Implicit Differentiation

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Sometimes functions are given not in the form $y = f(x)$ but in a more complicated form in which it is difficult or impossible to express y explicitly in terms of x. Such functions are called implicit functions. In this unit we explain how these can be differentiated using implicit differentiation.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- differentiate functions defined implicitly

Contents

1. Introduction

In this unit we look at how we might differentiate functions of y with respect to x.

Consider an expression such as

$$x^2 + y^2 - 4x + 5y - 8 = 0$$

It would be quite difficult to re-arrange this so y was given explicitly as a function of x. We could perhaps, given values of x, use the expression to work out the values of y and thereby draw a graph. In general even if this is possible, it will be difficult.

A function given in this way is said to be defined **implicitly**. In this unit we study how to differentiate a function given in this form.

It will be necessary to use a rule known as the *the chain rule* or the rule for differentiating a *function of a function*. In this unit we will refer to it as the chain rule. There is a separate unit which covers this particular rule thoroughly, although we will revise it briefly here.

2. Revision of the chain rule

We revise the chain rule by means of an example.

Example

Suppose we wish to differentiate $y = (5 + 2x)^{10}$ in order to calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$.

We make a substitution and let $u = 5 + 2x$ so that $y = u^{10}$.
The chain rule states

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Now

$$\text{if } y = u^{10} \quad \text{then} \quad \frac{dy}{du} = 10u^9$$

and

$$\text{if } u = 5 + 2x \quad \text{then} \quad \frac{du}{dx} = 2$$

hence

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 10u^9 \times 2 \\ &= 20u^9 \\ &= 20(5 + 2x)^9 \end{aligned}$$

So we have used the chain rule in order to differentiate the function $y = (5 + 2x)^{10}$.

In quoting the chain rule in the form $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ note that we write y in terms of u, and u in terms of x. i.e.

$$y = y(u) \qquad \text{and} \qquad u = u(x)$$

We will need to work with different variables. Suppose we have z in terms of y, and y in terms of x, i.e.

$$z = z(y) \qquad \text{and} \qquad y = y(x)$$

The chain rule would then state:

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

Example

Suppose $z = y^2$. It follows that $\frac{dz}{dy} = 2y$. Then using the chain rule

$$\begin{aligned} \frac{dz}{dx} &= \frac{dz}{dy} \times \frac{dy}{dx} \\ &= 2y \times \frac{dy}{dx} \\ &= 2y\frac{dy}{dx} \end{aligned}$$

Notice what we have just done. In order to differentiate y^2 with respect to x we have differentiated y^2 with respect to y, and then multiplied by $\frac{dy}{dx}$, i.e.

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dy}\left(y^2\right) \times \frac{dy}{dx}$$

We can generalise this as follows:

to differentiate a function of y with respect to x, we differentiate with respect to y and then multiply by $\frac{dy}{dx}$.

Key Point

$$\frac{d}{dx}\left(f(y)\right) = \frac{d}{dy}\left(f(y)\right) \times \frac{dy}{dx}$$

We are now ready to do some implicit differentiation. Remember, every time we want to differentiate a function of y with respect to x, we differentiate with respect to y and then multiply by $\frac{dy}{dx}$.

3. Implicit differentiation

Example

Suppose we want to differentiate the implicit function

$$y^2 + x^3 - y^3 + 6 = 3y$$

with respect x.

We differentiate each term with respect to x:

$$\frac{d}{dx}\left(y^2\right) + \frac{d}{dx}\left(x^3\right) - \frac{d}{dx}\left(y^3\right) + \frac{d}{dx}(6) = \frac{d}{dx}(3y)$$

Differentiating functions of x with respect to x is straightforward. But when differentiating a function of y with respect to x we must remember the rule given in the previous keypoint. We find

$$\frac{d}{dy}\left(y^2\right) \times \frac{dy}{dx} \; + \; 3x^2 - \frac{d}{dy}\left(y^3\right) \times \frac{dy}{dx} \; + \; 0 = \frac{d}{dy}(3y) \times \frac{dy}{dx}$$

that is

$$2y\frac{dy}{dx} \; + \; 3x^2 - 3y^2\frac{dy}{dx} = 3\frac{dy}{dx}$$

We rearrange this to collect all terms involving $\frac{dy}{dx}$ together.

$$3x^2 = 3\frac{dy}{dx} - 2y\frac{dy}{dx} + 3y^2\frac{dy}{dx}$$

then

$$3x^2 = \left(3 - 2y + 3y^2\right)\frac{dy}{dx}$$

so that, finally,

$$\frac{dy}{dx} = \frac{3x^2}{3 - 2y + 3y^2}$$

This is our expression for $\dfrac{dy}{dx}$.

Example

Suppose we want to differentiate, with respect to x, the implicit function

$$\sin y + x^2y^3 - \cos x = 2y$$

As before, we differentiate each term with respect to x.

$$\frac{d}{dx}(\sin y) + \frac{d}{dx}\left(x^2y^3\right) - \frac{d}{dx}(\cos x) = \frac{d}{dx}(2y)$$

Recognise that the second term is a product and we will need the product rule. We will also use the chain rule to differentiate the functions of y. We find

$$\frac{d}{dy}(\sin y) \times \frac{dy}{dx} + \left\{x^2\frac{d}{dx}\left(y^3\right) + y^3\frac{d}{dx}\left(x^2\right)\right\} + \sin x = \frac{d}{dy}(2y) \times \frac{dy}{dx}$$

so that

$$\cos y\frac{dy}{dx} + \left\{x^2.\frac{d}{dy}\left(y^3\right)\frac{dy}{dx} + y^3.\,2x\right\} + \sin x = 2\frac{dy}{dx}$$

Tidying this up gives

$$\cos y\frac{dy}{dx} + x^2\,3y^2\frac{dy}{dx} + 2xy^3 + \sin x = 2\frac{dy}{dx}$$

We now start to collect together terms involving $\frac{dy}{dx}$.

$$2xy^3 + \sin x = 2\frac{dy}{dx} - \cos y\frac{dy}{dx} - 3x^2y^2\frac{dy}{dx}$$

$$2xy^3 + \sin x = (2 - \cos y - 3x^2y^2)\frac{dy}{dx}$$

so that, finally

$$\frac{dy}{dx} = \frac{2xy^3 + \sin x}{2 - \cos y - 3x^2y^2}$$

We have deliberately included plenty of detail in this calculation. With practice you will be able to omit many of the intermediate stages. The following two examples show how you should aim to condense the solution.

Example

Suppose we want to differentiate $y^2 + x^3 - xy + \cos y = 0$ to find $\frac{dy}{dx}$. The condensed solution may take the form:

$$\begin{aligned}
2y\frac{dy}{dx} + 3x^2 - \frac{d}{dx}(xy) - \sin y\frac{dy}{dx} &= 0 \\
(2y - \sin y)\frac{dy}{dx} + 3x^2 - \left\{x\frac{dy}{dx} + y.1\right\} &= 0 \\
(2y - \sin y - x)\frac{dy}{dx} + 3x^2 - y &= 0 \\
(2y - \sin y - x)\frac{dy}{dx} &= y - 3x^2
\end{aligned}$$

so that

$$\frac{dy}{dx} = \frac{y - 3x^2}{2y - \sin y - x}$$

Example

Suppose we want to differentiate

$$y^3 - x\sin y + \frac{y^2}{x} = 8$$

The solution is as follows:

$$3y^2\frac{dy}{dx} - \left\{x\cos y\frac{dy}{dx} + \sin y.1\right\} + \frac{x\;2y\frac{dy}{dx} - y^2.1}{x^2} = 0$$

Multiplying through by x^2 gives:

$$\begin{aligned} 3x^2y^2\frac{dy}{dx} - x^3\cos y\frac{dy}{dx} - x^2\sin y + 2xy\frac{dy}{dx} - y^2 &= 0 \\ \frac{dy}{dx}\left(3x^2y^2 - x^3\cos y + 2xy\right) &= x^2\sin y + y^2 \end{aligned}$$

so that

$$\frac{dy}{dx} = \frac{x^2\sin y + y^2}{3x^2y^2 - x^3\cos y + 2xy}$$

Exercises

1. Find the derivative, with respect to x, of each of the following functions (in each case y depends on x).

a) y b) y^2 c) $\sin y$ d) e^{2y} e) $x+y$

f) xy g) $y\sin x$ h) $y\sin y$ i) $\cos(y^2+1)$ j) $\cos(y^2+x)$

2. Differentiate each of the following with respect to x and find $\frac{dy}{dx}$.

a) $\sin y + x^2 + 4y = \cos x$.

b) $3xy^2 + \cos y^2 = 2x^3 + 5$.

c) $5x^2 - x^3\sin y + 5xy = 10$.

d) $x - \cos x^2 + \frac{y^2}{x} + 3x^5 = 4x^3$.

e) $\tan 5y - y\sin x + 3xy^2 = 9$.

Answers to Exercises on Implicit Differentiation

1.

a) $\dfrac{dy}{dx}$ b) $2y\dfrac{dy}{dx}$ c) $\cos y\dfrac{dy}{dx}$

d) $2\mathrm{e}^{2y}\dfrac{dy}{dx}$ e) $1+\dfrac{dy}{dx}$ f) $x\dfrac{dy}{dx}+y$

g) $y\cos x + \sin x\dfrac{dy}{dx}$ h) $(\sin y + y\cos y)\dfrac{dy}{dx}$ i) $-2y\sin(y^2+1)\dfrac{dy}{dx}$

j) $-\left(2y\dfrac{dy}{dx}+1\right)\sin(y^2+x)$

2.

a) $\dfrac{dy}{dx} = \dfrac{-\sin x - 2x}{4+\cos y}$ b) $\dfrac{dy}{dx} = \dfrac{6x^2-3y^2}{6xy - 2y\sin y^2}$

c) $\dfrac{dy}{dx} = \dfrac{10x - 3x^2\sin y + 5y}{x^3\cos y - 5x}$ d) $\dfrac{dy}{dx} = \dfrac{12x^4 - 15x^6 + y^2 - 2x^3\sin x^2 - x^2}{2xy}$

e) $\dfrac{dy}{dx} = \dfrac{y\cos x - 3y^2}{5\sec^2 5y - \sin x + 6xy}$