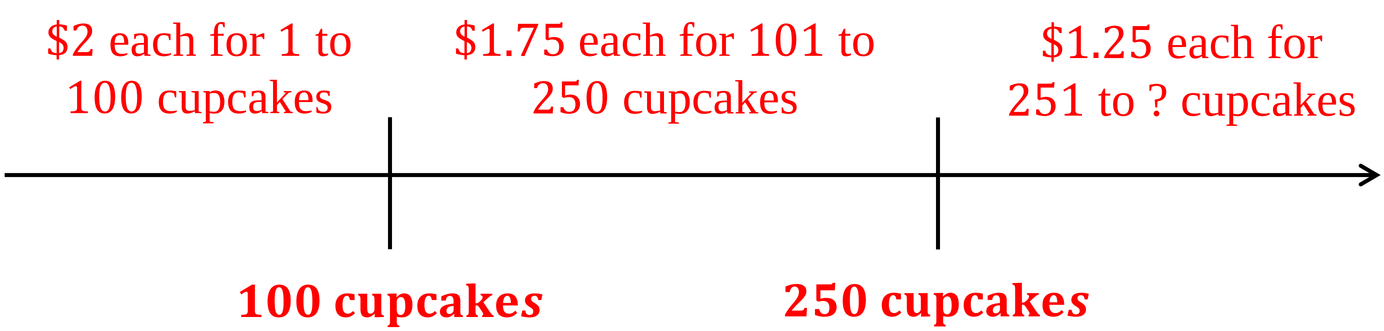
In this lesson we'll be covering how to set-up piecewise defined functions based on story problems. Keep in mind that each piece of a piecewise defined function has its own domain, so we'll also have to set-up an interval for each piece, just like the sample piecewise function f given below:

$$f(x) = \begin{cases} ax & ; \quad x \le f \\ bx + c & ; \quad f < x \le g \\ dx + e & ; \quad x > g \end{cases}$$

<u>Example 1:</u> A bakery has the following pricing for large orders of cupcakes. The first 100 cupcakes of any order cost \$2.00 each. Each of the next 150 cupcakes only cost \$1.75 each. And each cupcake ordered in excess of 250 costs \$1.25 each.



a. If a mom orders 10 dozen cupcakes for her child's birthday party, what would be the total cost for her order?

b. If a couple orders 300 cupcakes for their wedding reception, what would be the total cost of their order?

<u>Example 2:</u> A bakery has the following pricing for large orders of cupcakes. The first 100 cupcakes of any order cost $\$2.00$ each. Each of the next 150 cupcakes only cost $\$1.75$ each. And each cupcake ordered in excess of 250 costs $\$1.25$ each. The total cost C is a function of the number of cupcakes ordered x. Write the piecewise-defined function C.

\$2 each for 1 to 100 cupcakes	\$1.75 each for 101 to 250 cupcakes	\$1.25 each for 251 to ? cupcakes
$1 \leq x \leq 100$	$101 \leq x \leq 250$	$251 \leq x$

100 cupcakes **250 cupcakes**

$$C(x) = \begin{cases} \quad \\ \quad \\ \quad \end{cases}$$

Now that we have a piecewise function to determine the cost of any number of cupcakes, we use it to find the cost of any order.

a. If a school orders 15 dozen cupcakes for an event, what would be the total cost of their order?

b. If a couple orders 450 cupcakes for their wedding reception, what would be the total cost of their order?

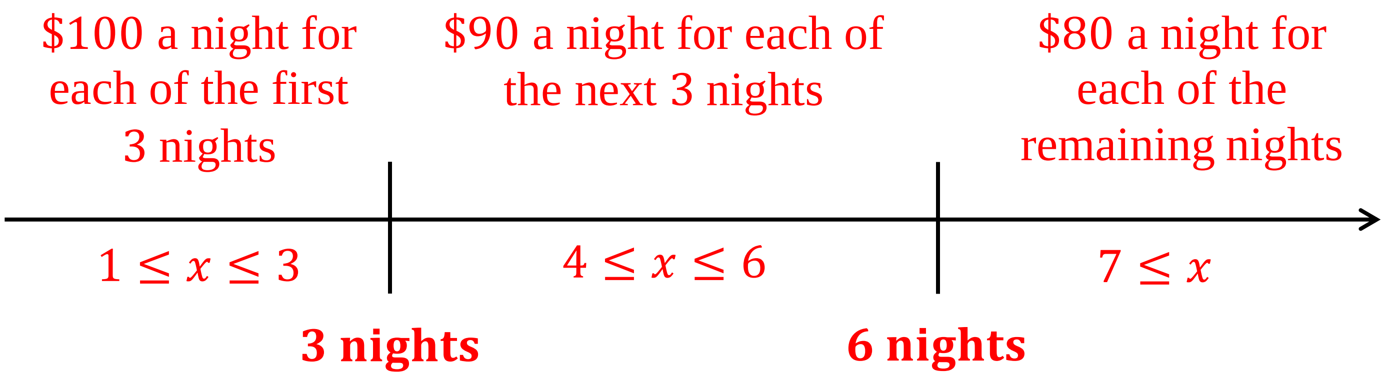
Keep in mind that every one of these story problems will have at least one threshold where we change from one piece to another. Once you exceed a threshold you must break inputs into separate parts, just like cupcake example above. In Examples 1 and 2, the thresholds were 100 cupcakes and 250 cupcakes, because there were price changes for each of those quantities.

$2x$ | $1.75x + 25$ | $1.25x + 150$

$1 \leq x \leq 100$ | $101 \leq x \leq 250$ | $251 \leq x$

100 cupcakes **250 cupcakes**

<u>Example 3:</u> A rental home on Airbnb rents for \$100 a night for the first three nights, \$90 a night for the next three nights, and \$80 a night for each remaining night. The total cost T is a function of the number of nights x that a guest stays. Write the piecewise-defined function T.



The first piece of our piece-wise defined function is $\mathbf{100x}$, where $\mathbf{x \leq 3}$. This is because when someone stays for 3 nights or fewer the rate is simply \$100 a night.

$$\boldsymbol{T(x) = \begin{cases} 100x & \text{if} \quad 1 \leq x \leq 3 \\ & \\ & \end{cases}}$$

The second interval is $4 \leq x \leq 6$

Cost of the first 3 nights + Cost of the remaining nights

$$100(3) + 90(x - 3)$$

$$300 + 90x - 270$$

$$90x + 30$$

So the second piece of our piece-wise defined function is $\mathbf{90x + 30}$, where $\mathbf{4 \leq x \leq 6}$.

$$\boldsymbol{T(x) = \begin{cases} 100x & \text{if} \quad 1 \le x \le 3 \\ 90x + 30 & \text{if} \quad 4 \le x \le 6 \\ & \end{cases}}$$

The third and final interval is $x \geq 7$.

Cost of the first 3 nights + Cost of the next 3 nights + Cost of the remaining nights

$$100(3) + 90(3) + 80(x - 6)$$

$$300 + 270 + 80x - 480$$

$$80x + 90$$

So the final piece of our piece-wise defined function is $\boldsymbol{80x + 90}$, where $\boldsymbol{x \geq 7}$.

$$\boldsymbol{T(x) = \begin{cases} 100x & \text{if} \quad 1 \le x \le 3 \\ 90x + 30 & \text{if} \quad 4 \le x \le 6 \\ 80x + 90 & \text{if} \quad x \ge 7 \end{cases}}$$

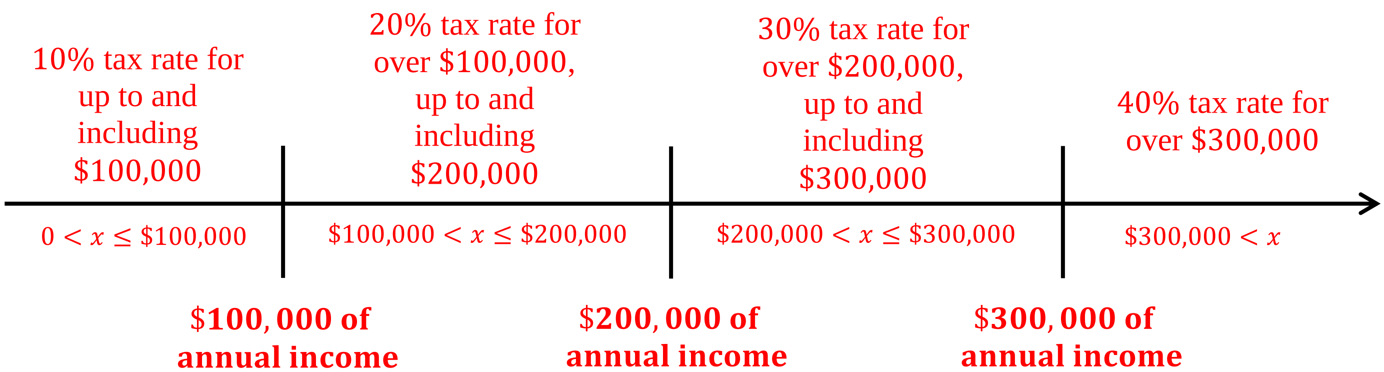
Keep in mind that whenever cross a threshold, such as going from

$1 \leq x \leq 3$ to $4 \leq x \leq 6$, you must take your total and subtract what you've already found. For instance when finding the expression for the second piece of the function $T(x)$, we took the total nights stayed (x) and subtracted 3 since we already knew the first 3 nights were \$100 each. When finding the expression for the third piece of the function $T(x)$, we took the total nights stayed (x) and subtracted 6 since we already knew the first 6 nights cost \$570 (\$100 for each of the first 3 nights plus \$90 for each of the next 3 nights).

<u>Example 4:</u> Below is a proposed alternative to the current federal income tax system based on annual income for all taxpayer's:

- for the first \$100,000 of income, every dollar is taxed at a rate of 10%
- each additional dollar over \$100,000 is taxed at a rate of 20%, for the next \$100,000 of income
- each additional dollar over \$200,000 is taxed at a rate of 30%, for the next \$100,000 of income
- every dollar over \$300,000 is taxed at a rate of 40%

Find a piecewise-defined function T that specifies the yearly federal income tax for a person earning x dollars per year.



Piece 4: $x > \$300{,}000$

Tax on the first \$100,000 of income	+	Tax on the next \$100,000 of income	+	Tax on the next \$100,000 of income	+	Tax on the remaining income

$$0.1(100{,}000) + 0.2(100{,}000) + 0.3(100{,}000) + 0.4(x - 300{,}000)$$

$$10{,}000 + 20{,}000 + 30{,}000 + 0.4x - 120{,}000$$

$$\mathbf{0.4x - 60{,}000}$$

$$T(x) = \begin{cases} \boxed{} \\ \boxed{} \\ \boxed{} \\ \boxed{} \end{cases}$$

a. How much federal income tax would someone with an annual income of $150,000 pay?

b. How much federal income tax would someone with an annual income of $225,000 pay?

Since $225,000 falls within the third interval ($200{,}000 < x \leq 300{,}000$), I need to use the third piece of the function to determine the tax:

$$0.3(225{,}000) - 30{,}000$$

$$67{,}500 - 30{,}000$$

$$\mathbf{37{,}500}$$

Someone with an income of $225,000 would owe $37,500 in taxes.

c. How much federal income tax would someone with an annual income of $1,000,000 pay?

Since $1,000,000 is part of the last interval ($x > 300{,}000$), I need to use the last piece of the function to determine the tax:

$$0.4(1{,}000{,}000) - 60{,}000$$

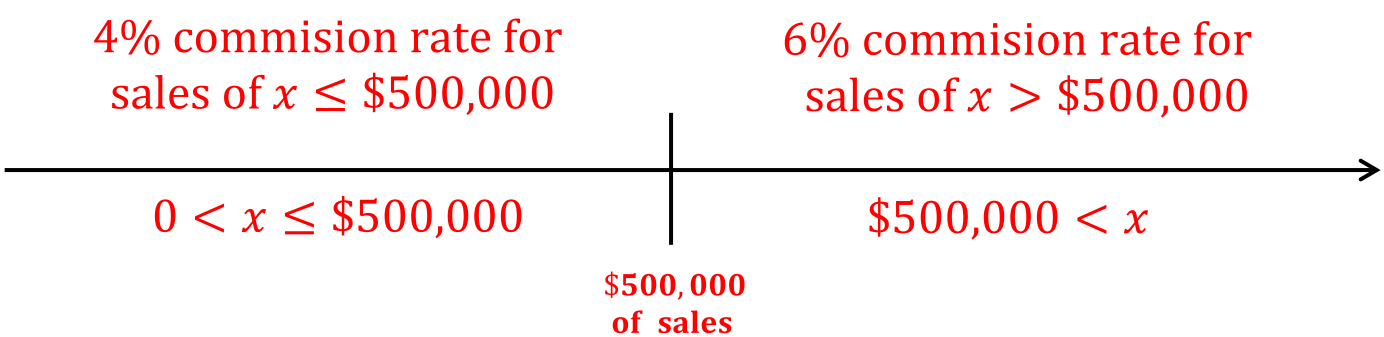
$$400{,}000 - 60{,}000$$

$$\mathbf{340{,}000}$$

Someone with an income of $1,000,000 would owe $340,000 in taxes.

<u>Example 5:</u> A salesperson makes \$35,000 a year plus 4% commission on all sales up to (and including) \$500,000. If they exceed \$500,000 in sales for a calendar year, the salesperson's commission jumps to 6% for all their remaining sales over \$500,000. The salesperson's total salary S for a given year is based on their total sales x. Write a piecewise defined function $S(x)$.

(hint: don't forget about the base salary of \$35,000 when finding each piece of the function)



$S(x) =$

Answers to Examples:

1a. \$235

1b. \$525

2. $C(x) = \begin{cases} 2x & \text{if } 1 \le x \le 100 \\ 1.75x + 25 & \text{if } 101 \le x \le 250 \\ 1.25x + 150 & \text{if } x \ge 251 \end{cases}$;

2a. \$340

2b. \$712.50

3. $T(x) = \begin{cases} 100x & \text{if } 1 \le x \le 3 \\ 90x + 30 & \text{if } 4 \le x \le 6 \\ 80x + 90 & \text{if } x \ge 6 \end{cases}$;

4. $T(x) = \begin{cases} 0.1x & \text{if } 0 < x \le 100{,}000 \\ 0.2x - 10{,}000 & \text{if } 100{,}000 < x \le 200{,}000 \\ 0.3x - 30{,}000 & \text{if } 200{,}000 < x \le 300{,}000 \\ 0.4x - 60{,}000 & \text{if } x > 300{,}000 \end{cases}$;

4a. \$20,000

4b. \$37,500

4c. \$340,000

5. $S(x) = \begin{cases} 0.04x + 35{,}000 & \text{if } 0 < x \le 500{,}000 \\ 0.06x + 25{,}000 & \text{if } x > 500{,}000 \end{cases}$;