

Improper Integrals

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Abstract

Notes on improper integrals.

1 Improper Integrals

1.1 Introduction

In Calculus II, students defined the integral $\int_a^b f(x) dx$ over a finite interval $[a, b]$. The function f was assumed to be continuous, or at least bounded, otherwise the integral was not guaranteed to exist. Assuming an antiderivative of f could be found, $\int_a^b f(x) dx$ always existed, and was a number. In this section, we investigate what happens when these conditions are not met.

Definition 1 (Improper Integral) *An integral is an improper integral if either the interval of integration is not finite (improper integral of type 1) or if the function to integrate is not continuous (not bounded) in the interval of integration (improper integral of type 2).*

Example 2 $\int_0^{\infty} e^{-x} dx$ is an improper integral of type 1 since the upper limit of integration is infinite.

Example 3 $\int_0^1 \frac{dx}{x}$ is an improper integral of type 2 because $\frac{1}{x}$ is not continuous at 0.

Example 4 $\int_0^{\infty} \frac{dx}{x-1}$ is an improper integral of types 1 since the upper limit of integration is infinite. It is also an improper integral of type 2 because $\frac{1}{x-1}$ is not continuous at 1 and 1 is in the interval of integration.

Example 5 $\int_{-2}^2 \frac{dx}{x^2-1}$ is an improper integral of type 2 because $\frac{1}{x^2-1}$ is not continuous at -1 and 1.

Example 6 $\int_0^\pi \tan x dx$ is an improper integral of type 2 because $\tan x$ is not continuous at $\frac{\pi}{2}$.

We now look how to handle each type of improper integral.

1.2 Improper Integrals of Type 1

These are easy to identify. Simply look at the interval of integration. If either the lower limit of integration, the upper limit of integration or both are not finite, it will be an improper integral of type 1.

Definition 7 (improper integral of type 1) *Improper integrals of type 1 are evaluated as follows:*

1. If $\int_a^t f(x) dx$ exists for all $t \geq a$, then we define

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

provided the limit exists as a finite number. In this case, $\int_a^\infty f(x) dx$ is said to be **convergent** (or to **converge**). Otherwise, $\int_a^\infty f(x) dx$ is said to be **divergent** (or to **diverge**).

2. If $\int_t^b f(x) dx$ exists for all $t \leq b$, then we define

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

provided the limit exists as a finite number. In this case, $\int_{-\infty}^b f(x) dx$ is said to be **convergent** (or to **converge**). Otherwise, $\int_{-\infty}^b f(x) dx$ is said to be **divergent** (or to **diverge**).

3. If both $\int_a^\infty f(x) dx$ and $\int_{-\infty}^b f(x) dx$ converge, then we define

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx$$

The integrals on the right are evaluated as shown in 1. and 2..

1.3 Improper Integrals of Type 2

These are more difficult to identify. One needs to look at the interval of integration, and determine if the integrand is continuous or not in that interval. Things to look for are fractions for which the denominator becomes 0 in the interval of integration. Keep in mind that some functions do not contain fractions explicitly like $\tan x$, $\sec x$.

Definition 8 (improper integral of type 2) *Improper integrals of type 2 are evaluated as follows:*

1. if f is continuous on $[a, b)$ and not continuous at b then we define

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

provided the limit exists as a finite number. In this case, $\int_a^b f(x) dx$ is said to be **convergent** (or to **converge**). Otherwise, $\int_a^b f(x) dx$ is said to be **divergent** (or to **diverge**).

2. if f is continuous on $(a, b]$ and not continuous at a then we define

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

provided the limit exists as a finite number. In this case, $\int_a^b f(x) dx$ is said to be **convergent** (or to **converge**). Otherwise, $\int_a^b f(x) dx$ is said to be **divergent** (or to **diverge**).

3. If f is not continuous at c where $a < c < b$ and both $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$ converge then we define

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

The integrals on the right are evaluated as shown in 1. and 2..

We now look at some examples.

1.4 Examples

- Evaluating an improper integral is really two problems. It is an integral problem and a limit problem. It is best to do them separately.
- When breaking down an improper integral to evaluate it, make sure that each integral is improper at only one place, that place should be either the lower limit of integration, or the upper limit of integration.

Example 9 $\int_1^{\infty} \frac{dx}{x^2}$

This is an improper integral of type 1. We evaluate it by finding $\lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x^2}$.

First, $\int_1^t \frac{dx}{x^2} = \left(1 - \frac{1}{t}\right)$ and $\lim_{t \rightarrow \infty} \left(1 - \frac{1}{t}\right) = 1$ hence $\int_1^{\infty} \frac{dx}{x^2} = 1$.

Example 10 $\int_1^{\infty} \frac{dx}{x}$

This is an improper integral of type 1. We evaluate it by finding $\lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x}$

First, $\int_1^t \frac{dx}{x} = \ln t$ and $\lim_{t \rightarrow \infty} \ln t = \infty$ hence $\int_1^{\infty} \frac{dx}{x}$ diverges.

Example 11 $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

This is an improper integral of type 1. Since both limits of integration are infinite, we break it into two integrals.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$$

Note that since the function $\frac{1}{1+x^2}$ is even, we have $\int_{-\infty}^0 \frac{dx}{1+x^2} = \int_0^{\infty} \frac{dx}{1+x^2}$;

we only need to do $\int_0^{\infty} \frac{dx}{1+x^2}$.

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{t \rightarrow \infty} \int_0^t \frac{dx}{1+x^2}$$

and

$$\begin{aligned} \int_0^t \frac{dx}{1+x^2} &= \tan^{-1} x \Big|_0^t \\ &= \tan^{-1} t - \tan^{-1} 0 \\ &= \tan^{-1} t \end{aligned}$$

Thus

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{t \rightarrow \infty} (\tan^{-1} t) \\ = \frac{\pi}{2}$$

It follows that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} \\ = \pi$$

Example 12 $\int_0^{\frac{\pi}{2}} \sec x dx$

This is an improper integral of type 2 because $\sec x$ is not continuous at $\frac{\pi}{2}$. We

evaluate it by finding $\lim_{t \rightarrow \frac{\pi}{2}^-} \int_0^t \sec x dx$.

First,

$$\int_0^t \sec x dx = \ln |\sec x + \tan x|_0^t \\ = \ln |\sec t + \tan t|$$

and $\lim_{t \rightarrow \frac{\pi}{2}^-} (\ln |\sec t + \tan t|) = \infty$ hence $\int_0^{\frac{\pi}{2}} \sec x dx$ diverges.

Example 13 $\int_0^{\pi} \sec^2 x dx$

This is an improper integral of type 2, $\sec^2 x$ is not continuous at $\frac{\pi}{2}$. Thus,

$$\int_0^{\pi} \sec^2 x dx = \int_0^{\frac{\pi}{2}} \sec^2 x dx + \int_{\frac{\pi}{2}}^{\pi} \sec^2 x dx$$

First, we evaluate $\int_0^{\frac{\pi}{2}} \sec^2 x dx$.

$$\int_0^{\frac{\pi}{2}} \sec^2 x dx = \lim_{t \rightarrow \frac{\pi}{2}^-} \int_0^t \sec^2 x dx \\ \int_0^t \sec^2 x dx = \tan t - \tan 0 \\ = \tan t$$

Therefore,

$$\int_0^{\frac{\pi}{2}} \sec^2 x dx = \lim_{t \rightarrow \frac{\pi}{2}^-} (\tan t) \\ = \infty$$

It follows that $\int_0^{\frac{\pi}{2}} \sec^2 x dx$ diverges, therefore, $\int_0^{\pi} \sec^2 x dx$ also diverges.

Remark 14 If we had failed to see that the above integral is improper, and had evaluated it using the Fundamental Theorem of Calculus, we would have obtained a completely different (and wrong) answer.

$$\begin{aligned}\int_0^\pi \sec^2 x dx &= \tan \pi - \tan 0 \\ &= 0 \text{ (this is not correct)}\end{aligned}$$

Example 15 $\int_{-\infty}^{\infty} \frac{dx}{x^2}$

This integral is improper for several reasons. First, the interval of integration is not finite. The integrand is also not continuous at 0. To evaluate it, we break it so that each integral is improper at only one place, that place being one of the limits of integration, as follows: $\int_{-\infty}^{\infty} \frac{dx}{x^2} = \int_{-\infty}^{-1} \frac{dx}{x^2} + \int_{-1}^0 \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2} + \int_1^{\infty} \frac{dx}{x^2}$. We then evaluate each improper integral. The reader will verify that it diverges.

Several important results about improper integrals are worth remembering, they will be used with infinite series. The proof of these results is left as an exercise.

Theorem 16 $\int_1^{\infty} \frac{dx}{x^p}$ converges if $p > 1$, it diverges otherwise.

Proof. This is an improper integral of type 1 since the upper limit of integration is infinite. Thus, we need to evaluate $\lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x^p}$. When $p = -1$, we have already seen that the integral diverges. Let us assume that $p \neq -1$. First, we evaluate the integral.

$$\begin{aligned}\int_1^t \frac{dx}{x^p} &= \int_1^t x^{-p} dx \\ &= \left. \frac{x^{1-p}}{1-p} \right|_1^t \\ &= \frac{t^{1-p}}{1-p} - \frac{1}{1-p}\end{aligned}$$

The sign of $1-p$ is important. When $1-p > 0$ that is when $p < 1$, t^{1-p} is in the numerator. Therefore, $\lim_{t \rightarrow \infty} \left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right) = \infty$ thus the integral diverges. When $1-p < 0$ that is when $p > 1$, t^{1-p} is really in the denominator so that $\lim_{t \rightarrow \infty} \left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right) = \frac{1}{p-1}$ and therefore $\int_1^{\infty} \frac{dx}{x^p}$ converges. In conclusion, we have looked at the following cases:

Case 1: $p = -1$. In this case, the integral diverges.

Case 2: $p < 1$. In this case, the integral diverges.

Case 3: $p > 1$. In this case, the integral converges.

■

Theorem 17 $\int_0^1 \frac{dx}{x^p}$ converges if $p < 1$, it diverges otherwise.

Proof. See problems. ■

1.5 Comparison Theorem for Improper Integrals

Sometimes an improper integral is too difficult to evaluate. One technique is to compare it with a known integral. The theorem below shows us how to do this.

Theorem 18 Suppose that f and g are two continuous functions for $x \geq a$ such that $0 \leq g(x) \leq f(x)$. Then, the following is true:

1. If $\int_a^\infty f(x) dx$ converges then $\int_a^\infty g(x) dx$ also converges.
2. If $\int_a^\infty g(x) dx$ diverges, then $\int_a^\infty f(x) dx$ also diverges.

The theorem is not too difficult to understand if we think about the integral in terms of areas. Since both functions are positive, the integrals simply represent the area of the region below their graph. Let A_f be the area of the region below the graph of f . Use a similar notation for A_g . If $0 \leq g(x) \leq f(x)$, then $A_g \leq A_f$. Part 1 of the theorem is simply saying that if A_f is finite, so is A_g ; this should be obvious from the inequality. Part 2 says that if A_g is infinite, so is A_f .

Example 19 Study the convergence of $\int_1^\infty e^{-x^2} dx$

We cannot evaluate the integral directly, e^{-x^2} does not have an antiderivative. We note that

$$\begin{aligned}x \geq 1 &\iff x^2 \geq x \\ &\iff -x^2 \leq -x \\ &\iff e^{-x^2} \leq e^{-x}\end{aligned}$$

Now,

$$\begin{aligned}\int_1^{\infty} e^{-x} dx &= \lim_{t \rightarrow \infty} \int_1^t e^{-x} dx \\ &= \lim_{t \rightarrow \infty} (e^{-1} - e^{-t}) \\ &= e^{-1}\end{aligned}$$

and therefore converges. It follows that $\int_1^{\infty} e^{-x^2} dx$ converges by the comparison theorem.

Remark 20 When using the comparison theorem, the following inequalities are useful:

$$x^2 \geq x \geq \sqrt{x} \geq 1$$

and

$$\ln x \leq x \leq e^x$$

1.6 Things to know

- Be able to tell if an integral is improper or not and what type it is.
- Be able to tell if an improper integral converges or diverges. If it converges, be able to find what it converges to.
- Be able to write an improper integral as a limit of definite integral(s).
- Related problems assigned:
 - Page 436: # 1, 3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 25, 29, 39, 41, 57.
 - Discuss the convergence of $\int_0^{\infty} e^{ax} dx$
 - Prove theorem 17.