Cross Product

Academic Resource Center

In This Presentation…

- We will give a definition
- Look at the properties
- Do some examples

Cross Product

<u>Definition:</u>

If **a** = <a_1, a_2, a_3> and **b** = <b_1, b_2 , b_3 >, then the cross product of **a** and **b** is the vector,

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

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Note the result is a **vector** and NOT a scalar value.

For this reason, it is also called the vector product.

To make this definition easer to remember, we usually use determinants to calculate the cross product.

Determinants

- Determinant of order 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- Determinant of order 3:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

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We can now rewrite the definition for the cross product using these determinants:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

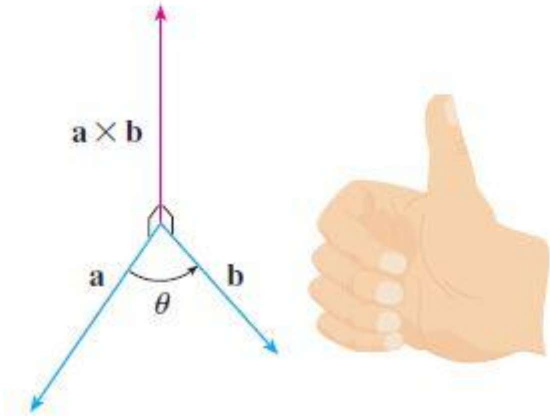
If you perform the determinants, you will obtain exactly what was stated in the definition.

Orthogonal Vectors

When you take the cross product of two vectors **a** and **b**,

The resultant vector, (**a** x **b**), is orthogonal to <u>BOTH</u> **a** and **b**.

We can use the right hand rule to determine the direction of **a** x **b**



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Parallel Vectors

Two nonzero vectors **a** and **b** are parallel if and only if,

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

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Examples

<u>Find **a** x **b**:</u>

1. Given $a = <1,4,-1>$ and $b = <2,-4,6>$,

   $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

   $= (4*6 - (-1)(-4))\mathbf{i} + ((-1)*2 - 1*6)\mathbf{j} + (1*(-4) - 4*2)\mathbf{k}$

   $= (24 - 4)\mathbf{i} + (-2 - 6)\mathbf{j} + (-4 - 8)\mathbf{k}$

   $= 20\mathbf{i} - 8\mathbf{j} - 12\mathbf{k} = <20,-8,-12>$

2. Given $\mathbf{a} = \mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$,

   $\mathbf{a} \times \mathbf{b} = ((1)(0) - 6(1))\mathbf{i} + (6*1 - 0*0)\mathbf{j} + (0*1 - 1*1)\mathbf{k}$

   $= - 6\mathbf{i} + 6\mathbf{j} - \mathbf{k}$

   $= -6\mathbf{i} + 6\mathbf{j} - \mathbf{k} = <-5,6,-1>$

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Alternative Method

Another way to calculate the cross product of two vectors is to multiply their components with each other. (Similar to the distributive property) But first we need to know,

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

An easier way to memorize this is to draw a circle with the **i**, **j**, and **k** vectors. Clockwise relates to the positive orientation and counter clockwise is the negative orientation.

i

k j

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Alternative Method (cont.)

Note that,

$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$

Therefore, the cross product is not commutative and the associative law does not hold.

$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

Cross Product

Properties of the cross product:

If **a**, **b**, and **c** are vectors and *c* is a scalar, then

1. **a** x **b** = –**b** x **a**
2. (*c***a**) x **b** = *c*(**a** x **b**) = **a** x (*c***b**)
3. **a** x (**b** + **c**) = **a** x **b** + **a** x **c**
4. (**a** + **b**) x **c** = **a** x **c** + **b** x **c**
5. **a** · (**b** x **c**) = (**a** x **b**) · **c**
6. **a** x (**b** x **c**) = (**a** · **c**)**b** – (**a** · **b**)**c**

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Example

If $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (3)(1)(\mathbf{i} \times \mathbf{i}) + (3)(4)(\mathbf{i} \times \mathbf{j}) + (3)(6)(\mathbf{i} \times \mathbf{k}) + \\ &\quad (2)(1)(\mathbf{j} \times \mathbf{i}) + (2)(4)(\mathbf{j} \times \mathbf{j}) + (2)(6)(\mathbf{j} \times \mathbf{k}) + \\ &\quad (5)(1)(\mathbf{k} \times \mathbf{i}) + (5)(4)(\mathbf{k} \times \mathbf{j}) + (5)(6)(\mathbf{k} \times \mathbf{k}) \\ &= 12\mathbf{k} - 18\mathbf{j} - 2\mathbf{k} + 12\mathbf{i} + 5\mathbf{j} - 20\mathbf{i} \\ &= -8\mathbf{i} - 13\mathbf{j} + 10\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \times \mathbf{a} &= 2\mathbf{k} - 5\mathbf{j} - 12\mathbf{k} + 20\mathbf{i} + 18\mathbf{j} - 12\mathbf{i} \\ &= 8\mathbf{i} + 13\mathbf{j} - 10\mathbf{k}\end{aligned}$$

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Cross Product

If the angle between the two vectors

a and **b** is θ, then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

(Note that $|\mathbf{a} \times \mathbf{b}|$ is the magnitude of the vector $\mathbf{a} \times \mathbf{b}$)

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Example

Find the angle between **a** and **b**:

1. Given **a** = <-4,3,0> and **b** = <2,0,0>

   |**a**| = 5

   |**b**| = 2

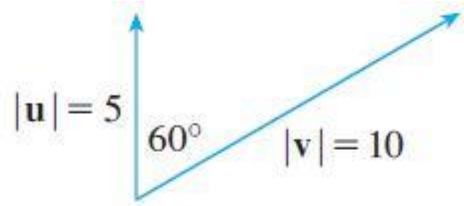
   a x **b** = <0,0,-6>

   |**a** x **b**| = 6

   $\sin^{-1}(6/(5*2)) = \sin^{-1}(3/5) =$ **36.87°**

Example

Find |**u** x **v**| and determine whether **u** x **v** is directed into the page or out of the page.

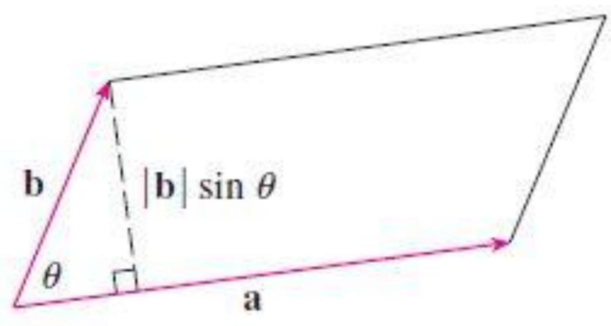


|**u** x **v**| = |**u**||**v**| sin θ = 5*10*sin 60° = **43.3**

From the right hand rule, going from vector **u** to **v**, the resultant vector **u** x **v** is directed into the page.

Cross Product

The length of the cross product **a** x **b**, |a x b|, is equal to the area of the parallelogram determined by **a** and **b**.



Example

Find the area of the triangle with the vertices P(0,1,4), Q(-5,9,2), and R(7,2,8):

PQ = <-5,8,-2> and **PR** = <7,1,4>

PQ x **PR** = <34,6,-61>

Area of parallelogram = |**PQ** x **PR**| = $\sqrt{34^2 + 6^2 + (-61)^2}$ = $17\sqrt{17}$

Area of the triangle = $17\sqrt{17}/2 \approx$ **35**

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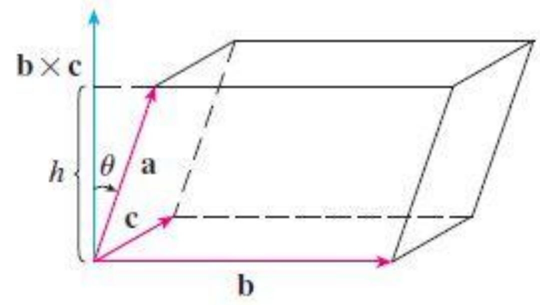
Cross Product

The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

where, $$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



If the triple scalar product is 0, then the vectors must lie in the same plane, meaning they are coplanar.

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Example

Find the volume of the parallelepiped determined by the vectors **a**, **b**, and **c**.

a = <6,3,-4>, **b** = <0,2,1>, **c** = <5,-1,2>

b x **c** = <5,5,-10>

a · (**b** x **c**) = <6,3,-4> · <5,5,-10> = 85

Volume = |**a** · (**b** x **c**)| = **85**

References

- Calculus – Stewart 6th Edition, Section 13.4 “The Cross Product”