Force Method for Analysis of Indeterminate Structures

(Ref: Chapter 10)

For determinate structures, the force method allows us to find internal forces (using equilibrium *i.e.* based on Statics) irrespective of the material information. Material (stress-strain) relationships are needed only to calculate deflections.

However, for indeterminate structures , Statics (equilibrium) alone is not sufficient to conduct structural analysis. Compatibility and material information are essential.

Indeterminate Structures

Number of unknown Reactions or Internal forces > Number of equilibrium equations
Note: Most structures in the real world are statically indeterminate.

Advantages	Disadvantages
• Smaller deflections for similar members • Redundancy in load carrying capacity (redistribution) • Increased stability	• More material => More Cost • Complex connections • Initial / Residual / Settlement Stresses

Methods of Analysis

Structural Analysis requires that the equations governing the following physical relationships be satisfied:

(i) Equilibrium of forces and moments
(ii) Compatibility of deformation among members and at supports
(iii) Material behavior relating stresses with strains
(iv) Strain-displacement relations
(v) Boundary Conditions

Primarily two types of methods of analysis:

Force (Flexibility) Method

- Convert the indeterminate structure to a determinate one by removing some unknown forces / support reactions and replacing them with (assumed) known / unit forces.
- Using superposition, calculate the force that would be required to achieve compatibility with the original structure.
- Unknowns to be solved for are usually redundant forces
- Coefficients of the unknowns in equations to be solved are "flexibility" coefficients.

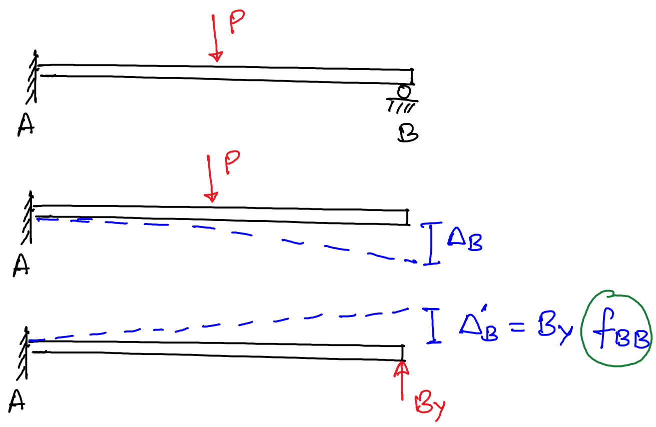
$[A]x = b$

Displacement (Stiffness) Method

- Express local (member) force-displacement relationships in terms of unknown member displacements.
- Using equilibrium of assembled members, find unknown displacements.
- Unknowns are usually displacements
- Coefficients of the unknowns are "Stiffness" coefficients.

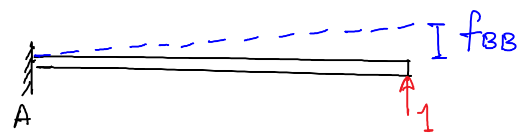
$[K]d = f$

Example:



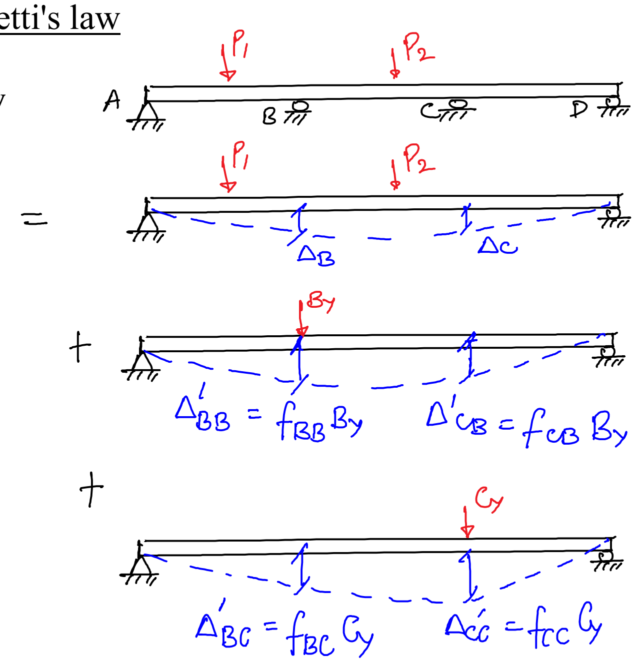
$\Delta_B + (-\Delta'_B) = 0$

$\Rightarrow \quad \Delta_B - \underbrace{[f_{BB}]}\,(B_y) = 0$



Maxwell's Theorem of Reciprocal displacements; Betti's law

For structures with multiple degree of indeterminacy



Compatibility:

$$\Delta_B + [f_{BB}]\, B_y + [f_{BC}]\, C_y = 0$$

$$\Delta_C + [f_{CB}]\, B_y + [f_{CC}]\, C_y = 0$$

$$\Rightarrow \begin{bmatrix} \Delta_B \\ \Delta_C \end{bmatrix} + \begin{bmatrix} f_{BB} & f_{BC} \\ f_{CB} & f_{CC} \end{bmatrix} \begin{bmatrix} B_y \\ C_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

displacement at B

Note: $\quad f_{BC} = \int_0^L \frac{m_B(x)\; m_C(x)}{EI}\, dx$

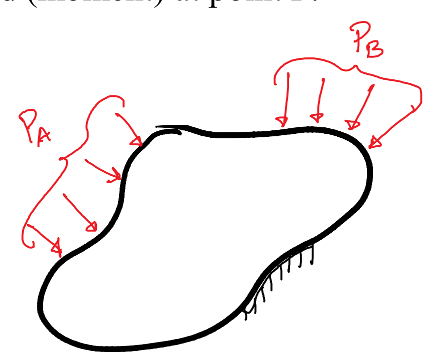
unit load @ C

$$f_{CB} = \int \frac{m_C(x)\; m_B(x)\, dx}{EI} \quad \Rightarrow \quad \boxed{f_{BC} = f_{CB}}$$

The displacement (rotation) at a point P in a structure due a UNIT load (moment) at point Q is equal to displacement (rotation) at a point Q in a structure due a UNIT load (moment) at point P.

Betti's Theorem

Virtual Work done by a system of forces $\boldsymbol{P}_B$ while undergoing displacements due to system of forces $\boldsymbol{P}_A$
is equal to the
Virtual Work done by the system of forces $\boldsymbol{P}_A$ while undergoing displacements due to the system of forces $\boldsymbol{P}_B$

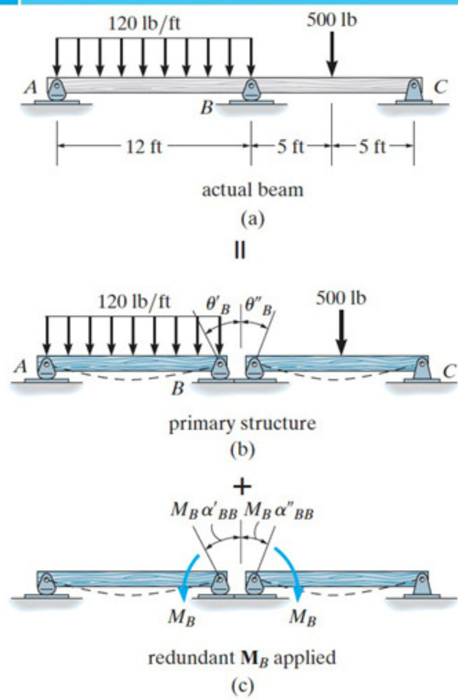


Force Method of Analysis for (Indeterminate) Beams and Frames

Example: Determine the reactions.

EXAMPLE 10.4 CONTINUED

$$\theta_B' + [\alpha_{BB}']M_B \qquad -\left[\theta_B'' + [\alpha_{BB}'']M_B\right]$$



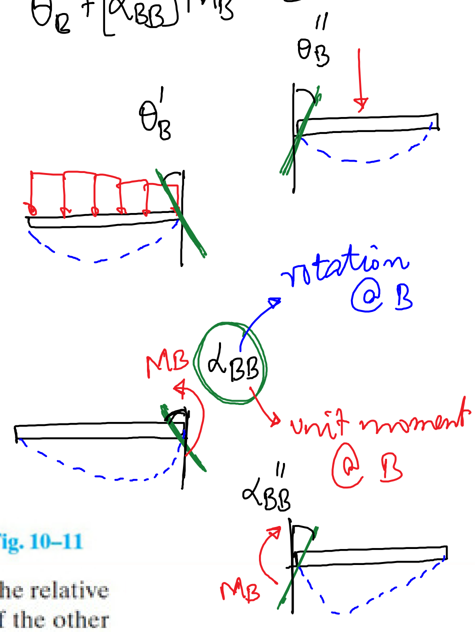


Fig. 10–11

Compatibility Equations. From Fig. 10–11*a* we require the relative rotation of one end of one beam with respect to the end of the other beam to be zero, that is,

$$(\curvearrowleft +) \qquad \theta_B + M_B\alpha_{BB} = 0$$

where

$$\theta_B = \theta_B' + \theta_B''$$

and

$$\alpha_{BB} = \alpha_{BB}' + \alpha_{BB}''$$

$$\theta_B' + [\alpha_{BB}']M_B = -\left[\theta_B'' + [\alpha_{BB}'']M_B\right]$$

$$\Rightarrow \quad (\theta_B' + \theta_B'') + [\alpha_{BB}' + \alpha_{BB}'']M_B = 0$$

EXAMPLE 10.4 CONTINUED

The slopes and angular flexibility coefficients can be determined from the table on the inside front cover, that is,

$$\theta_B' = \frac{wL^3}{24EI} = \frac{120(12)^3}{24EI} = \frac{8640\ \text{lb}\cdot\text{ft}^2}{EI}$$

$$\theta_B'' = \frac{PL^2}{16EI} = \frac{500(10)^2}{16EI} = \frac{3125\ \text{lb}\cdot\text{ft}^2}{EI}$$

$$\alpha_{BB}' = \frac{ML}{3EI} = \frac{1(12)}{3EI} = \frac{4\ \text{ft}}{EI}$$

$$\alpha_{BB}'' = \frac{ML}{3EI} = \frac{1(10)}{3EI} = \frac{3.33\ \text{ft}}{EI}$$

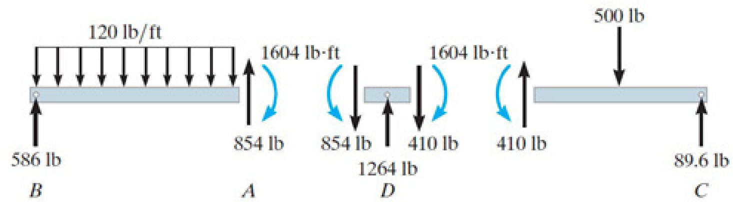
Thus

$$\frac{8640\ \text{lb}\cdot\text{ft}^2}{EI} + \frac{3125\ \text{lb}\cdot\text{ft}^2}{EI} + M_B\left(\frac{4\ \text{ft}}{EI} + \frac{3.33\ \text{ft}}{EI}\right) = 0$$

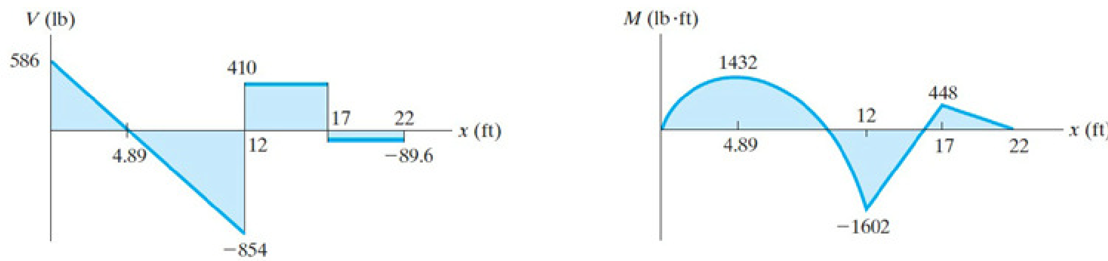
$$M_B = -1604\ \text{lb}\cdot\text{ft}$$

EXAMPLE 10.4 CONTINUED

The negative sign indicates M_B acts in the opposite direction to that shown in Fig. 10–11*c*. Using this result, the reactions at the supports are calculated as shown in Fig. 10–11*d*. Furthermore, the shear and moment diagrams are shown in Fig. 10–11*e*.



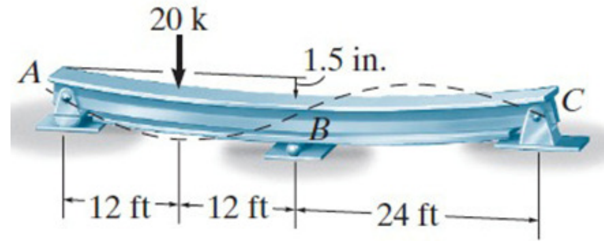
(d)



(e)

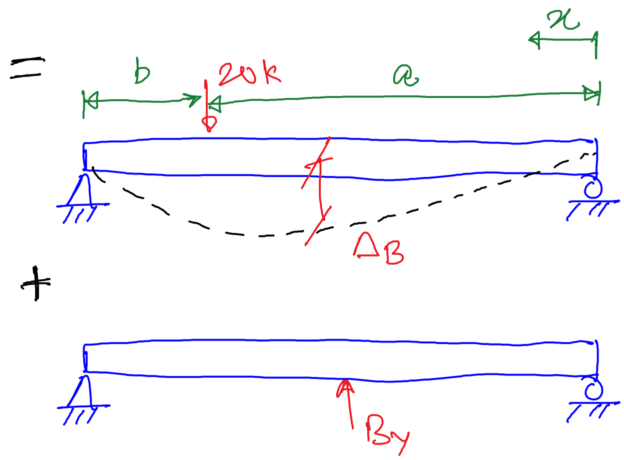
Examples

Support B settles by 1.5 in.
Find the reactions and draw the Shear Force and Bending Moment Diagrams of the beam.



$$(+\uparrow) \quad v(x) = -\frac{Pbx}{6L\,EI}\left(L^2 - b^2 - x^2\right)$$

=



$$\Delta_B = v(L/2) = -\frac{20 \times 12 \times 24\left(48^2 - 12^2 - 24^2\right)}{6 \times 48 \times EI}$$

$$= \frac{-31\,680}{EI} \quad K\text{-}ft^3$$

$$\Delta_B' = f_{BB}\,B_y = \left(\frac{L^3}{48EI}\right) B_y = \frac{2304}{EI}\,K\text{-}ft^3$$

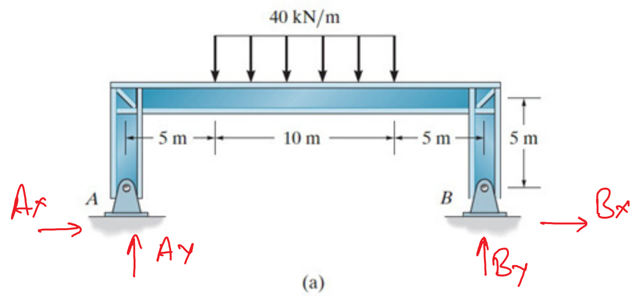
Compatibility equation: $\boxed{\Delta_B + \Delta_B' = -1.5}$

$$\Rightarrow \quad B_y = \frac{-1.5 - \Delta_B}{f_{BB}} \quad \Rightarrow \quad \underline{B_y = +5.56\ k}$$

Example: Frames

EXAMPLE 10.5

The frame, or bent, shown in the photo is used to support the bridge deck. Assuming EI is constant, a drawing of it along with the dimensions and loading is shown in Fig. 10–12*a*. Determine the support reactions.



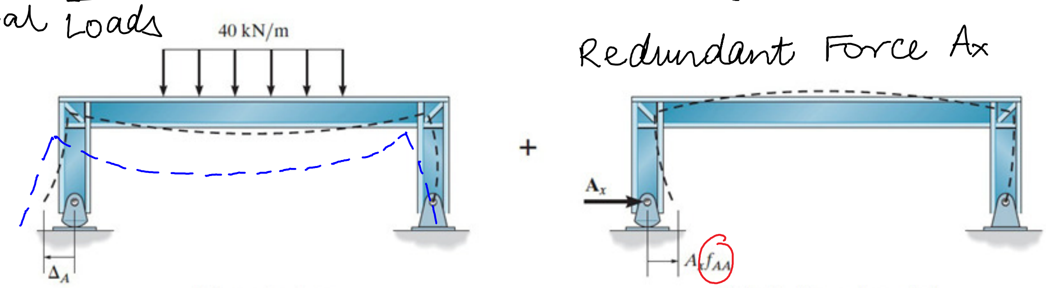
(a)

Fig. 10–12

Indeterminate Degree = 1.

PART-1: Actual Loads

PART-2: Redundant Force Ax



Primary structure

Redundant force $\mathbf{A}_x$ applied

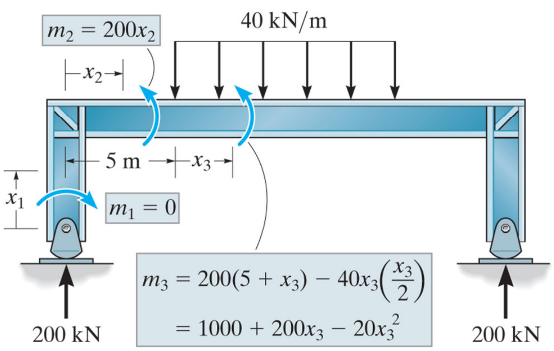
(b)

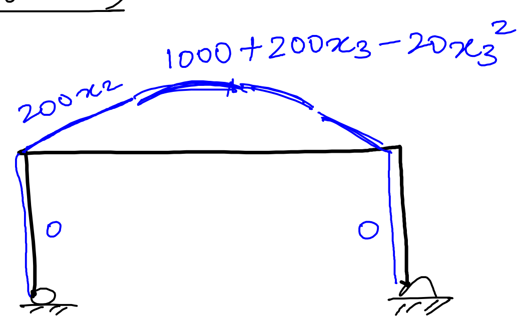
Compatibility Equation. Reference to point A in Fig. 10–12*b* requires

$$(\overset{+}{\rightarrow}) \qquad \boxed{0 = \Delta_A + A_x f_{AA}} \qquad (1)$$

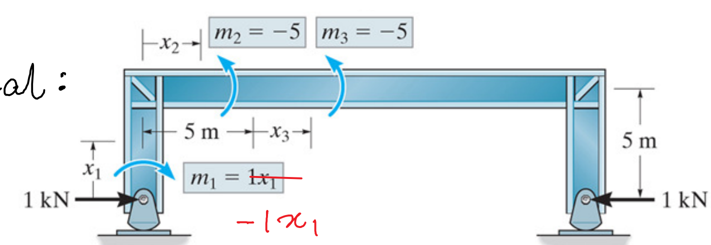
PART-1 (Δ_A Deflection under Actual loads)

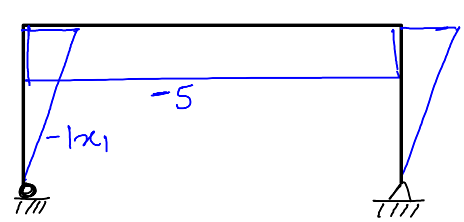
1(a) Real:





1(b) Virtual:





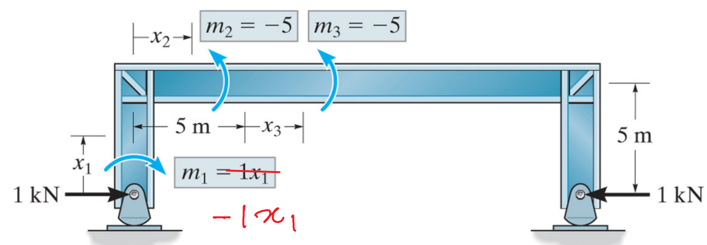
Principle of virtual work $\Rightarrow$

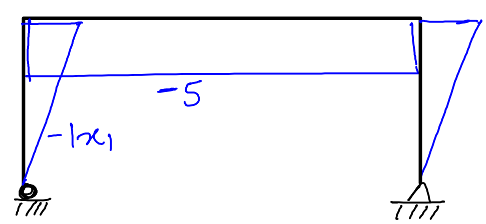
$d\theta/dx$ $\quad v$

$$\Delta_A = \int_0^L \frac{Mm}{EI}dx = 2\int_0^5 \frac{(0)(1x_1)dx_1}{EI} + 2\int_0^5 \frac{(200x_2)(-5)dx_2}{EI}$$
$$+ 2\int_0^5 \frac{(1000 + 200x_3 - 20x_3^2)(-5)dx_3}{EI}$$
$$= 0 - \frac{25000}{EI} - \frac{66\,666.7}{EI} = \boxed{-\frac{91\,666.7}{EI}}$$

PART-2: (f_{AA} Deflection under Redundant force):

2 a) Real:

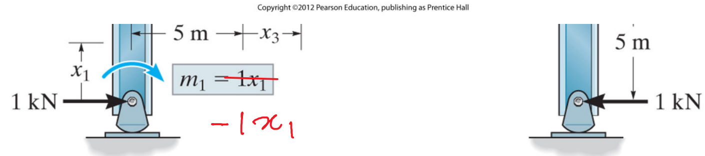


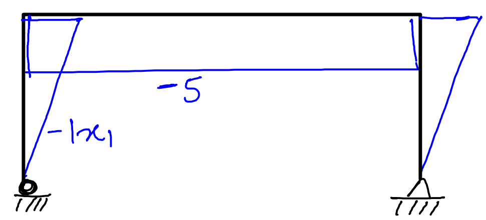


(d)

Figure: 10_12dEX05

2(b) Virtual:





EXAMPLE 10.5 CONTINUED

For f_{AA} we require application of a real unit load and a virtual unit load acting at A, Fig. 10–12d. Thus,

Principle of virtual work $\Rightarrow$

$d\theta/dx$ $\quad v$

$$f_{AA} = \int_0^L \frac{mm}{EI}dx = 2\int_0^5 \frac{(1x_1)^2 dx_1}{EI} + 2\int_0^5 (5)^2 dx_2 + 2\int_0^5 (5)^2 dx_3$$
$$= \frac{583.33}{EI}$$

Substituting the results into Eq. (1) and solving yields

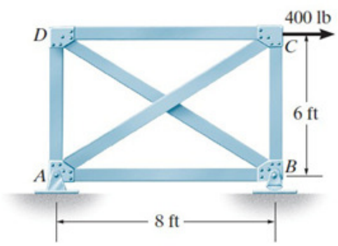
$$\boxed{\Delta_A + f_{AA} A_x = 0}$$

$$0 = \frac{-91\,666.7}{EI} + A_x\left(\frac{583.33}{EI}\right)$$

$$\boxed{A_x = 157 \text{ kN}}$$ *Ans.*

Force Method of Analysis for (Indeterminate) Trusses

EXAMPLE 10.7



(a)

Fig. 10–14

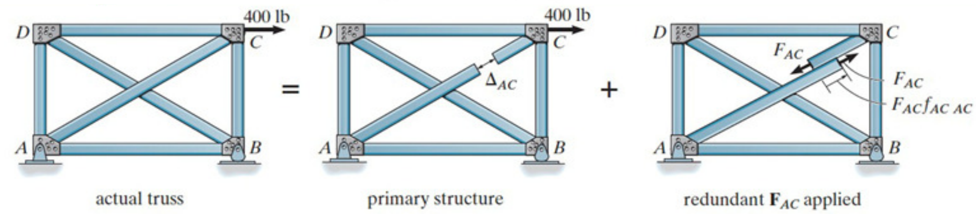
Determine the force in member AC of the truss shown in Fig. 10–14*a*. AE is the same for all the members.

SOLUTION

Principle of Superposition. By inspection the truss is indeterminate to the first degree.* Since the force in member AC is to be determined, member AC will be chosen as the redundant. This requires "cutting" this member so that it cannot sustain a force, thereby making the truss statically determinate and stable. The principle of superposition applied to the truss is shown in Fig. 10–14*b*.

Compatibility Equation. With reference to member AC in Fig. 10–14*b*, we require the relative displacement Δ_{AC}, which occurs at the ends of the cut member AC due to the 400-lb load, plus the relative displacement $F_{AC} f_{AC\,AC}$ caused by the redundant force acting alone, to be equal to zero, that is,

$$\boxed{0 = \Delta_{AC} + F_{AC} f_{AC\,AC}} \qquad (1)$$



(b)

*Applying Eq. 3–1, $b + r > 2j$ or $6 + 3 > 2(4)$, $9 > 8$, $9 - 8 = 1$st degree.

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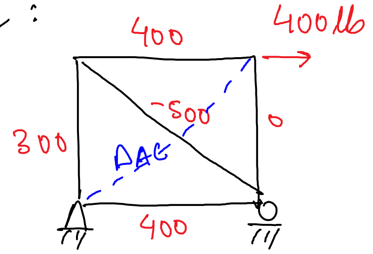
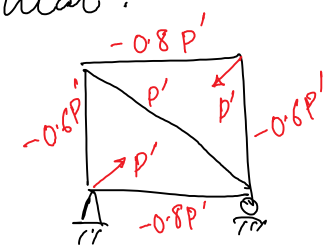
PART-1 (Δ_{AC} Relative Deflection of A & C due to Actual loads)

1(a) Real:



1(b) Virtual:



Principle of virtual work:

$$W'_E = U_I$$

$$P' \Delta_{AC} = \sum_{m=1}^{M} N' \left(\frac{NL}{AE}\right)$$

$$\Delta_{AC} = \sum \frac{nNL}{AE}$$

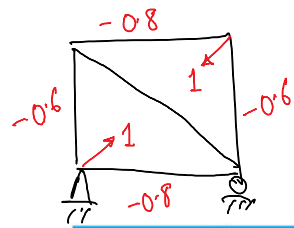
$$= 2\left[\frac{(-0.8)(400)(8)}{AE}\right] + \frac{(-0.6)(0)(6)}{AE} + \frac{(-0.6)(300)(6)}{AE}$$

$$+ \frac{(1)(-500)(10)}{AE} + \frac{(1)(0)(10)}{AE}$$

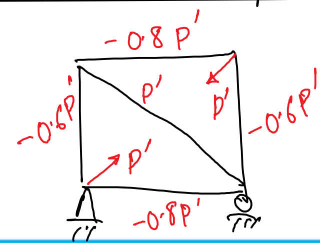
$$= -\frac{11\,200}{AE}$$

PART - 2 ($f_{AC\,AC}$ Relative Deflection of A & C due to Redundant force)

2 a) Real:



2b) Virtual:



Principle of Virtual Work

$W_E' = U_I'$

EXAMPLE 10.7 CONTINUED

$$f_{AC\,AC} = \sum \frac{n^2 L}{AE}$$

$$= 2\left[\frac{(-0.8)^2(8)}{AE}\right] + 2\left[\frac{(-0.6)^2(6)}{AE}\right] + 2\left[\frac{(1)^2 10}{AE}\right]$$

$$= \frac{34.56}{AE}$$

Substituting the data into Eq. (1) and solving yields

$$0 = -\frac{11\,200}{AE} + \frac{34.56}{AE} F_{AC}$$

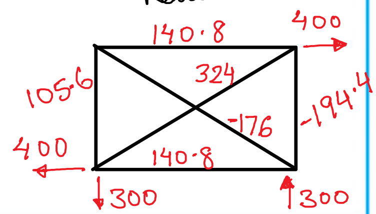
$$F_{AC} = 324 \text{ lb (T)} \qquad \textit{Ans.}$$

Since the numerical result is positive, *AC* is subjected to tension as assumed, Fig. 10–14b. Using this result, the forces in the other members can be found by equilibrium, using the method of joints.

Compatibility:

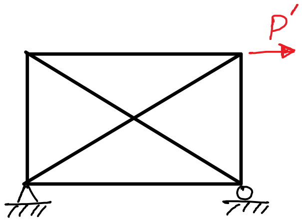
$$\underbrace{\Delta_{AC}}_{\text{Real loads}} + \underbrace{[f_{ACAC}](F_{AC})}_{\text{Redundant}} = 0$$

$\Rightarrow$

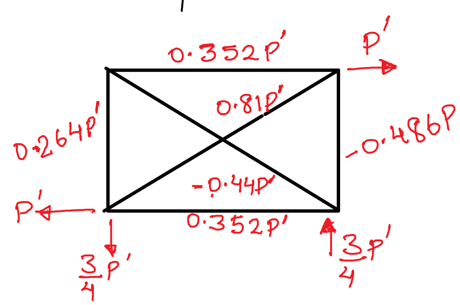


Aside: If we also want to find the actual horizontal displacement of C, Then we can use the method of virtual work:

In addition to the above "real" problem, we solve the following "virtual" problem:



(By scaling the response by $\frac{P'}{400}$)



Using the principle of virtual work:

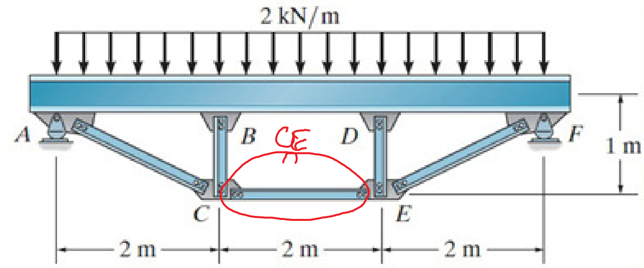
$$\Delta_c = \sum_{m=1}^{M} N' \frac{NL}{AE} = \frac{1}{AE}\Big[0.352 \times 140.8 \times 8 + 0.264 \times 105.6 \times 6 + 0.81 \times 324 \times 10 + (-0.44) \times (-176) \times 10 + (-0.486) \times (-194.4) \times 6 + 0.352 \times 140.8 \times 8\Big]$$

$$= \boxed{\frac{4925.93}{EA}}$$

Force Method of Analysis for (Inderminate) Composite Structures

EXAMPLE 10.9

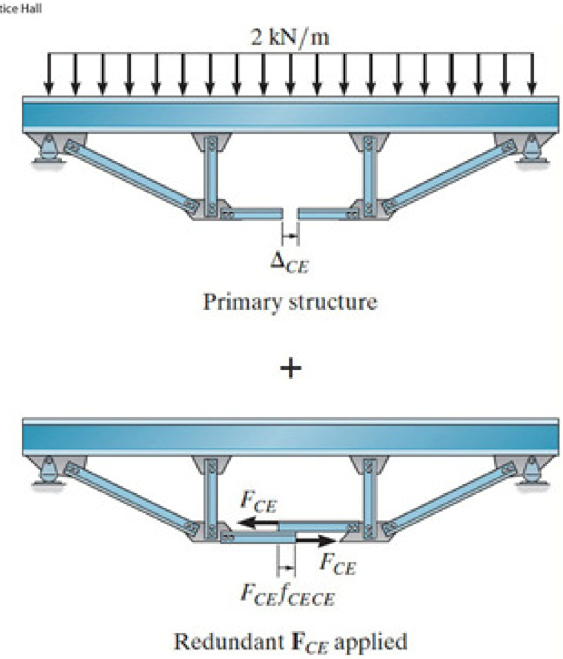
The simply supported queen-post trussed beam shown in the photo is to be designed to support a uniform load of 2 kN/m. The dimensions of the structure are shown in Fig. 10–16*a*. Determine the force developed in member *CE*. Neglect the thickness of the beam and assume the truss members are pin connected to the beam. Also, neglect the effect of axial compression and shear in the beam. The cross-sectional area of each strut is 400 mm^2, and for the beam $I = 20(10^6)$ mm^4. Take $E = 200$ GPa.



Actual structure

(a)

Fig. 10–16



(b)

SOLUTION

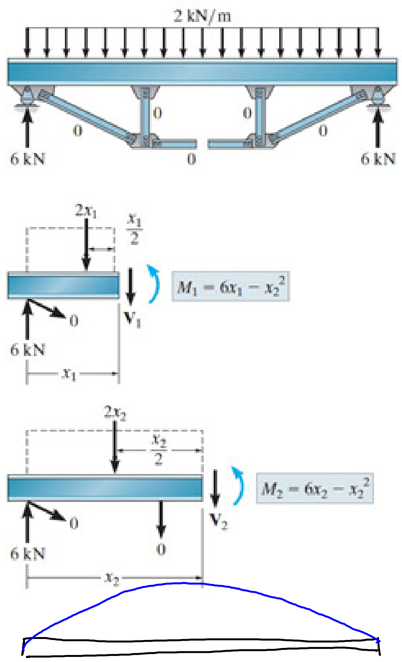
Principle of Superposition. If the force in one of the truss members is known, then the force in all the other members, as well as in the beam, can be determined by statics. Hence, the structure is indeterminate to the first degree. For solution the force in member *CE* is chosen as the redundant. This member is therefore sectioned to eliminate its capacity to sustain a force. The principle of superposition applied to the structure is shown in Fig. 10–16*b*.

Compatibility Equation. With reference to the relative displacement of the cut ends of member *CE*, Fig. 10–16*b*, we require

$$0 = \Delta_{CE} + F_{CE} f_{CE\,CE} \qquad (1)$$

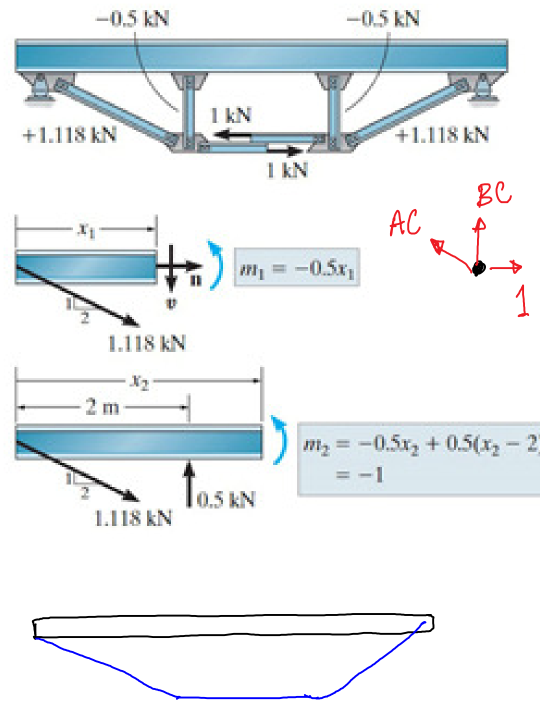
Part 1

Displacement Δ_{CE}
Real loads



Part 2

flexibility coeff
f_{CECE}



$$\Delta_{CE} = f_{CECE}\, F_{CE}$$

$d\theta/dx$ m

$$\Delta_{CE} = \int_0^L \frac{Mm}{EI}dx + \sum \frac{nNL}{AE} = 2\int_0^2 \frac{(6x_1 - x_1^2)(-0.5x_1)dx_1}{EI}$$

$$+ 2\int_2^3 \frac{(6x_2 - x_2^2)(-1)dx_2}{EI} + 2\left(\frac{(1.118)(0)(\sqrt{5})}{AE}\right)$$

$$+ 2\left(\frac{(-0.5)(0)(1)}{AE}\right) + \left(\frac{1(0)2}{AE}\right)$$

$$= -\frac{12}{EI} - \frac{17.33}{EI} + 0 + 0 + 0$$

$$= \frac{-29.33(10^3)}{200(10^9)(20)(10^{-6})} = \boxed{-7.333(10^{-3})\text{ m}}$$

$\left(\frac{m}{EI}\right)$ m

$$f_{CECE} = \int_0^L \frac{m^2dx}{EI} + \sum \frac{n^2L}{AE} = 2\int_0^2 \frac{(-0.5x_1)^2dx_1}{EI} + 2\int_2^3 \frac{(-1)^2dx_2}{EI}$$

$$+ 2\left(\frac{(1.118)^2(\sqrt{5})}{AE}\right) + 2\left(\frac{(-0.5)^2(1)}{AE}\right) + \left(\frac{(1)^2(2)}{AE}\right)$$

$$= \frac{1.3333}{EI} + \frac{2}{EI} + \frac{5.590}{AE} + \frac{0.5}{AE} + \frac{2}{AE}$$

$$= \frac{3.333(10^3)}{200(10^9)(20)(10^{-6})} + \frac{8.090(10^3)}{400(10^{-6})(200(10^9))}$$

$$= \boxed{0.9345(10^{-3})\text{ m/kN}}$$

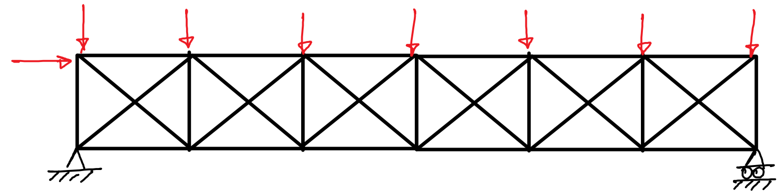
Substituting the data into Eq. (1) yields

$$0 = -7.333(10^{-3})\text{ m} + F_{CE}(0.9345(10^{-3})\text{ m/kN})$$

$$F_{CE} = 7.85\text{ kN} \qquad \textit{Ans.}$$

Systematic Analysis using the Force (Flexibility) Method

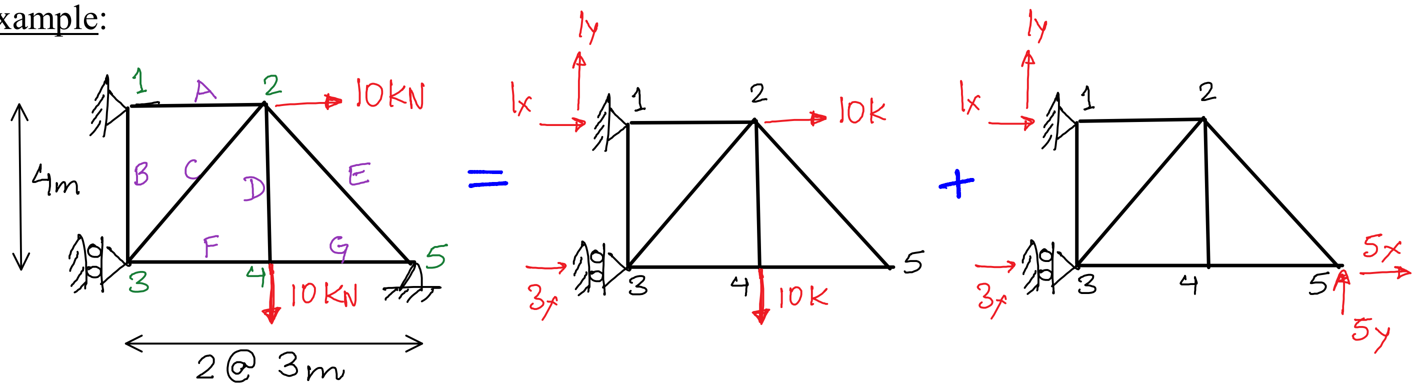
For structures with large number of redundant unknowns



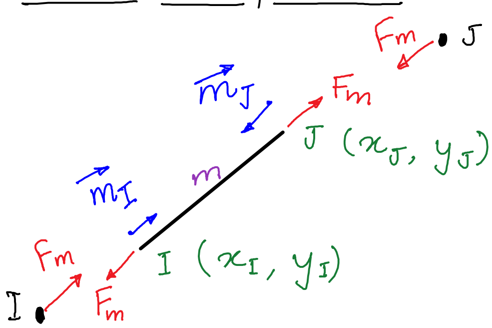
$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix} = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix}$$

Note: Maxwell's Theorem (Betti's Law)
=> Flexibility matrix is symmetric!

Example:



Member Properties : For any member "m" connected nodes I and J



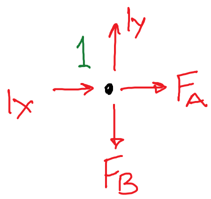
$$\vec{m}_I = \frac{(x_J - x_I)}{\ell}\hat{i} + \frac{(y_J - y_I)}{\ell}\hat{j} = m_{Ix}\hat{i} + m_{Iy}\hat{j}$$

$$\vec{m}_J = \frac{(x_I - x_J)}{\ell}\hat{i} + \frac{(y_I - y_J)}{\ell}\hat{j} = m_{Jx}\hat{i} + m_{Jy}\hat{j}$$

$$\ell = \sqrt{(x_J - x_I)^2 + (y_J - y_I)^2}$$

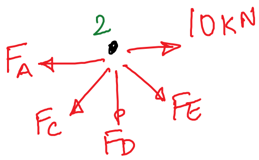
m	I	J	m_{Ix} $(-m_{Jx})$	m_{Iy} $(-m_{Jy})$
A	1	2	1	0
B	1	3	0	-1
C	3	2	3/5	4/5
D	2	4	0	-1
E	2	5	3/5	-4/5
F	3	4	1	0
G	4	5	1	0

Equilibrium of nodes



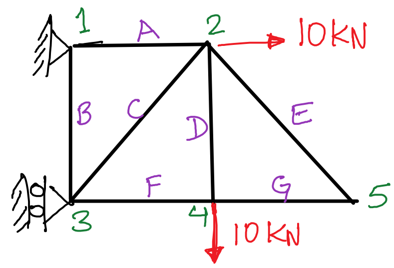
$\Rightarrow$ $\sum f_x = 0 \Rightarrow \quad 1_x + A_{1x} F_A + B_{1x} F_B = 0$

$\sum f_y = 0 \Rightarrow \quad 1_y + A_{1y} F_A + B_{1y} F_B = 0$



$\Rightarrow$ $\sum f_x = 0 \Rightarrow \quad A_{2x} F_A + C_{2x} F_C + D_{2x} F_D + E_{2x} F_E + 10 = 0$

$\sum F_y = 0 \Rightarrow \quad A_{2y} F_A + C_{2y} F_C + D_{2y} F_D + E_{2y} F_E = 0$



Similarly for all the other nodes:

$$
\begin{array}{cc|cccccccccc}
 & & \multicolumn{7}{c}{M} & \multicolumn{3}{c}{r} \\
 & & A & B & C & D & E & F & G & 1_x & 1_y & 3_x \\
1 & 1 & 1 & 0 & & & & & & 1 & & \\
 & 2 & 0 & -1 & & & & & & & 1 & \\
2 & 3 & -1 & & -3/5 & 0 & 3/5 & & & & & \\
 & 4 & 0 & & -4/5 & -1 & -4/5 & & & & & \\
3 & 5 & & 0 & 3/5 & & & 1 & & & & 1 \\
 & 6 & & 1 & 4/5 & & & 0 & & & & \\
4 & 7 & & & & 0 & & -1 & 1 & & & \\
 & 8 & & & & 1 & & 0 & 0 & & & \\
5 & 9 & & & & & -3/5 & & -1 & & & \\
 & 10 & & & & & 4/5 & & 0 & & & \\
N & & & & & & & & & & &
\end{array}
\begin{Bmatrix} F_A \\ F_B \\ F_C \\ F_D \\ F_E \\ F_F \\ F_G \\ 1_x \\ 1_y \\ 3_x \end{Bmatrix}
+
\begin{Bmatrix} 0 \\ 0 \\ 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10 \\ 0 \\ 0 \end{Bmatrix}
= \underline{0}
$$

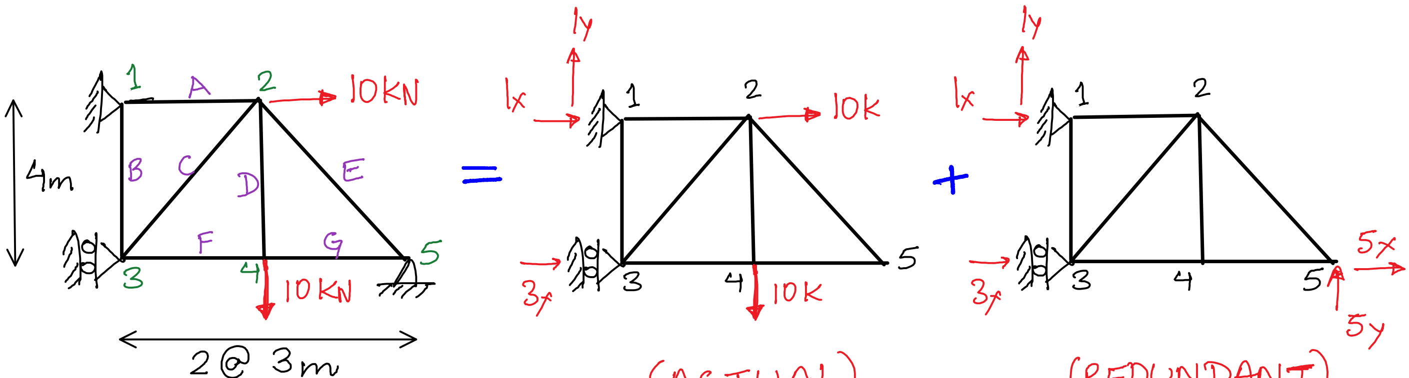
$$[A]_{2N \times 2N} \qquad \{N\}_{2N\times 1} + \{f\}_{2N\times 1} = \{0\}$$

i.e.

$$\boxed{[A]\ \{N\} = -\{f\}}$$

Compatibility equations for original indeterminate truss:

$$N_d \begin{cases} \Delta_{5x} = 0 \quad \Rightarrow \quad \Delta'_{5x\,\text{ACTUAL}} + \Delta''_{5x\,\text{REDUNDANT}} = 0 \\ \Delta_{5y} = 0 \quad \Rightarrow \quad \Delta'_{5y\,\text{ACTUAL}} + \Delta''_{5y\,\text{REDUNDANT}} = 0 \end{cases}$$



(ACTUAL) (REDUNDANT)

$[A]\{N_A\} = -\{f_A\}$

$$\boxed{\begin{aligned} [A]\{N_{R_1}\} &= -\{f_{R_1}\} \\ [A]\{N_{R_2}\} &= -\{f_{R_2}\} \\ &\vdots \\ [A]\{N_{R_N}\} &= -\{f_{R_N}\} \end{aligned}}$$

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}_{2N\times 2N} \overbrace{\underbrace{\begin{bmatrix} N_{R1} & N_{R2} & \cdots & N_{RN} \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{2N\times N_d}}^{[NR]} = - \overbrace{\underbrace{\begin{bmatrix} 0 & 0 & \cdots \\ \vdots & \vdots & \\ 1 & 0 & \\ 0 & 1 & \cdots \end{bmatrix}}_{2N\times N_d}}^{[Rd]} \overbrace{\underbrace{\begin{bmatrix} f_{R1} & & \\ & f_{R2} & \\ & & \ddots \;\; f_{RN} \end{bmatrix}}_{N_d\times N_d}}^{[fR]}$$

$$\underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}_{10\times 10} \underbrace{\begin{bmatrix} 0 & -1.5000 \\ 0 & -1.0000 \\ 0 & 1.2500 \\ 0 & 0 \\ 0 & -1.2500 \\ 1.0000 & 0.7500 \\ 1.0000 & 0.7500 \\ 0 & 1.5000 \\ 0 & -1.0000 \\ -1.0000 & -1.5000 \end{bmatrix}}_{10\times 2} = - \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{10\times 2} \underbrace{\begin{bmatrix} 5x & \\ & 5y \end{bmatrix}}_{2\times 2}$$

Using the Principle of virtual work:

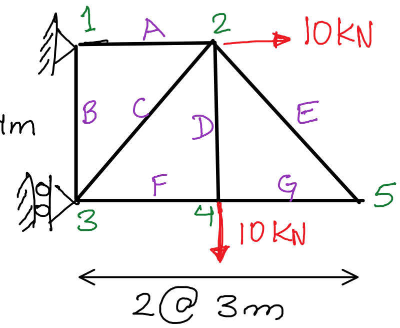
$$\underset{1\times Nd}{\Delta_A} = \sum_{m=1}^{M} \boxed{N_{R1_m}} \boxed{N_{A_m} \frac{L_m}{A_m E_m}} = \underset{1\times M}{\{N_A\}^T} \underset{M\times M}{[F_m]} \underset{M\times Nd}{\{N_{R1}\}}$$

$$\underset{Nd\times Nd}{\Delta_R} = \sum_{m=1}^{M} \boxed{N_{R1}} \boxed{N_{R1} \frac{L_m}{A_m E_m}} = \underset{Nd\times M}{\{N_{R1}\}^T} \underset{M\times M}{[F_m]} \underset{M\times Nd}{\{N_{R1}\}}$$

where

$$[F_m] = \frac{1}{AE} \begin{array}{c|ccccccc} & A & B & C & D & E & F & G \\ \hline A & 3 & & & & & & \\ B & & 4 & & & & & \\ C & & & 5 & & & & \\ D & & & & 4 & & & \\ E & & & & & 5 & & \\ F & & & & & & 3 & \\ G & & & & & & & 3 \end{array}_{\substack{7\times 7 \\ (M\times M)}}$$

global (unassembled) flexibility matrix.

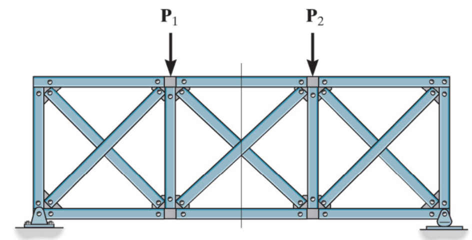


Thus from the compatibility equation:

$$\underset{Nd\times 1}{\{\Delta_A\}^T} + \underset{Nd\times Nd}{[\Delta_R]^T} \underset{Nd\times 1}{\{diag(fR)\}} = \underline{0}$$

$$\frac{1}{AE}\begin{Bmatrix} 0 \\ -196.8750 \end{Bmatrix}_{2\times 1} + \frac{1}{AE}\begin{bmatrix} 6.0000 & 4.5000 \\ 4.5000 & 29.7500 \end{bmatrix}_{2\times 2} \begin{Bmatrix} 5_x \\ 5_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \boxed{\begin{Bmatrix} 5_x \\ 5_y \end{Bmatrix} = \begin{Bmatrix} -5.5983 \\ 7.4645 \end{Bmatrix} \text{kN}}$$



axis of symmetry

(a)

Figure: 10_17a

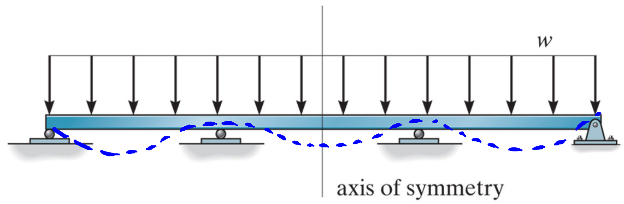
Analysis of Symmetric structures

Symmetry: Structure, Boundary Conditions, and Loads are symmetric.

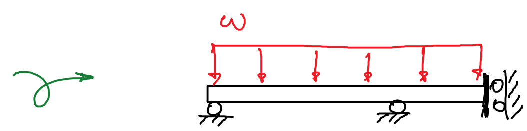
Anti-symmetric: Structure, Boundary Conditions are symmetric, Loads are anti-symmetric.

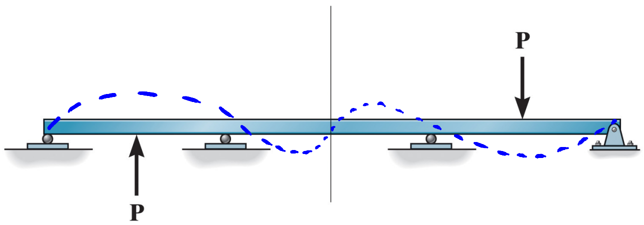
Symmetry helps in reducing the number of unknowns to solve for.

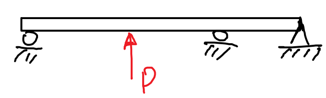
Examples:

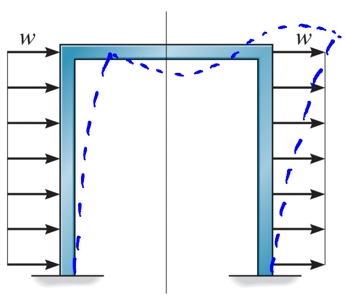


(b)



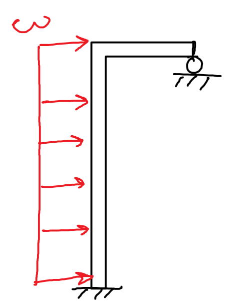


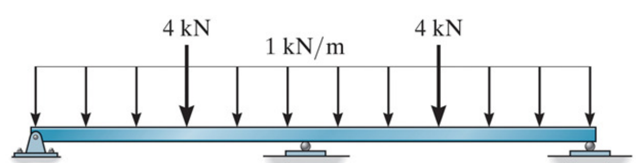




Antisymmetric loading

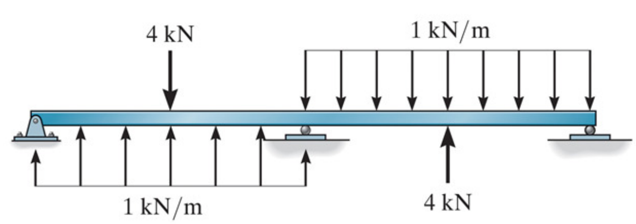
Figure: 10_18





symmetric loading

+



antisymmetric loading

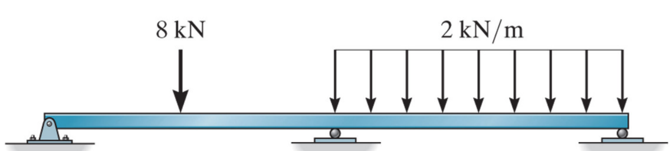


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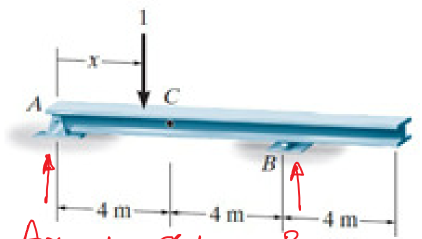
Figure 10_19a

Influence lines for *Determinate* structures

(Ref: Chapter 6)

Influence line is a diagram that shows the variation for a particular force/moment at specific location in a structure as a unit load moves across the entire structure.



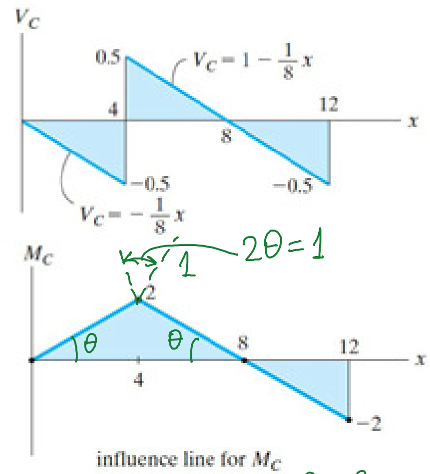
$A_y = 1 - x/8$ $\quad B_y$

$\sum M_B = 0$

$\Rightarrow A_y \times 8 = 1(8-x)$

$$V_C = \begin{cases} -x/8 & 0<x<4 \\ 1 - x/8 & 4<x<12 \end{cases}$$

$$M_C = \begin{cases} \left(1-\frac{x}{8}\right)4 - 1(8-x) = (x/2) & 0<x<4 \\ \left(1-\frac{x}{8}\right)4 = \left(4-\frac{x}{2}\right) & 4<x<12 \end{cases}$$

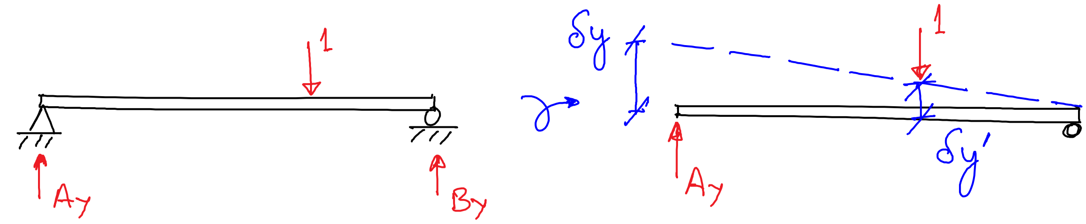


$\theta = \frac{2}{4}$

Müller-Breslau Principle

The influence of a certain force (or moment) in a structure is given by (*i.e.* it is equal to) the deflected shape of the structure in the absence of that force (or moment) and when given a corresponding unit displacement (or rotation).

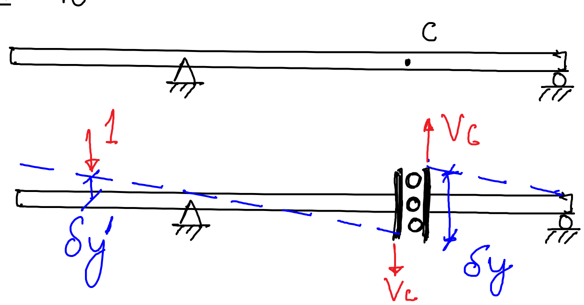
Example:



Principle of virtual work $\Rightarrow A_y\,\delta y - 1\,\delta y' = 0$

$\Rightarrow$ For influence line of A_y choose $\delta y = 1$ $\quad \Rightarrow \boxed{A_y = \delta y'}$

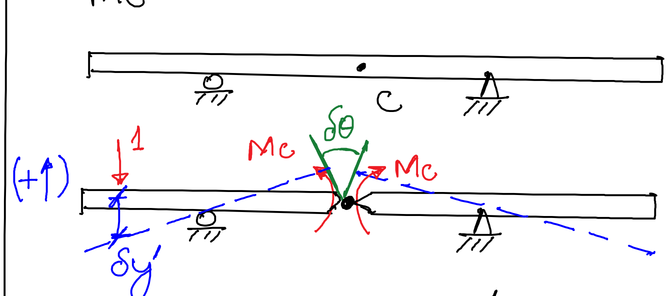
Examples: V_C



$V_C\,\delta y - 1\,\delta y' = 0$

$\delta y = 1 \Rightarrow \boxed{V_C = \delta y'}$

M_C



$M_C\,\delta\theta - 1(+\delta y') = 0$

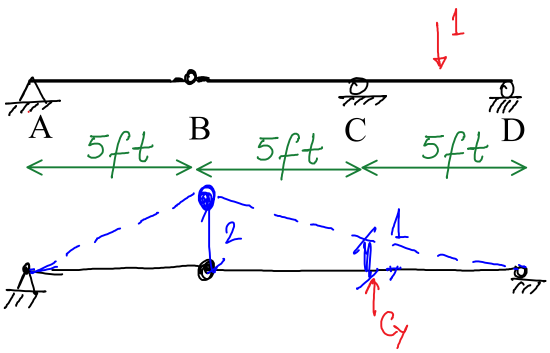
$\delta\theta = 1 \Rightarrow \boxed{M_C = \delta y'}$

Example

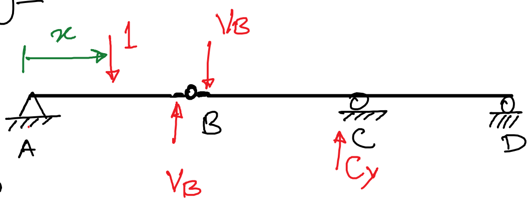
Draw the influence lines for the reaction and bending-moment at point C for the following beam.

Using the Müller-Breslau principle:

Reaction at C:



Verify:

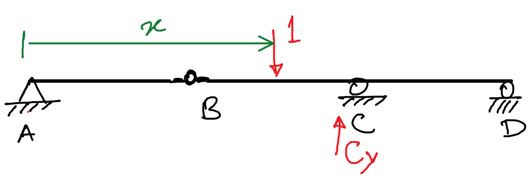


$\sum M_A = 0$

$\Rightarrow (V_B \times 5) - 1 \times x = 0 \Rightarrow \underline{V_B = x/5}$

$\sum M_D = 0$

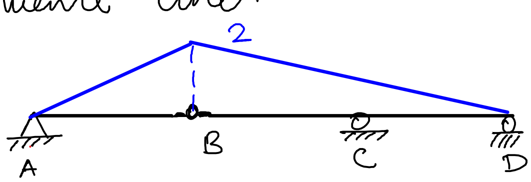
$\Rightarrow (V_B \times 10) - (C_y \times 5) = 0 \Rightarrow \underline{C_y = \frac{2x}{5}}$



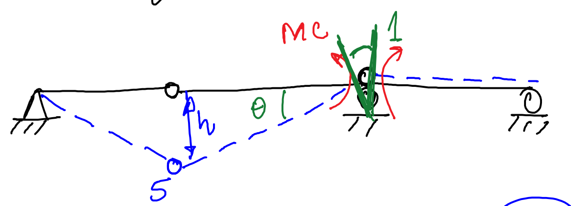
$\sum M_D = 0$

$\Rightarrow 1 \times (15 - x) - C_y \times 5 = 0 \Rightarrow \underline{C_y = 3 - x/5}$

$\Rightarrow$ Influence line:

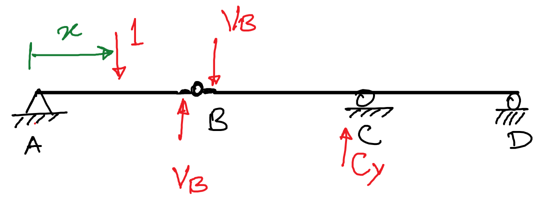


Bending Moment at C:

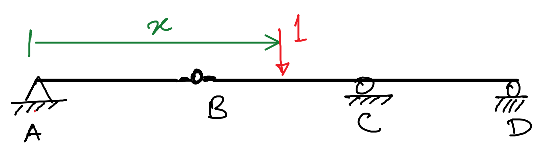


$\theta = 1 = \frac{h}{5} \Rightarrow h = 5$

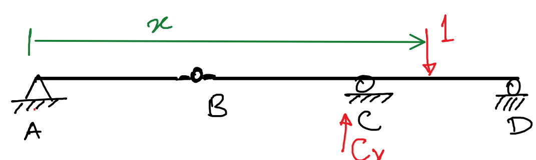
Verify:



$M_C = -V_B \times 5 = -x$

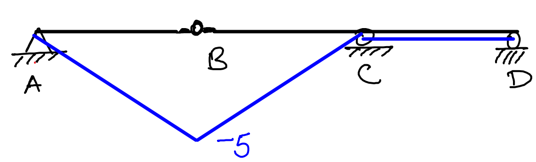


$M_C = -1(10 - x) = x - 10$



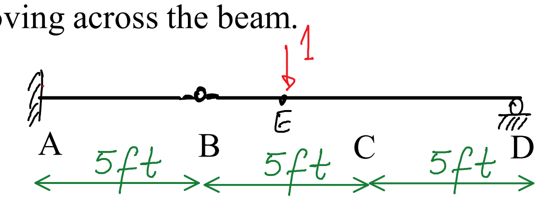
$M_C = 0$

$\Rightarrow$ Influence line:



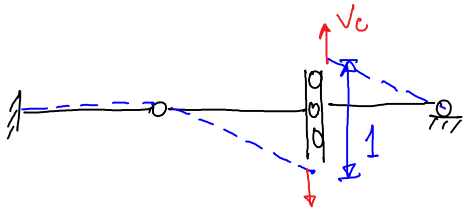
Example

- Draw the influence lines for the shear-force and bending-moment at point C for the following beam.
- Find the maximum bending moment at C due to a 400 lb load moving across the beam.

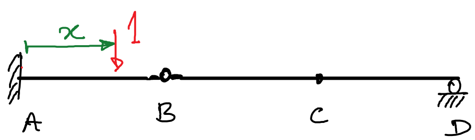


Using Müller - Breslau principle:

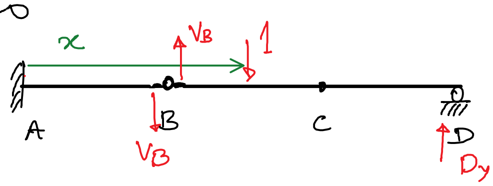
V_C:



Verify:



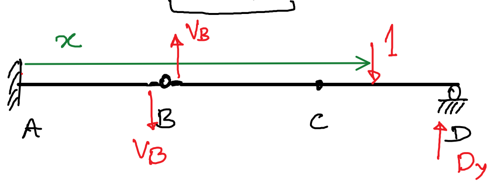
$V_C = 0$



$\sum M_B = 0$

$\Rightarrow \; -1 \times (x-5) + D_y \times 10 \Rightarrow D_y = \frac{x}{10} - \frac{1}{2}$

$V_C = -D_y = \underbrace{\frac{1}{2} - \frac{x}{10}}$

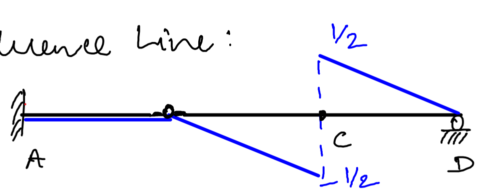


$\sum M_D = 0$

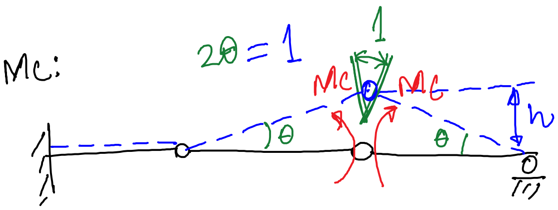
$\Rightarrow \; 1(15-x) - V_B \times 10 = 0 \Rightarrow V_B = \frac{3}{2} - \frac{x}{10}$

$V_C = V_B = \underbrace{\frac{3}{2} - \frac{x}{10}}$

$\Rightarrow$ Influence Line:



M_C:

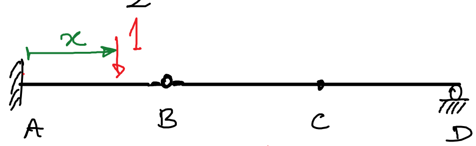


$\theta = \frac{h}{5} \Rightarrow h = 5\theta$

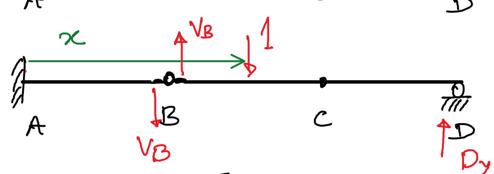
$h = \frac{5}{2}$

$M_{C\,max} = \frac{5}{2} \times 400$

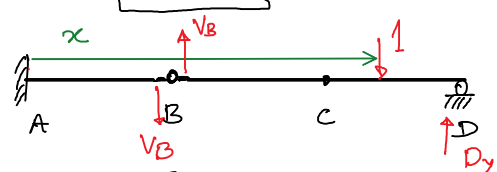
Verify:



$M_C = 0$

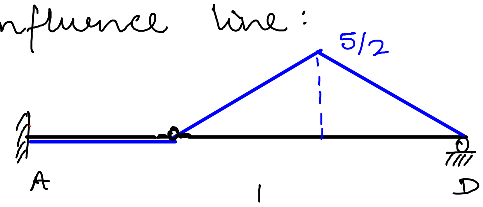


$M_C = D_y \times 5 = \underbrace{\frac{x}{2} - \frac{5}{2}}$



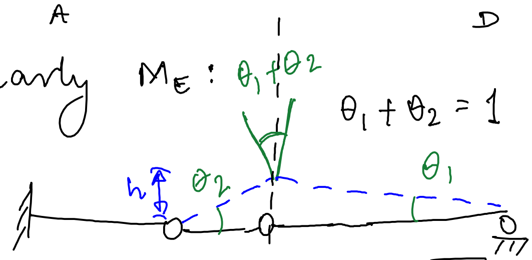
$M_C = V_B \times 5 = \underbrace{\frac{15}{2} - \frac{x}{2}}$

$\Rightarrow$ Influence line:



Similarly M_E: $\theta_1 + \theta_2$

$\theta_1 + \theta_2 = 1$



$\frac{h}{7.5} + \frac{3h}{3 \times 2.5} = 1 \Rightarrow h = \frac{7.5}{4} = \boxed{\frac{15}{8}}$

Influence lines for *Indeterminate* structures

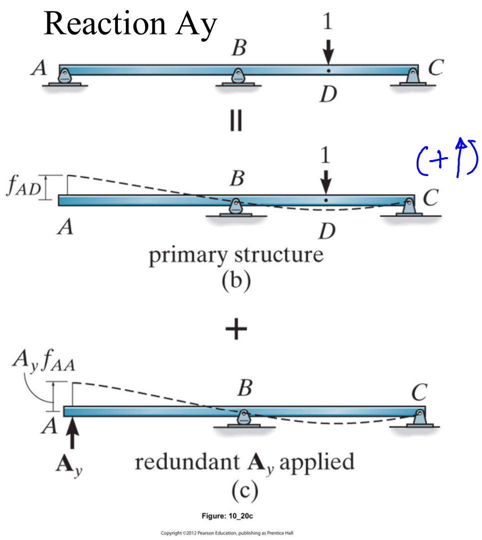
The Müller-Breslau principle also holds for indeterminate structures.

For statically determinate structures, influence lines are straight.

For statically indeterminate structures, influence lines are usually curved.

Examples:

Reaction Ay

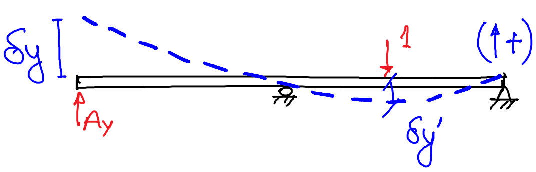


Note: compatibility

$\Rightarrow (-1) f_{AD} + A_y f_{AA} = 0$

$\Rightarrow \boxed{A_y = \dfrac{f_{AD}}{f_{AA}}} = \dfrac{f_{DA}}{f_{AA}}$

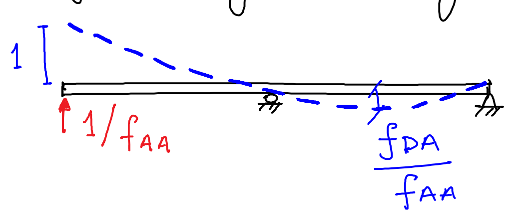
Alternatively, using Müller-Breslau principle:



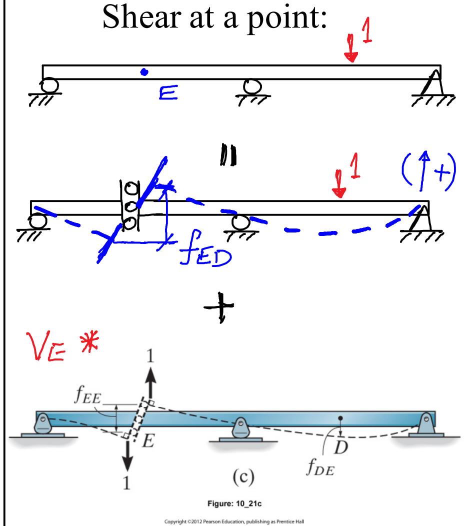
PVW $\Rightarrow A_y \delta y - 1 \cdot \delta y' = 0$

choose $\delta y = 1 \Rightarrow \boxed{A_y = \delta y'}$

To find $\delta y'$ when $\delta y = 1$



Shear at a point:

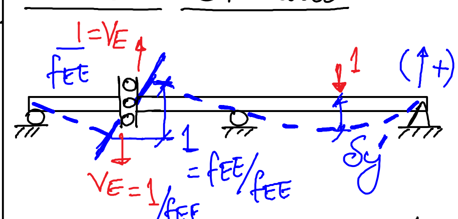


Compatibility

$\Rightarrow (-1) f_{ED} + V_E f_{EE} = 0$

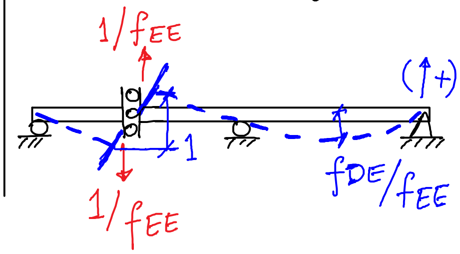
$\Rightarrow \boxed{V_E = \dfrac{f_{ED}}{f_{EE}}}$

Müller-Breslau:

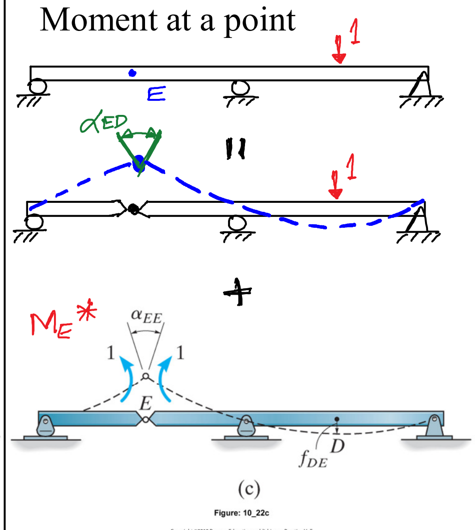


PVW $\Rightarrow V_E \times 1 = 1\, \delta y'$

$\boxed{V_E = \delta y'}$



Moment at a point

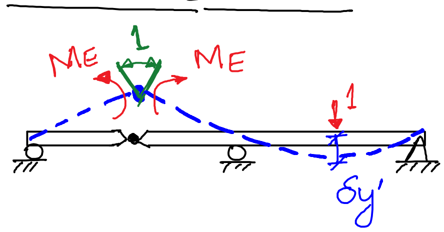


Compatibility

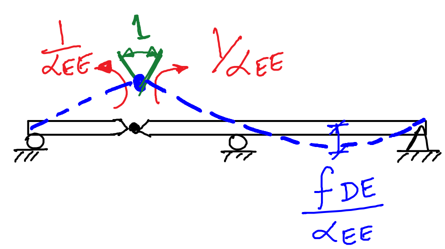
$\Rightarrow (-1) \alpha_{ED} + M_E \alpha_{EE} = 0$

$\Rightarrow \boxed{M_E = \dfrac{\alpha_{ED}}{\alpha_{EE}}}$

Muller-Breslau:



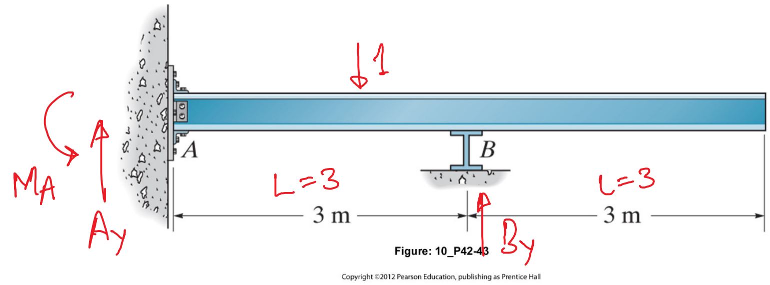
PVW $\Rightarrow \boxed{M_E = \delta y'}$



Example

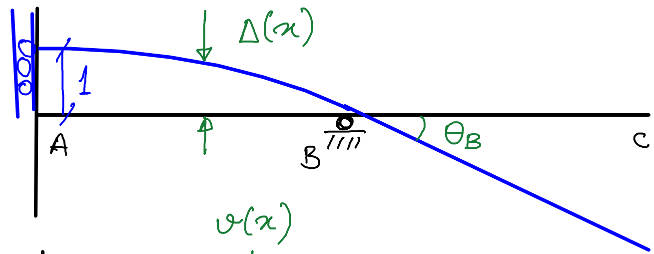
Draw the influence line for

- Vertical reaction at A
- Moment at A



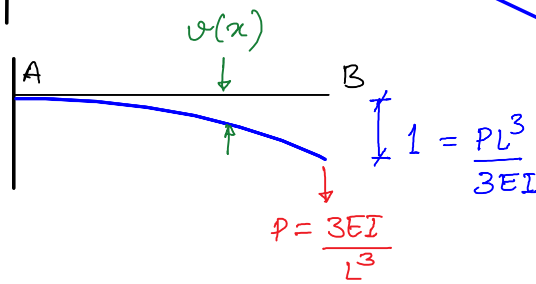
Using Müller-Breslau principle:

(i) Remove Vertical reaction at A and give unit displacement



(ii) To find the equation of the deflected curve:

Consider AB as a cantilever



Using table in the book:

$$v(x) = \frac{-P}{6EI}\left(x^3 - 3Lx^2\right) = -\frac{1}{2}\left(\frac{x^3}{L^3} - \frac{3x^2}{L^2}\right)$$

For $0 < x < 3$:

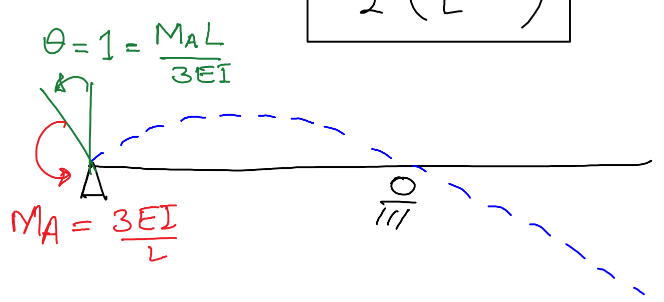
$$\Rightarrow \text{Influence line} = \Delta(x) = 1 - v(x) = \boxed{1 - \tfrac{1}{2}\left(\frac{x^3}{L^3} - \frac{3x^2}{L^2}\right)}$$

For $3 < x < 6$:

$$\text{Influence line} = \Delta(x) = -\theta_B\,(x-L) = -\frac{PL^2}{2EI}(x-L)$$

$$= \boxed{-\frac{3}{2}\left(\frac{x}{L} - 1\right)}$$

Similarly for Moment at A:



For $0 < x < 3$

$$\Delta(x) = \boxed{\frac{(L-x)}{2}\frac{(x^2)}{L^2}}$$

For $3 < x < 6$

$$\Delta(x) = -\theta_B\,(x-L) = \boxed{-\tfrac{1}{2}(x-L)}$$