

Continuous compounding

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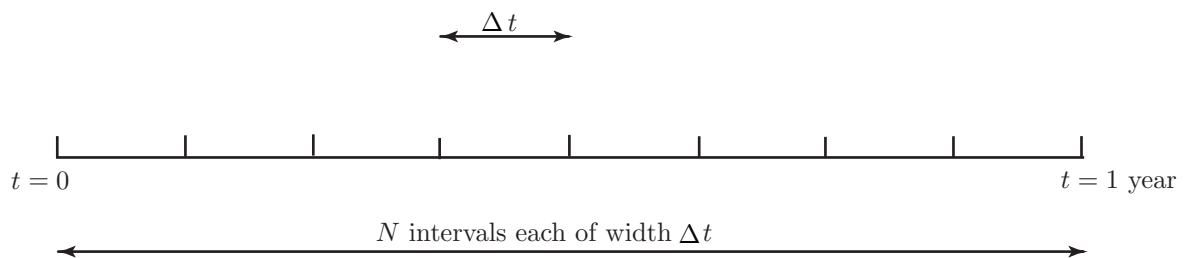
Introduction

Interest earned on an investment, or due on a loan, is usually compounded. Compound interest was described on leaflet *Interest* where we described how to calculate the interest when the compounding period was annual. On occasions interest is compounded continuously which has the effect of increasing the amount of interest. This leaflet gives details of continuously compounded interest.

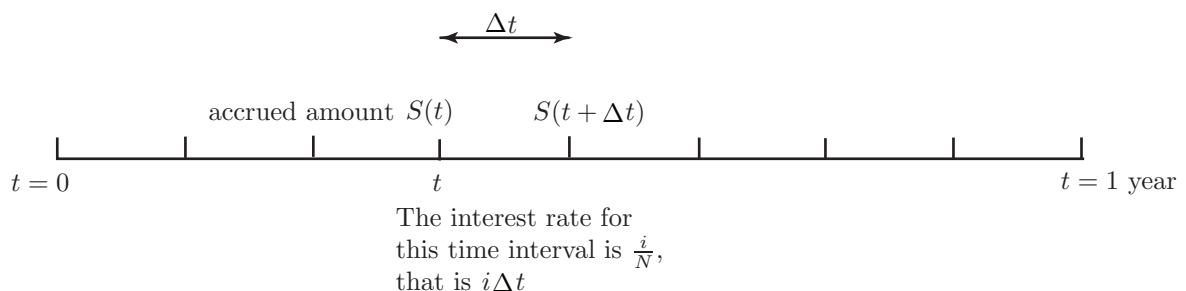
Continuous compounding - with a constant interest rate

Suppose the annual interest rate per annum is constant and equal to i .

Suppose we divide the year up into N intervals each of length Δt (e.g. 4 quarters, or twelve months). Then $N \times \Delta t = 1$. It follows that $N = \frac{1}{\Delta t}$.



Suppose at some time t during the year the accrued amount of an investment is $S(t)$. We write $S(t)$ to show that the accrued amount, S , depends upon the time t . This becomes the principal for the next time interval, and at the end of this interval the accrued amount is $S(t + \Delta t)$. The interest rate for this interval is $\frac{i}{N} = i \Delta t$.



Then, using the compound interest formula,

$$S(t + \Delta t) = S(t) \left(1 + \frac{i}{N}\right) = S(t)(1 + i \Delta t)$$

Then, rearranging

$$S(t + \Delta t) - S(t) = S(t) i \Delta t$$

that is

$$\frac{S(t + \Delta t) - S(t)}{\Delta t} = S(t) i$$

As we let the compounding interval Δt become very small, and equivalently the number of intervals N become very large, techniques of calculus enable us to write this equation as

$$\frac{dS}{dt} = S(t) i$$

This is an example of a **differential equation**. The solution of such equations is beyond the scope of these leaflets, but it can be shown that the accrued amount at any time t is $S(t)$ given by:

$$S(t) = Pe^{it}$$

where P is the principal invested right at the start, and e is the exponential constant (see leaflet *The exponential constant e*).

This formula tells us the accrued amount at any time t , given the principal P and the constant interest rate i . When using this formula the units of time must be consistent throughout, so for example, if i is the annual interest rate, t must be measured in years.

Example

A principal of £ 1000 is invested at a constant annual rate of 8%. Interest earned is compounded continuously. Find the accrued amount after 25 years.

Solution

With $P = 1000$, $i = 0.08$ and $t = 25$ we have

$$S = 1000e^{(0.08)(25)} = 1000e^2 = £7389.06$$

Exercises

In each of the following calculate

(a) the accrued amount after investment for the given number of years where interest is to be compounded continuously (b) the difference between your result in (a) and the amount that would be earned by compounding annually.

1. £ 500 invested at an annual rate of 4% for 20 years
2. £ 3 000 invested at an annual rate of 3.2% for 5 years
3. £ 25 000 invested at an annual rate of 6.5% for 3 years.

Answers

1. (a) £ 1 112.77 (b) £ 17.21
2. (a) £ 3 520.53 (b) £ 8.81
3. (a) £ 30 382.77 (b) £ 184.03.