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logical Equivalence

Two statements have the same truth table

De Morgan's Law $$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

De Morgan's Law $$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Constructing New Logical Equivalences

- Example: Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

  Solution:

  $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$ *by example on earlier slide*

  $\equiv \neg(\neg p) \wedge \neg q$ *by the second De Morgan law*

  $\equiv p \wedge \neg q$ *by the double negation law*

- *Example: Show that* $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

  Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

  $(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ *by example on earlier slides*

  $\equiv (\neg p \vee \neg q) \vee (p \vee q)$ *by the first De Morgan law*

  $\equiv (\neg p \vee p) \vee (\neg q \vee q)$ *by the associative and communicative law for disjunction*

  $\equiv T \vee T$

  $\equiv T$

- *Note: The above examples can also be done using truth tables.*

Important Logic Equivalence

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Predicates

- Statements involving variables are neither true nor false.
- E.g. "$x > 3$", "$x = y + 3$", "$x + y = z$"
- "x is greater than 3"
  - "x": subject of the statement
  - "is greater than 3": the ***predicate***
- We can denote the statement "x is greater than 3" by $P(x)$, where P denotes the predicate and x is the variable.
- Once a value is assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

Predicates

- Example: Let *P(x)* denote the statement "$x > 3$." What are the truth values of *P*(4) and *P*(2)?

  Solution: *P*(4) – "4 > 3", *true*
  P(3) – "2 > 3", *false*

- Example: Let *Q*(x,y) denote the statement "$x = y + 3$." What are the truth values of the propositions *Q*(1,2) and *Q*(3,0)?

  Solution: *Q*(1,2) – "1 = 2 + 3" , *false*
  Q(3,0) – "3 = 0 + 3", *true*

Predicates

- Example: Let *A(c,n)* denote the statement "Computer *c* is connected to network *n*", where *c* is a variable representing a computer and *n* is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of A(MATH1, CAMPUS1) and A(MATH1, CAMPUS2)?

Solution: *A(MATH1, CAMPUS1) – "MATH1 is connect to CAMPUS1", false*

A(MATH1, CAMPUS2) – "MATH1 is connect to CAMPUS2", true

Propositional Function (Predicate)

- A statement involving *n* variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.
- A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function *P* at the *n*-tuple $(x_1, x_2, \ldots, x_n)$, and *P* is also called a ***n*-place predicate** or a ***n*-ary predicate**.

Quantifiers

- Quantification: express the extent to which a predicate is true over a range of elements.
- **Universal quantification**: a predicate is true for every element under consideration
- **Existential quantification**: a predicate is true for one or more element under consideration
- A domain must be specified.

Universal Quantifier

The *universal quantification* of *P(x)* is the statement
"*P(x)* for all values of *x* in the domain."
The notation $\forall xP(x)$ denotes the universal quantification of *P(x)*. Here $\forall$ is called the **Universal Quantifier**. We read $\forall xP(x)$ as "for all *xP(x)*" or "for every *xP(x)*." An element for which *P(x)* is false is called a **counterexample** of *xP(x)*.

Example: Let *P(x)* be the statement "*x* + 1 > *x*." What is the truth value of the quantification $\forall xP(x)$, where the domain consists of all real numbers?

Solution: *Because P(x) is true for all real numbers, the quantification is true.*

Universal Quantification

- A statement $\forall xP(x)$ is false, if and only if $P(x)$ is not always true where x is in the domain. One way to show that is to find a counterexample to the statement $\forall xP(x)$.
- Example: Let $Q(x)$ be the statement "x < 2". What is the truth value of the quantification $\forall xQ(x)$, where the domain consists of all real numbers?

  Solution: *Q(x) is not true for every real numbers, e.g. Q(3) is false. x = 3 is a counterexample for the statement* $\forall xQ(x)$*. Thus the quantification is false.*

- $\forall xP(x)$ is the same as the conjunction

  $P(x_1) \wedge P(x_2) \wedge .... \wedge P(x_n)$

Universal Quantifier

- Example: What does the statement $\forall x N(x)$ mean if $N(x)$ is "Computer x is connected to the network" and the domain consists of all computers on campus?

Solution: *"Every computer on campus is connected to the network."*

Existential Quantification

DEFINITION 2

The *existential quantification* of *P(x)* is the statement

"There exists an element x in the domain such that *P(x)*."

We use the notation $\exists x P(x)$ for the existential quantification of *P(x)*. Here

$\exists$ is called the **Existential Quantifier**.

- The existential quantification $\exists x P(x)$ is read as

  "There is an *x* such that *P(x)*," or

  "There is at least one *x* such that *P(x)*," or

  "For some *x*, *P(x)*."

Existential Quantification

- Example: Let *P(x)* denote the statement "*x* > 3". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

  Solution: *"x > 3" is sometimes true – for instance when x = 4. The existential quantification is true.*

- $\exists x P(x)$ is false if and only if *P*(x) is false for every element of the domain.
- Example: Let *Q*(x) denote the statement "x = x + 1". What is the true value of the quantification $\exists x Q(x)$, where the domain consists for all real numbers?

  Solution: *Q(x) is false for every real number. The existential quantification is false.*

Existential Quantification

- If the domain is empty, $\exists x Q(x)$ is false because there can be no element in the domain for which $Q(x)$ is true.
- The existential quantification $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \dots \vee P(x_n)$

Quantifiers		
Statement	***When True?***	***When False?***
$\forall x P(x)$	*xP(x) is true for every x.*	*There is an x for which xP(x) is false.*
$\exists x P(x)$	*There is an x for which P(x) is true.*	*P(x) is false for every x.*

∃

Uniqueness Quantifier

- Uniqueness quantifier $\exists!$ or $\exists_1$
  - $\exists! x P(x)$ or $\exists_1 P(x)$ states "There exists a unique x such that $P(x)$ is true."
- Quantifiers with restricted domains
  - Example: What do the following statements mean? The domain in each case consists of real numbers.
    - $\forall x < 0\ (x^2 > 0)$: For every real number x with $x < 0$, $x^2 > 0$. "The square of a negative real number is positive." It's the same as $\forall x(x < 0 \rightarrow x^2 > 0)$
    - $\forall y \neq 0\ (y^3 \neq 0)$: For every real number y with $y \neq 0$, $y^3 \neq 0$. "The cube of every non-zero real number is non-zero." It's the same as $\forall y(y \neq 0 \rightarrow y^3 \neq 0)$.
    - $\exists z > 0\ (z^2 = 2)$: There exists a real number z with $z > 0$, such that $z^2 = 2$. "There is a positive square root of 2." It's the same as $\exists z(z > 0 \land z^2 = 2)$:

Precedence of Quantifiers

- Precedence of Quantifiers
  - $\forall$ and $\exists$ have higher precedence than all logical operators.
  - E.g. $\forall x P(x) \lor Q(x)$ is the same as $(\forall x P(x)) \lor Q(x)$

Translating from English into Logical Expressions

- Example: Express the statement “Every student in this class has studied calculus” using predicates and quantifiers.

Solution:

If the domain consists of students in the class –

$$\forall x C(x)$$

where $C(x)$ is the statement “x has studied calculus.

If the domain consists of all people –

$$\forall x (S(x) \rightarrow C(x))$$

where $S(x)$ represents that person x is in this class.

If we are interested in the backgrounds of people in subjects besides calculus, we can use the two-variable quantifier $Q(x,y)$ for the statement “student x has studies subject y.” Then we would replace $C(x)$ by $Q(x, \text{calculus})$ to obtain $\forall x Q(x, \text{calculus})$ or $\forall x (S(x) \rightarrow Q(x, \text{calculus}))$

Translating from English into Logical Expressions

- Example: Consider these statements. The first two are called *premises* and the third is called the *conclusion*. The entire set is called an *argument*.

  "All lions are fierce."

  "Some lions do not drink coffee."

  "Some fierce creatures do not drink coffee."

Solution: Let $P(x)$ be "x is a lion."

$Q(x)$ be "x is fierce."

$R(x)$ be "x drinks coffee."

$\forall x(P(x) \rightarrow Q(x))$

$\exists x(P(x) \land \neg R(x))$

$\exists x(Q(x) \land \neg R(x))$