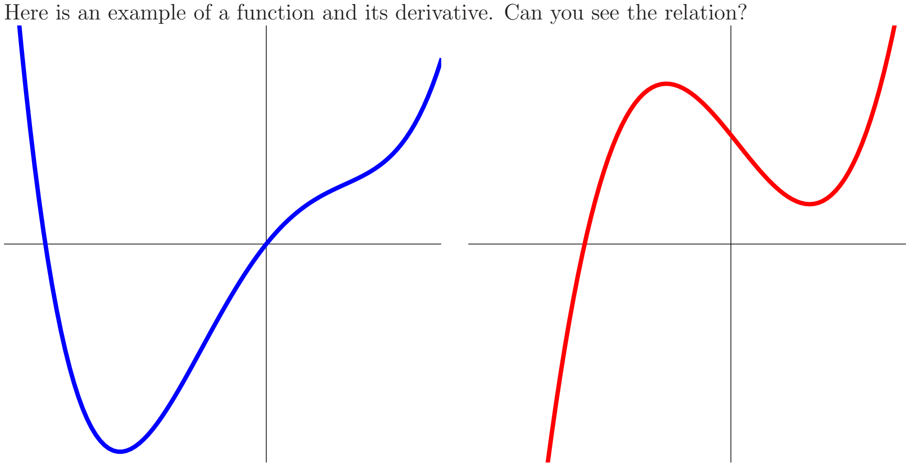
Lecture 8: The derivative function

In the last lecture, we have introduced the derivative $f'(x) = \frac{d}{dx}f(x)$ as a limit of $Df(x)$ for $h \to 0$. We have seen that $\frac{d}{dx}x^n = nx^{n-1}$ holds for integer n. We also know already that $\sin' = \cos, \cos' = -\sin$ and $\exp' = \exp$. We can already differentiate a lot of functions and evaluate the derivative $f'(x)$ at some point x. This is the slope of the curve at x.

1 Find the derivative $f'(x)$ of $f(x) = \sin(\pi x) + \cos(\pi x) - \sqrt{x} + 1/x + x^4$ and evaluate it at $x = 1$. **Solution**: $f'(x) = \pi\cos(\pi x) - \pi\sin(\pi x) - 1/(2\sqrt{x} - 1/x^2 + 4x^3$. Plugging in $x = 1$ gives $-\pi - 1/2 - 1 + 4$.

Taking the derivative at every point defines a new function, the **derivative function**. For example, for $f(x) = \sin(x)$, we get $f'(x) = \cos(x)$. In this lecture, we want to understand the new function and its relation with f. What does it mean if $f'(x) > 0$. What does it mean that $f'(x) < 0$. Do the roots of f tell something about f' or do the roots of f' tell something about f?

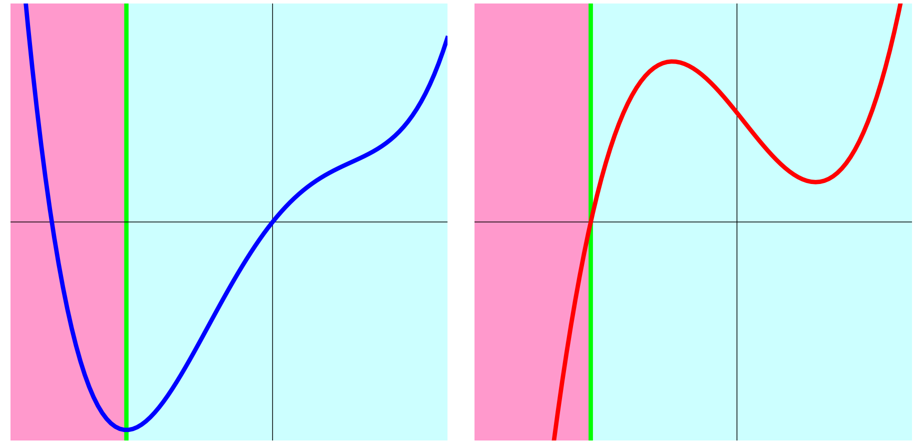
Here is an example of a function and its derivative. Can you see the relation?



To understand the relation, it is good to distinguish intervals, where $f(x)$ is increasing or decreasing. This are the intervals where $f'(x)$ is positive or negative.

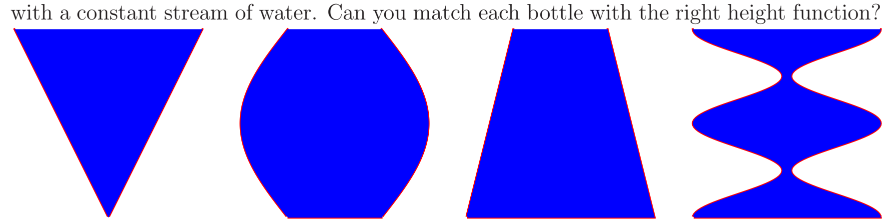
A function is called **monotonically increasing** on an interval $I = (a, b)$ if $f'(x) > 0$ for all $x \in (a, b)$. It is **monotonically decreasing** if $f'(x) < 0$ for all $x \in (a, b)$.

Lets look at the previous example again.

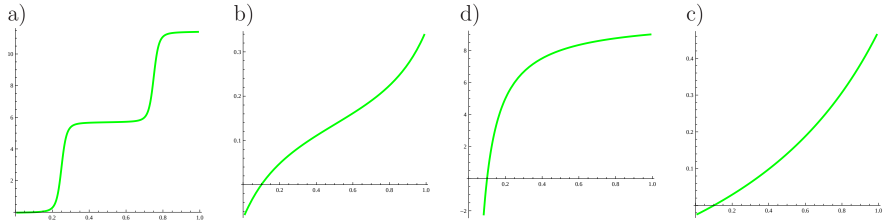


Here is an interesting inverse problem called **bottle calibration problem**. We fill a circular bottle or glass with constant amount of fluid. Plot the height of the fluid in the bottle at time t. Assume the radius of the bottle is $f(z)$ at height z. Can you find a formula for the height $g(t)$ of the water? This is not so easy. But we can find the rate of change $g'(t)$. Assume for example that f is constant, then the rate of change is constant and the height of the water increases linearly like $g(t) = t$. If the bottle gets wider, then the height of the water increases slower. There is definitely a relation between the rate of change of g and f. Before we look at this more closely, lets try to match the following cases of bottles with the graphs of the functions g qualitatively.

2 In each of the bottles, we call g the height of the water level at time t, when filling the bottle with a constant stream of water. Can you match each bottle with the right height function?



a) b) d) c)



1) 2) 3) 4)

The key is to look at $g'(t)$, the rate of change of the height function. Because $[g(t+h) - g(t)]$ times the area πf^2 is a constant times the time difference $h = dt$, we have

$$g' = \frac{1}{\pi f^2} .$$

This formula relates the derivative function of g with the thickness $f(t)$ of the bottle at height g. It tells that if f is large, then g' is small and if f is small, then g' is large. Finding g from f is possible but we are not doing this now.

3 Can you find a function f which is bounded $|f(x)| \leq 1$ and such that $f'(x)$ is unbounded? s

> Given the function $f(x)$, we can define $g(x) = f'(x)$ and then take the derivative g' of g. This second derivative $f''(x)$ is called the **acceleration**. It measures the rate of change of the tangent slope. For $f(x) = x^4$, for example we have $f''(x) = 12x^2$. If $f''(x) > 0$ on some interval the function is called **concave up**, if $f''(x) < 0$, it is **concave down**.

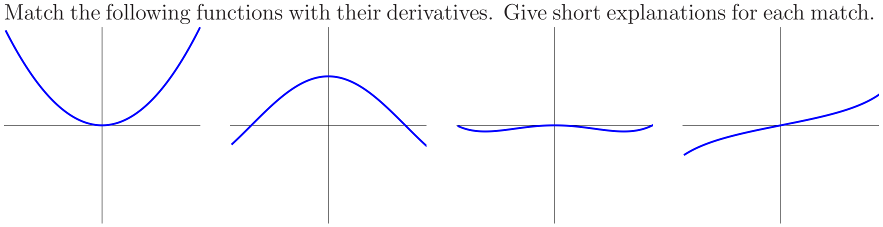
4 Find a function f which has the property that its acceleration is constant equal to 10.

5 Can you find a function f which is bounded $|f(x)| \leq 1$ and such that $f''(x)$ is positive everywhere?

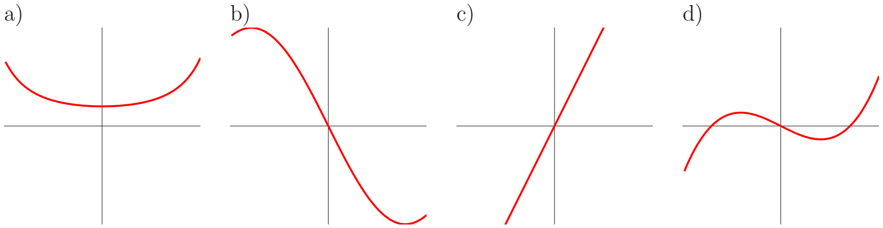
Homework

1 For the following functions, determine on which intervals the function is monotonically increasing or decreasing.
a) $f(x) = x^3 - x$.
b) $f(x) = \sin(\pi x)$.
c) $f(x) = e^{2x} - 2e^x$

2 Match the following functions with their derivatives. Give short explanations for each match.



a) b) c) d)



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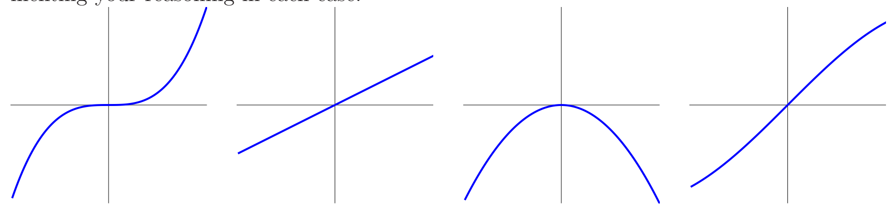
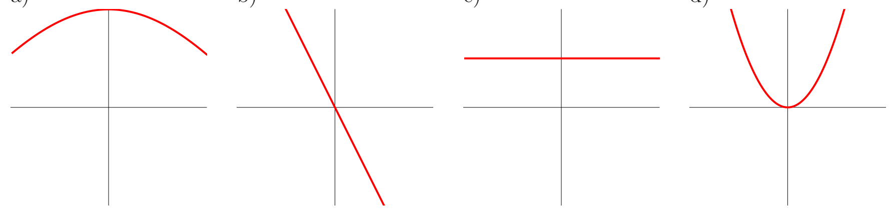
3 Match also the following functions with their derivatives. Give short explanations documenting your reasoning in each case.



a) b) c) d)



1) 2) 3) 4)

4 Draw for the following functions the graph of the function $f(x)$ as well as the graph of its derivative $f'(x)$. You do not have to compute the derivative analytically as a formula here since we do not have all tools yet to compute the derivatives. The derivative function you draw needs to have the right qualitative shape however.
a) The Gaussian bell curve or the **"To whom the bell tolls"** function

$$f(x) = e^{-x^2}$$

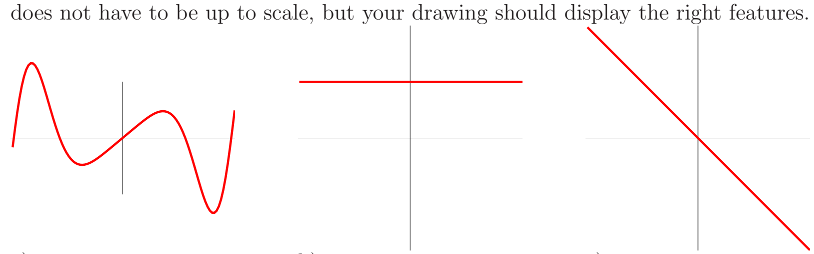
b) The **witch of Maria Agnesi**.

$$f(x) = \frac{1}{1+x^2}$$

c) The **three gorges function**

$$f(x) = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} .$$

5 Below you the graphs of three derivative functions $f'(x)$. In each case you are told that $f(0) = 1$. Your task is to draw the function $f(x)$ in each of the cases a),b),c). Your picture does not have to be up to scale, but your drawing should display the right features.



a) b) c)